Reflected Backward SPDEs and Optimal Stopping Problems

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American Option Problem

Dividend paying stock:

$$X_t^{s,x} = x + \int_s^t (r-d) X_\theta^{s,x} d\theta + \int_s^t \sigma X_\theta^{s,x} dW_\theta, \ t \in [s,T];$$

Arbitrage-free value of American option:

$$V(s,x) = \sup_{s \le \tau \le T} E e^{-r(\tau - s)} g(X_\tau^{s,x}), \ (\, g(x) = (x-K)^+ \,).$$

Reflected BSDE (El Karoui-Kapoudjian-Pardoux-Peng-Quenes, 1997,Ann. Prob.):

$$\begin{cases} Y_t^{s,x} = g(X_T^{s,x}) - \int_t^T r Y_\theta^{s,x} d\theta + K_T^{s,x} - K_t^{s,x} - \int_t^T Z_\theta^{s,x} dW_\theta, \ t \in [s,T], \\ Y_t^{s,x} \ge g(X_t^{s,x}), \quad t \in [s,T], \\ K^{s,x} \text{ is continuously increasing, } K_s^{t,x} = 0, \ \int_s^T (Y_t^{s,x} - g(X_t^{s,x}))\, dK_t^{s,x} = 0. \end{cases}$$

Benth-Karlsen-Reikvam (2003), Klimsiak-Rozkosz (2010)

$$V(s,x) = Y_s^{s,x}, \ K_t^{s,x} = \int_s^t (dX_\theta^{s,x} - rK)^+ 1_{Y_\theta^{s,x} = g(X_\theta^{s,x})}\, d\theta$$

quasi-variational inequality:

$$\begin{cases} \min\{u(s,x)-g(x),\ -\mathcal{L}_{BS}u(s,x)+ru(s,x)\}=0, \\ u(T,x)=g(x). \\ \mathcal{L}_{BS}=\partial_s u+(r-d)x\partial_x u+\dfrac{1}{2}\sigma^2x^2\partial_{xx}^2u. \end{cases}$$

$$\Leftrightarrow \begin{cases} \mathcal{L}_{BS}u(s,x)=ru(s,x)-\mu, \\ u(T)=g,\ u\geq g,\ \displaystyle\int_{[0,T]\times\mathbb{R}}(u-g)\varrho^2 d\mu=0. \end{cases}$$

$$(Y_t^{s,x},Z_t^{s,x})=(u(s,X_t^{s,x}),\sigma x\partial_x u(t,X_t^{s,x})),$$

$$\mu(dt,dx)=q(x,u(t,x))dtdx, \quad q(x,y)=(dx-rK)^+1_{(-\infty,g(x)]}(y).$$

- How about Non-Markovian case ?

Introduction

Starting point: quasi-linear Backward SPDE (BSPDE):

$$\begin{cases} -du(t,x) = \Big[\frac{1}{2}\Delta u(t,x) + \big(f + \operatorname{div} g\big)(t,x,u(t,x),\nabla u(t,x),v(t,x))\Big]\,dt \\ \qquad\qquad \sum_{r=1}^{m} v^r(t,x)\,dW_t^r,\ \ (t,x)\in[0,T]\times\mathbb{R}^d; \\ u(T,x) = G(x), \quad x\in\mathbb{R}^d. \end{cases}$$

- $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$ complete probability space with filtration;
- W: m-dimensional BM;
- $G\in L^2(\Omega,\mathscr{F}_T;L^2(\mathbb{R}^d))$;
- f and g satisfy Lipschitz conditions.

Introduction continued

Known results:

- Existence and uniqueness of the weak solution;
- The solution satisfies Itô's formula;
- Comparison Theorem;
- Maximum principles for Backward SPDEs on bounded domains;
- ...

Our aims

The obstacle problem for BSPDEs leads to reflected BSPDE (RBSPDE)

$$
\begin{cases}
-du(t,x) = \left[\frac{1}{2}\Delta u(t,x) + \big(f + \operatorname{div} g\big)(t,x,u,\nabla u, v)\right] dt \\
\qquad\qquad + \mu(dt,dx) - v^r(t,x)\, dW_t^r, \ (t,x) \in [0,T] \times \mathbb{R}^d; \\
u(t,x) \geq \xi(t,x), \quad \mathbb{P} \otimes \mathrm{dtdx\text{-}a.e.}; \\
u(T,x) = G(x); \int_0^T \int_{\mathbb{R}^d} (\bar{u}(s,x) - \xi(s,x))\, \mu(dx,ds) = 0, \ a.s..
\end{cases} \tag{1}
$$

- Unique solvability of RBSPDE (1), unknown is the triple (u, v, μ);
- Its connections with optimal stopping problems.

