FBSDEs for expected utility maximization

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The financial market:

Fix $T > 0$. Let $W := (W_t)_{t\in[0,T]}$ and $W^o := (W_t^o)_{t\in[0,T]}$ be two independent Brownian motions defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. And let $(\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by (W, W^o).

Market:

- a risk less bond B with interest rate 0,
- A risky process $S := (S_t)_{t\in[0,T]}$ with dynamics $dS_t = S_t(dW_t + \theta_t dt)$ on which the agent can invest,
- A risky process $S^o := (S_t^o)_{t\in[0,T]}$ with dynamics $dS_t^o = S_t^o(\beta(t, S_t)dW_t^o + \gamma(t, S_t)dW_t + \delta(t, S_t)dt)$ which the agent has not access to.

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Example:

- S: heating oil, S^o: jet fuel
- W^o is an exogenous source of risk like a temperature process.

The investment problem:

Consider:

$$V(0,x) := \sup_{\pi \in \mathcal{A}} \mathbb{E}[U(X_T^\pi + H)]$$

where:

- $U : \mathbb{R}_+ \to \mathbb{R}$ is a general utility function,
- $$X_t^\pi := x + \int_0^t \pi_r X_r^\pi \frac{dS_r}{S_r}$$
  denotes the wealth process associated to a $(\mathcal{F}_t)_{t\in[0,T]}$ self-financing trading strategy $\pi := (\pi_t)_{t\in[0,T]}$,
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Usually one is interested in showing that:

1) $\exists \pi^*$ admissible such that $V(0,x) = \mathbb{E}[U(X_T^{\pi^*} + H)]$,
2) simulate the optimal strategy π^*,
3) simulate the value function

$$V(t,x) := \text{esssup}_\pi \, \mathbb{E}\left[U\left(x + \int_t^T \pi_r X_r^\pi dS_r + H \right) \middle| \mathcal{F}_t \right],$$

These methods have been introduced and studied by Bismut, Cvitanić, Hugonnier, Karatzas, Kramkov, Schachermayer, Wang,....

Introduce a dual problem:

$$v(y) := \inf_{Y_T \in \mathcal{Y}} \mathbb{E}[\mathcal{U}(yY_T)], \; y > 0$$

where $\mathcal{U}(y) := \sup_{x>0}\{U(x) - xy\}, \; y > 0$.

Under some kind of "growth type" condition on U, one can find a solution to the dual problem, namely

$$\exists Y_T^*, \quad v(y) = \mathbb{E}[V(yY_T^*)].$$

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What about 2) and 3)? To simulate π^* and $V(t,x)$ we need an equation.

(F)BSDE techniques

We would like to mimic the method of Hu, Imkeller and Müller or of El Karoui and Rouge for a general utility function with a general endowment.

The idea: combine martingale optimality principle with BSDEs to reduce the optimization problem to solving a BSDE of the form:

$$Y_t = \xi - \int_t^T Z_s dW_s - \int_t^T f(s, Z_s) ds, \quad t \in [0, T].$$

Then:

- $f \sim z^2$,
- $V(t, x)$ is given as $\phi(x, Y_t)$,
- π^* is completely characterized by Z,
- But: this is restricted to the case $U(x) := x^\gamma$ and $H = 0$.

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  As an example what can we do for:
- $U(x) := x^{\gamma_1} + x^{\gamma_2}$ and $H = 0$? or for
- $U(x) := x^\gamma$ and $H \neq 0$?

Mania and Tevzadze have derived a verification theorem for $H = 0$.

They have obtained a BSPDE

$$V(t,x) = U(x) - \int_t^T \varphi(s,x)dW_s - \int_t^T \frac{|\varphi_x(s,x)|^2}{V_{xx}(s,x)}ds, \quad t \in [0,T]$$

for the value function.

Let U be smooth enough and $H > 0$.

Theorem (HHIRZ)

Let (X, Y, Z) be an adapted solution of the FBSDE

$$\begin{cases} X_t = x - \int_0^t \frac{U'(X_s)}{U''(X_s)}(Z_s + \theta_s)dW_s - \int_0^t \frac{U'(X_s)}{U''(X_s)}(Z_s + \theta_s)\theta_s ds, \\ Y_t = \log\left(\frac{U'(X_T+H)}{U'(X_T)}\right) - \int_t^T \left[(|Z_s + \theta_s|^2)\left(1 - \frac{1}{2}\frac{U^{(3)}(X_s)U'(X_s)}{|U''(X_s)|^2}\right) - \frac{1}{2}|Z_s + Z_s^o|^2\right] ds \\ \qquad - \int_t^T Z_s dW_s - \int_t^T Z_s^o dW_s^o \end{cases} \tag{1}$$

such that (Z, Z^o) is an element of $\mathbb{H}^2(\mathbb{R}^2)$ and the positive local martingale $XU'(X)\exp(Y)$ is a true martingale. Then

$$\pi_t^* := -\frac{U'(X_t)}{X_t U''(X_t)}(Z_t + \theta_t), \quad t \in [0, T]$$

is an optimal solution to the original optimization problem.

Approach with FBSDEs

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Main ingredients: variational approach + the fact that $X^{\pi^*} U'(X^{\pi^*})\exp(Y)$ is a martingale.

What is the role of the process Y?

Let (X, Y, Z) be an adapted solution of the FBSDE above. Then $X^{\pi^*} U'(X^{\pi^*}) \exp(Y)$ is a martingale and the process $D_t := U'(X_t^{\pi^*}) \exp(Y_t)$ is given by

$$D = cst. \times \mathcal{E}\left(- \int_0^{\cdot} \theta_r dW_r + \int_0^{\cdot} Z_r^o dW_r^o \right).$$

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So basically $D = Y^*$.

Let $U(x) := x^\gamma$, $\gamma \in (0,1)$, $H > 0$ be a bounded $\mathcal{F}_T$-measurable random variable.

Theorem (HHIRZ)

There exists $x_0 > 0$, such that for every $x > x_0$, the system

$$\begin{cases} X_t = x - \int_0^t \frac{X_s(Z_s+\theta_s)}{1-\gamma} dW_s - \int_0^t \frac{X_s(Z_s+\theta_s)}{1-\gamma}\theta_s ds, \\ Y_t = (\gamma - 1)\log\left(1 + \frac{H}{X_T}\right) - \int_t^T \left[\frac{\gamma}{2(\gamma-1)}|Z_s + \theta_s|^2 - \frac{|Z_s|^2+|Z_s^o|^2}{2}\right] ds \\ \qquad - \int_t^T Z_s dW_s - \int_t^T Z_s^o dW_s^o \end{cases} \tag{2}$$

admits a solution. If in addition, Z belongs to $\mathbb{H}^2(\mathbb{R})$ then

$$\pi^* := \frac{1}{1-\gamma}(Z + \theta)$$

is the optimal solution to the maximization problem.