Backward SDEs Driven by G-Brownian Motion

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G-normal distribution

◇ Normal distribution D^σ with variance σ: By Feynman-Kac formula, we know that $D^\sigma[\varphi] = v(1,0)$. Here v is the solution of the heat equation:

$$\partial_t v - \frac{\sigma^2}{2}\partial_{xx} v = 0, v(0,x) = \varphi(x).$$

◇ G-Normal distribution:

$$\partial_t u - G(\partial_{xx} u) = 0, u(0,x) = \varphi(x),$$

where $G(a) = \frac{1}{2}\sup_{\sigma\in[\underline{\sigma},\overline{\sigma}]}(\sigma^2 a)$.

Define $D^G(\varphi) = u(1,0)$. Then

$$D^G : C_{b,Lip}(R) \to R$$

is called G-Normal distribution.

Properties of G-normal distribution

$\diamond$ $D^G[\varphi] = D^{\bar{\sigma}}[\varphi]$, if φ is convex; $D^G[\varphi] = D^{\underline{\sigma}}[\varphi]$, if φ is concave.

$\diamond$ Assume X is G-normally distributed and $\bar{X}$ is an independent copy of X, i.e., $\bar{X} \overset{d}{=} X$ and $\bar{X} \perp X$. Then we have, for each $a, b \geq 0$,

$$aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2} X. \tag{1}$$

G-expectation

Definition 1 $\Omega_T = C_0([0,T];\mathbb{R})$, the space of real valued continuous functions on $[0,T]$ with $\omega_0 = 0$;

$B_t(\omega) = \omega_t$: the canonical process;

Set $L_{ip}(\Omega_T) := \{\varphi(B_{t_1},...,B_{t_n}) : n \geq 1, t_1,...,t_n \in [0,T], \varphi \in C_{b,Lip}(\mathbb{R}^n)\}$. G-expectation is a sublinear expectation defined by

$$\hat{\mathbb{E}}[X] = \tilde{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0}\xi_1, \cdots, \sqrt{t_m - t_{m-1}}\xi_m)],$$

for all $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})$, where $\xi_1, \cdots, \xi_n$ are i.i.d G-normally distributed random variables in a sublinear expectation space $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})$.

Conditional G-expectation

Definition 2

Let us define the conditional G-expectation $\hat{\mathbb{E}}_t$ of $\xi \in \mathcal{H}_T^0$ knowing $\mathcal{H}_t^0$, for $t \in [0, T]$. Without loss of generality we can assume that ξ has the representation $\xi = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})$ with $t = t_i$, for some $1 \leq i \leq m$, and we put

$$\hat{\mathbb{E}}_{t_i}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})]$$

$$= \tilde{\varphi}(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_i} - B_{t_{i-1}}),$$

where

$$\tilde{\varphi}(x_1, \cdots, x_i) = \hat{\mathbb{E}}[\varphi(x_1, \cdots, x_i, B_{t_{i+1}} - B_{t_i}, \cdots, B_{t_m} - B_{t_{m-1}})].$$

Representation of G-expectation

Theorem 3[DHP11] There exists a tight subset $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$, the set of probability measures on $(\Omega_T, \mathcal{B}(\Omega_T))$, such that

$$\hat{\mathbb{E}}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi] \quad \text{for} \quad \text{all } \xi \in L_{ip}(\Omega_T).$$

G-martingales

Definition 4 A process $\{M_t\}$ with values in $L_G^1(\Omega_T)$ is called a G-martingale if $\hat{E}_s(M_t) = M_s$ for any $s \leq t$. If $\{M_t\}$ and $\{-M_t\}$ are both G-martingales, we call $\{M_t\}$ a symmetric G-martingale.

$\{M_t\}$ is symmetric $\Longleftrightarrow \hat{E}(M_T) + \hat{E}(-M_T) = 0$.

• For any $Z \in M_G^2(0, T)$, $M_t = \int_0^t Z_s dB_s$ is a symmetric G-martingale.

⋄ Problem : Does any symmetric G-martingale have the above representation?