Two existing results

- Optimal stopping problems with random coefficients :
  Chang-Pang-Yong, SCION, (2008);
- Singular control problems of SPDEs:
  ∅ksendal-Sulem-Zhang, INRIA, (2011).
- Results with $\mu(dt,dx)=k(t,x)dt$.

Notations

- Continuous Hunt process $(\Omega', B_t, \theta_t, \mathscr{F}^0, \mathscr{F}_t^0, \mathbb{P}^x)$:
  $\Omega' := C([0,\infty);\mathbb{R}^d)$;
  $(B_t)_{t\geq 0}$: d-dim Brownian motion starting from distribution dx;
  $\mathbb{P}^{dx} := (B_\cdot)^{-1}(dx \otimes \mathbb{P}^0)$;
- $(L^2(\mathbb{R}^d), \langle\cdot,\cdot\rangle, \|\cdot\|_2)$, $(H^1(\mathbb{R}^d), \langle\cdot,\cdot\rangle_1, \|\cdot\|_{H^1})$;
  For each Banach space $(V, \|\cdot\|_V)$,
- $\mathcal{S}^2(V)$: V-valued, $(\mathscr{F}_t)$-adapted and continuous processes $(X_t)_{t\in[0,T]}$, s.t.

$$\|X\|_{\mathcal{S}^2(V)} := \left(E[\sup_{t\in[0,T]} \|X_t\|_V^2]\right)^{1/2} < \infty;$$

- $\mathcal{L}^2(V)$: $\|X\|_{\mathcal{L}^2(V)} := \left(E[\int_0^T \|X_t\|_V^2\, dt]\right)^{1/2} < \infty$;
- $\mathcal{H} =: \mathcal{S}^2(L^2(\mathbb{R}^d)) \cap \mathcal{L}^2(H^1(\mathbb{R}^d))$ equipped with norm

$$\|\phi\|_{\mathcal{H}} := \left(\|\phi\|^2_{\mathcal{S}^2(L^2(\mathbb{R}^d))} + \|\nabla\phi_t\|^2_{\mathcal{L}^2(L^2(\mathbb{R}^d))}\right)^{1/2}, \quad \phi \in \mathcal{H}.$$

Assumptions

$(\mathcal{A}1)$ *The pair of random functions*

$$f(\cdot,\cdot,\cdot,\vartheta,y,z):\ \Omega\times[0,T]\times\mathbb{R}^d\to\mathbb{R}\ \textit{and}\ g(\cdot,\cdot,\cdot,\vartheta,y,z):\ \Omega\times[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$$

are $\mathscr{P}\otimes\mathcal{B}(\mathbb{R}^d)$*-measurable for any* $(\vartheta,y,z)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m$*. There exist positive constants* $\kappa<1/2$ *and* L *such that for all* $(\vartheta_1,y_1,z_1),(\vartheta_2,y_2,z_2)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n\times m}$*,* $\varphi_1,\varphi_2\in L^2(\mathbb{R}^d)$*,* $\phi_1,\phi_2\in(L^2(\mathbb{R}^d))^m$ *and* $(\omega,t,x)\in\Omega\times[0,T]\times\mathbb{R}^d$

$$|f(\omega,t,x,\vartheta_1,y_1,z_1)-f(\omega,t,x,\vartheta_2,y_2,z_2)|\le L|\vartheta_1-\vartheta_2|+|y_1-y_2|+|z_1-z_2|;$$

$$|g(\omega,t,x,\vartheta_1,y_1,z_1)-g(\omega,t,x,\vartheta_2,y_2,z_2)|\le L(|\vartheta_1-\vartheta_2|+|y_1-y_2|+|z_1-z_2|);$$

$$-\langle\nabla(\varphi_1-\varphi_2),\,g(t,\varphi_1,\nabla\varphi_1,\phi_1)-g(t,\varphi_2,\nabla\varphi_2,\phi_2)\rangle$$

$$\le\ \kappa\left(\|\nabla(\varphi_1-\varphi_2)\|_2^2+\|\phi_1-\phi_2\|_2^2\right)+L\|\varphi_1-\varphi_2\|_2^2.$$

Assumptions continued

$(\mathcal{A}2)$ $G \in L^2(\Omega, \mathscr{F}_T; L^2(\mathbb{R}^d))$.

