Time discretization of quadratic and superquadratic Markovian BSDEs with unbounded terminal conditions

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framework

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(W_t)_{t \in \mathbb{R}^+}$ be a Brownian motion in $\mathbb{R}^d$, $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$ be his augmented natural filtration, T be a nonnegative real number. We consider an SDE

$$X_t = x + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s,$$

with standard assumptions on b and σ, and a Markovian BSDE

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s.$$

Time discretization

We consider a time discretization of the BSDE. We denote the time step by $h = T/n$ and $(t_k = kh)_{0 \leqslant k \leqslant n}$ stands for the discretization times. For X we take the Euler scheme :

$$\begin{aligned} X_0^n &= x \\ X_{t_{k+1}}^n &= X_{t_k}^n + hb(t_k, X_{t_k}^n) + \sigma(t_k, X_{t_k}^n)(W_{t_{k+1}} - W_{t_k}), \quad 0 \leqslant k \leqslant n. \end{aligned}$$

For (Y, Z) we use the classical dynamic programming equation

$$\begin{aligned} Y_{t_n}^n &= g(X_{t_n}^n) \\ Z_{t_k}^n &= \frac{1}{h}\mathbb{E}_{t_k}[Y_{t_{k+1}}^n(W_{t_{k+1}} - W_{t_k})], \quad 0 \leqslant k \leqslant n-1, \\ Y_{t_k}^n &= \mathbb{E}_{t_k}[Y_{t_{k+1}}^n] + h\mathbb{E}_{t_k}[f(t_k, X_{t_k}^n, Y_{t_{k+1}}^n, Z_{t_k}^n)], \quad 0 \leqslant k \leqslant n-1, \end{aligned}$$

where $\mathbb{E}_{t_k}$ stands for the conditional expectation given $\mathcal{F}_{t_k}$.

Remarks on simulation

- We need to compute conditional expectations.
- We have a speed of convergence.

Theorem (J. Zhang [2004], B. Bouchard, N. Touzi [2004])

Let us assume that g and f are Lipschitz functions with respect to x, y, z and t. We define the approximation error by

$$e(n) = \sup_{0 \leqslant k \leqslant n} \mathbb{E} \left| Y^n_{t_k} - Y_{t_k} \right|^2 + \mathbb{E} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left| Z^n_{t_k} - Z_t \right|^2 dt.$$

Then $e(n) = O(1/n)$.

Question

What happens when f has a quadratic or a superquadratic growth with respect to z?

Simplifications and restriction

- Simplifications :

$$
\begin{aligned}
X_t &= x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \\
Y_t &= g(X_T) + \int_t^T f(Z_s)ds - \int_t^T Z_s dW_s.
\end{aligned}
$$

- Restriction :

$$
\begin{aligned}
X_t &= x + \int_0^t b(X_s)ds + \int_0^t \sigma dW_s, \\
Y_t &= g(X_T) + \int_t^T f(Z_s)ds - \int_t^T Z_s dW_s.
\end{aligned}
$$

A simple lemma

Lemma

Let us assume that g is a Lipschitz function (not necessarily bounded) and f is locally Lipschitz :

$$\left|f(z) - f(z')\right| \leqslant C(1 + h(|z|) + h(\left|z'\right|)) \left|z - z'\right|,$$

with $h : \mathbb{R} \to \mathbb{R}$ a nondecreasing function. Then there exists a unique solution $(Y, Z) \in \mathcal{S}^2 \times \mathcal{M}^2$ such that Z is bounded.

Sketch of the proof

Let us introduce the BSDE

$$Y_t^M = g(X_T) + \int_t^T f(\rho_M(Z_s^M))ds - \int_t^T Z_s^M dW_s,$$

with ρ_M the projection on the centered euclidean ball of radius M. We have $Z_s^M = \nabla Y_s^M (\nabla X_s)^{-1}\sigma$ and

$$\begin{aligned}\nabla Y_t^M &= \nabla g(X_T)\nabla X_T + \int_t^T \nabla(f\circ\rho_M)(Z_s^M)\nabla Z_s^M ds - \int_t^T \nabla Z_s^M dW_s \\ &= \nabla g(X_T)\nabla X_T - \int_t^T \nabla Z_s^M (dW_s - \nabla(f\circ\rho_M)(Z_s^M)ds).\end{aligned}$$

We can use Girsanov :

$$\nabla Y_t^M = \mathbb{E}_t^{\mathbb{Q}^M}[\nabla g(X_T)\nabla X_T]$$

and

$$\left|Z_s^M\right| \leqslant \mathbb{E}_s^{\mathbb{Q}^M}\left[|\nabla g(X_T)|\left|\nabla X_T(\nabla X_s)^{-1}\right|\right]|\sigma| \leqslant C.$$

A remark for non markovian framework

Remark

Let us assume here that the terminal condition is $g(X)$ with g Lipschitz :

$$\left|g(x^1) - g(x^2)\right| \leqslant C \sup_{0 \leqslant s \leqslant T} \left|x_s^1 - x_s^2\right|.$$

Then the result stays true, Z is bounded.

Proof : we just have to approximate $g(X)$ by $g_n(X_{t_1}, X_{t_2}, ..., X_{t_n})$ (see J. Ma, J. Zhang [2002]).
Result already proved in P. Cheridito, M. Stadje [2012].