Representation of G-martingales

Theorem 5 ([P07]) For all $\xi = \varphi(B_{t_1} - B_{t_0}, \cdots, B_{t_n} - B_{t_{n-1}}) \in \mathcal{H}_T^0$,

we have the following representation:

$$\xi = \hat{E}(\xi) + \int_0^T Z_t dB_t + \int_0^T \eta_t d\langle B \rangle_t - \int_0^T 2G(\eta_t) dt.$$

where $Z \in M_G^2(0,T)$ and $\eta \in M_G^1(0,T)$.

◇ $G(a) = \frac{1}{2}[\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-]$;

◇ $K_t := \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds$ is continuous and nonincreasing!

◇ $K_t \equiv 0$ if the G-expectation reduces to the classical linear case($\overline{\sigma} = \underline{\sigma}$).

Decomposition of G-martingales

[STZ11] and [Song11] generalized Peng's result.

Theorem 6 [Song11]For $\xi \in L_G^{\beta}(\Omega_T)$ with some $\beta > 1$, $X_t = \hat{E}_t(\xi)$, $t \in [0, T]$ has the following decomposition:

$$X_t = X_0 + \int_0^t Z_s dB_s + K_t, \ q.s.$$

where $\{Z_t\} \in H_G^{\alpha}(0, T)$ and $\{K_t\}$ is a continuous decreasing G-martingale with $K_0 = 0$, $K_T \in L_G^{\alpha}(\Omega_T)$ for any $1 \leq \alpha < \beta$.

Theorem 7 [Song11]Let $\xi \in L_G^{\beta}(\Omega_T)$ for some $\beta > 1$ with $\hat{E}(\xi) + \hat{E}(-\xi) = 0$. Then there exists $\{Z_t\}_{t\in[0,T]} \in H_G^{\beta}(0, T)$ such that

$$\xi = \hat{E}(\xi) + \int_0^T Z_s dB_s.$$

Classical Backward SDES

A typical classical Backward SDE is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ in which $B_t(\omega) = \omega_t$ is a standard BM with its natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$. The problem is to find a solution consisting of a pair of $\mathbb{F}$-adapted processes (Y, Z) satisfying the following BSDE

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \tag{2}$$

where g is a given function, called the generator, and ξ is a given $\mathcal{F}_T$-measurable random variable called the terminal condition of the BSDE.

Linear BSDE was introduced by Bismut(1973) . The existence and uniqueness theorem of nonlinear BSDEs (with Lipschitz condition of g in (y, z) was obtained in Pardoux & Peng (1990).

BSDEs driven by G-BM(GBSDE for short)

To find processes (Y, Z, K) satisfying

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t), \quad (3)$$

where K is a decreasing G-martingale.

Why not consider BSDE in the following form?

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s. \quad (4)$$

Generally, the equation above does not have a solution.

$$Y_t^P = \xi + \int_t^T f(s, Y_s^P, Z_s^P)ds - \int_t^T Z_s^P dB_s, \ P - a.s.. \quad (5)$$

In general, there dose not exist a universal (Y, Z).

Assumptions on f

Assumptions on f:

$$f(t,\omega,y,z):[0,T]\times\Omega_T\times\mathbb{R}^2\to\mathbb{R}$$

satisfies the following properties: There exists some $\beta>1$ such that

(H1) for any y,z, $f(\cdot,\cdot,y,z)\in M_G^{\beta}(0,T)$;

(H2) $|f(t,\omega,y,z)-f(t,\omega,y',z')|\le L(|y-y'|+|z-z'|)$ for some $L>0$.

What is a solution?

For simplicity, we denote by $\mathcal{S}_G^{\alpha}(0,T)$ the collection of processes (Y,Z,K) such that $Y \in S_G^{\alpha}(0,T)$, $Z \in H_G^{\alpha}(0,T)$, K is a decreasing G-martingale with $K_0 = 0$ and $K_T \in L_G^{\alpha}(\Omega_T)$.

Definition 8 Let $\xi \in L_G^{\beta}(\Omega_T)$ with $\beta > 1$ and f satisfy (H1) and (H2). A triplet of processes (Y,Z,K) is called a solution of equation (3) if for some $1 < \alpha \leq \beta$ the following properties hold:

(a) $(Y,Z,K) \in \mathcal{S}_G^{\alpha}(0,T)$;

(b) $Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t)$.