$$f_0 := f(\cdot,\cdot,\cdot,0,0,0) \in \mathcal{L}^2(L^2(\mathbb{R}^d)),\ g_0 := g(\cdot,\cdot,\cdot,0,0,0) \in \mathcal{L}^2(L^2(\mathbb{R}^d)^m).$$

$(\mathcal{A}3)$ *The obstacle process* $\xi(\omega,t,x)$ *is a predictable random function with respect to filtration* $(\mathscr{F}_t)_{t\in[0,T]}$ *and* $t \mapsto \xi(\omega,t,B_t)$ *is* $\mathbb{P}\otimes\mathbb{P}^{dx}$*-a.s. continuous on* $[0,T]$ *and satisfies*

$$EE^{dx}\Big[\sup_{t\in[0,T]} |\xi^+(t,B_t)|^2\Big] < \infty \ and\ \xi(T,\omega) \leq G,\ \mathbb{P}\otimes dx\text{-}a.e..$$

Definition

Random function $u : \Omega \times [0,T] \times \mathbb{R}^d \to \bar{\mathbb{R}}$ is said to be stochastic quasi-continuous provided that for each $\varepsilon > 0$, there exists a predictable random set $D^\varepsilon \subset \Omega \times [0,T] \times \mathbb{R}^d$ such that $\mathbb{P}$-a.s. the section D^ε_ω is open and $u(\omega,\cdot,\cdot)$ is continuous on its complement $(D^\varepsilon_\omega)^c$ and

$$\mathbb{P} \otimes \mathbb{P}^{dx} \left((\omega,\omega') | \exists t \in [0,T] \text{ s.t. } (\omega, t, B_t(\omega')) \in D^\varepsilon \right) \leq \varepsilon.$$

Remark

If u is stochastic quasi-continuous, we can check that the process $u(t,B_t)_{t\in[0,T]}$ has continuous trajectories, $\mathbb{P} \otimes \mathbb{P}^{dx}$-a.s..

Quasi-continuity of the weak solutions for BSPDEs

Consider BSPDE

$$\begin{cases} -du(t,x) = \left[\frac{1}{2}\Delta u(t,x) + \big(f + \operatorname{div} g\big)(t,x,u(t,x),\nabla u(t,x),v(t,x))\right] dt \\ \qquad\qquad - \sum_{r=1}^{m} v^r(t,x)\, dW_t^r, \ (t,x) \in [0,T] \times \mathbb{R}^d; \\ u(T,x) = G(x), \quad x \in \mathbb{R}^d. \end{cases}$$

Theorem

Let $(\mathcal{A}1)$ and $(\mathcal{A}2)$ hold. Then the BSPDE above admits a unique weak solution pair

$$(u,v) \in \mathcal{H} \times \mathcal{L}^2((L^2(\mathbb{R}^6)^m))$$

Moreover, u admits a stochastic quasi-continuous version.

Definition

$u \in \mathcal{H}$ is called a stochastic potential, provided that u is stochastic quasi-continuous, $\lim_{t\to T} u(t,\cdot) = 0$ in $L^2(\mathbb{R}^d)$, a.s.,

$$EE^{dx}\left[\sup_{t\in[0,T]} |u(t,B_t)|^2\right] < \infty, \tag{2}$$

and

$$E\left[(\tilde{P}_s u)(t)\middle|\mathscr{F}_t\right] \leq u(t),\ \forall s>0, \forall t \in [0,T], a.s.. \tag{3}$$

where the conditional expectation is defined in the Hilbertian sense and

$$\tilde{P}_s u(t,x) := \begin{cases} \displaystyle\int_{\mathbb{R}^d} \rho_s(x-y)u(t+s,y)\,dy, & \text{if } s+t \leq T; \\ 0, & \text{otherwise,} \end{cases}$$

with $\rho_s(x) = (2\pi s)^{-d/2}\exp(-|x|^2/2s)$.