A generalization of the lemma

We have

$$|\nabla g| \leqslant C \quad \Rightarrow \quad |Z| \leqslant C,$$

that is to say : the growth of Z is linked with the growth of the derivative of the terminal condition.
Generalization : Could we have

$$|\nabla g| \leqslant C(1+|x|^r) \quad \Rightarrow \quad |Z| \leqslant C(1+|X|^r)?$$

Theorem

Let us assume

$$\begin{aligned} \left|g(x) - g(x')\right| &\leqslant C(1 + |x|^r + \left|x'\right|^r)\left|x - x'\right| \\ \left|f(z) - f(z')\right| &\leqslant C(1 + |z|^l + \left|z'\right|^l)\left|z - z'\right|, \end{aligned}$$

with $l \geqslant 1$ ($l = 1$: quadratic case).

Theorem

Let us assume

$$\begin{aligned}\left|g(x)-g(x')\right| &\leqslant C(1+|x|^r+\left|x'\right|^r)\left|x-x'\right| \\ \left|f(z)-f(z')\right| &\leqslant C(1+|z|^l+\left|z'\right|^l)\left|z-z'\right|,\end{aligned}$$

with $l \geqslant 1$ ($l = 1$: quadratic case).

- When $rl < 1$, there exists a solution (Y, Z) in $\mathcal{S}^2 \times \mathcal{M}^2$ s.t.

  $$|Z| \leqslant C(1+|X|^r).$$

  This solution is unique amongst solutions s.t. $Y \in \mathcal{S}^2$ and

  $$\mathbb{E}\left[e^{\tilde{C}\int_0^T |Z_s|^{2l} ds}\right] < +\infty.$$

- When $rl = 1$ the result stays true only when T is small enough.

Remarks

- In the quadratic case, the uniqueness result is new (f is not assume to be convex or concave).
- In the superquadratic case, existence and uniqueness results are new (in the paper of X. Bao, F. Delbaen and Y. Hu (2010), g is bounded).
- Uniqueness result allows us to obtain a Feynman-Kac formula.
- In the path-dependent framework, the result stays true (?) with the estimate

$$Z_t \leqslant C(1 + \sup_{0 \leqslant s \leqslant t} |X_s|^r).$$

Remark on the uniqueness

$$
\begin{aligned}
Y_t^1 - Y_t^2 &= \int_t^T f(Z_s^1) - f(Z_s^2) ds - \int_t^T Z_s^1 - Z_s^2 dW_s \\
&= -\int_t^T Z_s^1 - Z_s^2 (dW_s - \beta_s ds).
\end{aligned}
$$

"lemma"

We have always a uniqueness result in the class of processes such that we are allowed to apply Girsanov.

In our case, this class is not empty because we can apply Novikov's condition :

$$
\mathbb{E}\left[e^{C\int_0^T |Z_s|^{2l} ds}\right] \leqslant C\mathbb{E}\left[e^{C \sup 0 \leqslant s \leqslant T |X_s|^{2rl}}\right] < +\infty
$$

when $rl < 1$ or $rl = 1$ and C small enough.

Time approximation

Let us consider an initial approximation of (Y, Z) :

$$Y_t^M = g(\rho_M(X_T)) + \int_t^T f(Z_s^M)ds - \int_t^T Z_s^M dW_s.$$

Let us denote

$$(Y, Z) \xrightarrow{e_1(M)} (Y^M, Z^M) \xrightarrow{e_2(n,M)} (Y^{M,n}, Z^{M,n})$$

and

$$(Y, Z) \xrightarrow{e(n,M)} (Y^{M,n}, Z^{M,n}).$$

Time approximation

Theorem

We have

$$e(M,n) \leqslant e_1(M) + e_2(M,n) \leqslant \frac{C}{e^{C_1 M^2}} + \frac{Ce^{C_2 M^{2rl}}}{n}.$$

- When $rl < 1$, we can choose M such that $e(M,n) = o((1/n)^{1-\varepsilon})$ for all $\varepsilon > 0$.
- When $rl = 1$, we can choose M such that $e(M,n) = 0((1/n)^{\frac{C_1}{C_1+C_2}})$.

A partial result in the general case

When $l = 1$ and the terminal condition is bounded, we can threat the case

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s,$$

with the growth assumption

$$|\sigma(x)| \leqslant C(1 + |x|^\kappa).$$

Lemma

$$|Z| \leqslant C(1 + |X|^{r+\kappa}).$$

Time approximation

Theorem

When $2\kappa \leqslant 1 - r$, we have

$$e(M, n) \leqslant \frac{C}{e^{C_1 M^2}} + \frac{C e^{C_2 M^2}}{n}.$$

- When $2\kappa < 1 - r$, we can take C_1 as small as we want and we can choose M such that $e(M, n) = o((1/n)^{1-\varepsilon})$ for all $\varepsilon > 0$.
- When $2\kappa = 1 - r$, we can choose M such that $e(M, n) = 0((1/n)^{\frac{C_1}{C_1 + C_2}})$.

Two interesting (?) open questions

Is it possible to obtain a “good” speed of convergence for the time discretization scheme when

- g is locally Lipschitz, unbounded and $\sigma(x)$ is bounded ?
- g is Lipschitz and $\sigma(x)$ is Lipschitz with linear growth ?

For the second point it is already shown that we can choose M such that

$$e(M, n) \leqslant \frac{C}{(\log n)^k},$$

for all $k \in \mathbb{N}$. See P. Imkeller, G. dos Reis [2010], A. R. [2012].