Main results

Theorem 9 Assume that $\xi \in L_G^{\beta}(\Omega_T)$ for some $\beta > 1$ and f satisfies (H1) and (H2). Then equation (3) has a unique solution (Y, Z, K). Moreover, for any $1 < \alpha < \beta$ we have $Y \in S_G^{\alpha}(0, T)$, $Z \in H_G^{\alpha}(0, T)$ and $K_T \in L_G^{\alpha}(\Omega_T)$.

Compared to 2BSDE(STZ12)

Soner, Touzi and Zhang [2012] have obtained an existence and uniqueness theorem for a type of fully nonlinear BSDE, called 2BSDE. Their solution is $(Y, Z, K^P)_{P\in\mathcal{P}_H^\kappa}$, which solves, for each probability $P \in \mathcal{P}_H^\kappa$, the following BSDE

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s - K_T^P + K_t^P, \ P - a.s.$$

for which the following minimum condition is satisfied

$$K_t^{\mathbb{P}} = \operatorname{ess} \inf_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t+,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[K_T^{\mathbb{P}}], \quad \mathbb{P}\text{-a.s.}, \ \forall \mathbb{P} \in \mathcal{P}_H^\kappa, \ t \in [0, T].$$

In their paper the processes $(K^P)_{P\in\mathcal{P}_H^\kappa}$ are not able to be "aggregated" into a "universal" K.

Differences from the classical BSDEs

⋄ Since the structure of G-martingales is much more complicated than that of the classical ones, we can not establish a contraction mapping for equation (3).

⋄ We apply the partition of unity theorem to construct a new type of Galerkin approximation, in the place of the well-Known Picard approximation and the related fixed point approach frequently used in BSDE theory.

Main idea of the proof

$\diamond$ In order to prove the existence of equation (3), we start with the simple case $f(t, \omega, y, z) = h(y, z)$, $\xi = \varphi(B_T)$. Here $h \in C_0^\infty(\mathbb{R}^2)$, $\varphi \in C_{b.Lip}(\mathbb{R}^2)$. For this case, we can obtain the solution of equation (3) from the following nonlinear partial differential equation:

$$\partial_t u + G(\partial_{xx}^2 u) + h(u, \partial_x u) = 0, u(T, x) = \varphi(x). \tag{6}$$

$\diamond$ Based on some a priori estimates for equations (3) with different generating functions, we approximate the solution of equation (3) with more complicated f by those of equations (3) with much simpler $\{f_n\}$.

A priori estimates

$\diamond$ The following property for decreasing G-martingales is critical in the proof to the a priori estimates.

Lemma 10 Let $X \in S_G^\alpha(0,T)$ for some $\alpha > 1$ and $\alpha^* = \frac{\alpha}{\alpha - 1}$. Assume that K^j, $j = 1, 2$, are two decreasing G-martingales with $K_0^j = 0$ and $K_T^j \in L_G^{\alpha^*}(\Omega_T)$. Then the process defined by

$$\int_0^t X_s^+ dK_s^1 + \int_0^t X_s^- dK_s^2$$

is also a decreasing G-martingale.

A priori estimates

Proposition 11 Assume that $(Y, Z, K) \in \mathcal{S}_G^{\alpha}(0, T)$ for some $1 < \alpha < \beta$ is a solution of equation (3). Then there exists $C_\alpha := C(\alpha, T, \underline{\sigma}, L) > 0$ such that

$$\|Y\|_{S_G^\alpha}^{\alpha} + \|Z\|_{H_G^\alpha}^{\alpha} + \|K_T\|_{L_G^\alpha}^{\alpha} \leq C_\alpha\{\|f^0\|_{M_{\mathcal{E}}^\alpha}^{\alpha} + \|\xi\|_{L_{\mathcal{E}}^\alpha}^{\alpha}\},$$

where $f^0(s) = |f(s, 0, 0)|$.

A priori estimates

Assume $(Y^i, Z^i, K^i) \in \mathcal{S}_G^\alpha(0, T)$ for some $1 < \alpha < \beta$ such that

$$Y_t^i = \xi^i + \int_t^T f_i(s, Y_s^i, Z_s^i)ds - \int_t^T Z_s^i dB_s - K_T^i + K_t^i,$$

where $\xi^i \in L_G^\beta(\Omega_T)$, $f_i, i = 1, 2$ satisfy (H1) and (H2).