Theorem

Let $u \in \mathcal{H}$. Then u admits a version which is a stochastic potential if and only if there exist stochastic field $v \in \mathcal{L}^2((L^2(\mathbb{R}^d))^m)$ and a continuous increasing process $A = (A_t)_{t\in[0,T]}$ which is $\mathscr{F}_t \vee \mathscr{F}_t^0$-adapted and such that $A_0 = 0$, $EE^{dx}\left[A_T^2\right] < \infty$, and
(i)

$$u(t,B_t) = A_T - A_t - \sum_{i=1}^{d}\int_t^T \partial_{x^i} u(s,B_s)\,dB_s^i - \sum_{r=1}^{m}\int_t^T v^r(s,B_s)\,dW_s^r,\ \mathbb{P}\otimes\mathbb{P}^{dx} - a.s.$$

for each $t \in [0,T]$. The processes A and v are uniquely determined by those properties. Moreover, there hold the following relations:
(ii)

$$E\left[\|u(t)\|_2^2 + \int_t^T \left(\|\nabla u(s)\|_2^2 + \|v(s)\|_2^2\right)\,ds\right]$$
$$= EE^{dx}\left[(A_T - A_t)^2\right], \quad \forall t \in [0,T];$$

Theorem (continued)

(iii) for any $\varphi \in \mathcal{D}_T$,

$$\langle u(0), \varphi(0)\rangle + \int_0^T \left(\frac{1}{2} \langle \nabla u(s), \nabla \varphi(s)\rangle \right) + \langle u(s), \partial_s \varphi(s)\rangle \, ds + \sum_{r=1}^{m} \int_0^T \langle \varphi(s), v^r(s, B_s)$$

$$= \mu(\varphi) = \int_0^T \int_{\mathbb{R}^d} \varphi(s,x) \mu(dx, ds),$$

where μ is the random measure $\mu : \Omega \to \mathcal{M}([0,T] \times \mathbb{R}^d)$

(iv) $$\mu(\varphi) = E^{dx} \int_0^T \varphi(t, B_t) \, dA_t, \;\; \varphi \in \mathcal{D}_T, \; a.s.,$$

with $\mathcal{M}([0,T] \times \mathbb{R}^d)$ denoting the set of all the Radon measures on $[0,T] \times \mathbb{R}^d$.

Lemma

Let u be a stochastic potential and $\mu : \Omega \to \mathcal{M}([0,T] \times \mathbb{R}^d)$ a random Radon measure such that relations (iii) holds. Then one has

$$\langle \phi, u(t) \rangle = E\left[\int_t^T \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \phi(x) \rho_{s-t}(x,y) dx \right) \mu(dy, ds) \middle| \mathscr{F}_t \right], \tag{4}$$

for each $\phi \in L^2(\mathbb{R}^d)$ and $t \in [0,T]$.

Definition

A nonnegative random Radon measure $\mu : \Omega \to \mathcal{M}([0,T] \times \mathbb{R}^d)$ is called regular stochastic measure provided that there exists a stochastic potential u such that the relation (iii) from the above theorem is satisfied.

Remark

As $EE^{dx}\left[A_T^2\right] < \infty$, for any random field $\phi \in \mathcal{L}^2(L^2(\mathbb{R}^d))$ satisfying

$$\phi(t,B_t) \text{ is continuous } \mathbb{P}\otimes\mathbb{P}^{dx}\text{-a.s., and } EE^{dx}\left[\sup_{t\in[0,T]} |\phi(t,B_t)|^2\right] < \infty,$$

$\mu(\phi)$ makes sense by relation (iv).

Proposition A

Let $\{u^n; n\in\mathbb{N}\}$ be a sequence of stochastic potentials associated with $\{(v^n,\mu^n); n\in\mathbb{N}\}$ such that $u^n \to u$ in $\mathcal{H}$ and $v^n \to v$ in $\mathcal{L}^2((L^2(\mathbb{R}^d))^m)$ respectively. Then for some regular stochastic measure μ, u is a stochastic potential associated with (v,μ).

Proposition B

Let $\{u^n; n\in\mathbb{N}\}$ be a sequence of stochastic potential which converges up to some $u\in\mathcal{H}$. Assume moreover that u is quasi-continuous and $EE^{dx}\left[\sup_{t\in[0,T]}|u(t,B_t)|^2\right] < \infty$. Then u is a stochastic potential.

Definition

We say that a triple (u, v, μ) is a weak solution of the RBSPDE (1) associated to (G, f, g, ξ), if
(1) $u \in \mathcal{H}$, $u(t,x) \geq \xi(t,x)$, $\mathbb{P} \otimes dt \otimes dx$-a.e. and $u(T,x) = G$, $\mathbb{P} \otimes dx$-a.e.
(2) $\mu : \Omega \to \mathcal{M}([0,T] \times \mathbb{R}^d)$ is a regular stochastic measure;
(3) for each $\varphi \in \mathcal{D}_T$ and $t \in [0,T]$

$$
\begin{aligned}
&\langle u(t),\, \varphi(t)\rangle + \int_t^T \left[\langle u(s),\, \partial_s \varphi(s)\rangle + \frac{1}{2}\langle \nabla u(s),\, \nabla \varphi(s)\rangle \right] ds \\
&= \langle G, \varphi(T)\rangle + \int_t^T \left[\langle f(s,u,\nabla u, v),\, \varphi(s)\rangle - \langle g(s,u,\nabla u,v),\, \nabla \varphi(s)\rangle \right] ds \\
&\quad + \int_t^T \int_{\mathbb{R}^d} \varphi(s,x) \mu(ds,dx) - \sum_{r=1}^m \int_t^T \langle \varphi(s),\, v^r(s)\, dW_s^r \rangle;
\end{aligned}
$$

(4) u admits a stochastic quasi-continuous version $\bar{u}$ such that

$$
\int_0^T \int_{\mathbb{R}^d} (\bar{u}(s,x) - \xi(s,x))\, \mu(dx,ds) = 0, \quad a.s..
$$

Decomposition of the Solution

- Let (u, v, μ) be a weak solution of RBSPDE (1) and (u_μ, v_μ) corresponds to the regular stochastic measure μ with u_μ being the stochastic potential.
- Clearly, $(u_0, v_0) := (u - u_\mu, v - v_\mu)$ solves the following BSPDE without obstacle:

$$-du(t,x) = \left[\frac{1}{2}\Delta u(t,x) + \big(f + \operatorname{div} g\big)(t, x, u(t,x), \nabla u(t,x), v(t,x))\right] dt \\ - \sum_{r=1}^{m} v^r(t,x)\, dW_t^r;$$

- $u = u_0 + u_\mu$ must be stochastic quasi-continuous.

Existence and Uniqueness Theorem

Let assumptions $(\mathcal{A}1) - (\mathcal{A}3)$ hold. Then there exists a unique weak solution (u, v, μ) of RBSPDE (1) associated with (G, f, g, ξ).

Comparison theorem

Let $\tilde{G}, \tilde{f}, \tilde{\xi}$ satisfy the same hypothesis as G, f, ξ. And let (u, v, μ) be the weak solution of RBSPDE (1) associated with (G, f, g, ξ) and $(\tilde{u}, \tilde{v}, \tilde{\mu})$ the weak solution associated with $(\tilde{G}, \tilde{f}, g, \tilde{\xi})$. Moreover, we assume that there hold the following conditions:
(i) $G \leq \tilde{G}$, $\mathbb{P} \otimes dx$-a.e.;
(ii) $f(u, \nabla u, v) \leq \tilde{f}(u, \nabla u, v)$, $\mathbb{P} \otimes dtdx$-a.e.;
(iii) $\xi \leq \tilde{\xi}$, $\mathbb{P} \otimes dtdx$-a.e..
Then one has $u \leq \tilde{u}$, $\mathbb{P} \otimes dtdx$-a.e..

RBSPDEs and optimal stopping problems

Let (u, v, μ) be the weak solution of RBSPDE (1). Denote

$$(Y_t, Z_t, \tilde{Z}_t, \zeta_t) = (u, \nabla u, v, \xi)(t, B_t), \quad t \in [0, T].$$

$(K_t)_{t\in[0,T]}$ is the increasing process w.r.t. μ. Then $(Y, Z, \tilde{Z}, K)$ solves RBSDE:

$$\begin{cases} Y_t = G + \int_t^T f(s, B_s, Y_s, Z_s, \tilde{Z}_s)\, ds + \int_t^T g(s, B_s, Y_s, Z_s, \tilde{Z}_s) * dB_s \\ \qquad - \int_t^T Z_s\, dB_s - \int_t^T \tilde{Z}_s\, dW_s, \quad t \in [0, T]; \\ Y_t \geq \zeta_s, \quad t \in [0, T]; \quad \int_0^T (Y_s - \zeta_s)\, dK_s = 0. \end{cases}$$

Hence,

$$u(t, B_t) = \operatorname*{ess\,sup}_{\tau \in S_{t,T}} E E^{dx} \left[\int_t^\tau f_s\, ds + \int_t^\tau g_s * dB_s + \zeta_\tau 1_{\tau < T} + G(T, B_T) 1_{\tau = T} \middle| \mathscr{F}_t \right].$$

Future work

- Analytic approach to Reflected BSPDEs;
- Reflected BSPDE on domains;
- Degenerate case;
- regularity problems;
- Applications $\cdots$

Thank You !