Set $\hat{Y}_t = Y_t^1 - Y_t^2, \hat{Z}_t = Z_t^1 - Z_t^2$ and $\hat{K}_t = K_t^1 - K_t^2$.

A priori estimates

Proposition 12 (i) There exists $C_\alpha := C(\alpha, T, \underline{\sigma}, L_1) > 0$ such that

$$\|\hat{Z}\|_{H_G^\alpha}^\alpha \leq C_\alpha\{\|\hat{Y}\|_{S_G^\alpha}^\alpha + \|\hat{Y}\|_{S_G^\alpha}^{\alpha/2}[\|f_1^0\|_{M_\mathcal{E}^\alpha}^{\alpha/2} + \|\xi^1\|_{L_\mathcal{E}^\alpha}^{\alpha/2} + \|f_2^0\|_{M_\mathcal{E}^\alpha}^{\alpha/2} + \|\xi^2\|_{L_\mathcal{E}^\alpha}^{\alpha/2}]\}.$$

(ii) There exists a constant $C_\alpha := C(\alpha, T, \underline{\sigma}, L_1) > 0$ such that

$$|\hat{Y}_t|^\alpha \leq C_\alpha \hat{\mathbb{E}}_t[|\hat{\xi}|^\alpha + \int_t^T |\hat{f}_s|^\alpha ds], \qquad (7)$$

where $\hat{f}_s = |f_1(s, Y_s^2, Z_s^2) - f_2(s, Y_s^2, Z_s^2)|$.

A priori estimates

(iii) For any given α' with $\alpha < \alpha' < \beta$, there exists a constant $C_{\alpha,\alpha'}$ depending on α, α', T, $\underline{\sigma}$, L such that

$$\hat{\mathbb{E}}[\sup_{t\in[0,T]} |\hat{Y}_t|^{\alpha}] \leq C_{\alpha,\alpha'}\{\hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[|\hat{\xi}|^{\alpha}]] + (\hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[(\int_0^T \hat{f}_s ds)^{\alpha'}]])^{\frac{\alpha}{\alpha'}} + \hat{\mathbb{E}}[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t[(\int_0^T \hat{f}_s ds)^{\alpha'}]]\} \tag{8}$$

Sketch of the Proof to Theorem 9

By Proposition 12 we know that the solution is unique, and that for the existence of the solution it suffices to consider $\xi \in L_{ip}(\Omega_T)$.

Step 1. $f(t, \omega, y, z) = h(y, z)$ with $h \in C_0^\infty(\mathbb{R}^2)$.

Step 2. $f(t, \omega, y, z) = \sum_{i=1}^N f^i h^i(y, z)$ with $f^i \in M_G^0(0, T)$ and $h^i \in C_0^\infty(\mathbb{R}^2)$.

Step 3. $f(t, \omega, y, z) = \sum_{i=1}^N f^i h^i(y, z)$ with $f^i \in M_G^\beta(0, T)$ bounded and $h^i \in C_0^\infty(\mathbb{R}^2)$, $h^i \geq 0$ and $\sum_{i=1}^N h^i \leq 1$.

Choose $f_n^i \in M_G^0(0, T)$ such that $|f_n^i| \leq \|f^i\|_\infty$ and $\sum_{i=1}^N \|f_n^i - f^i\|_{M_G^\beta} < 1/n$. Set $f_n = \sum_{i=1}^N f_n^i h^i(y, z)$, which are uniformly Lipschitz.

Sketch of the Proof to Theorem 9 (continued)

Step 4. f is bounded, Lipschitz. $|f(t,\omega,y,z)| \leq CI_{B(R)}(y,z)$ for some $C, R > 0$. Here $B(R) = \{(y,z)|y^2+z^2 \leq R^2\}$.

For any n, by the partition of unity theorem, there exists $\{h_n^i\}_{i=1}^{N_n}$ such that $h_n^i \in C_0^\infty(\mathbb{R}^2)$, the radius of support r(supp(h_n^i))$< 1/n$, $0 \leq h_n^i \leq 1$, $I_{B(R)} \leq \sum_{i=1}^N h_n^i \leq 1$. Then

$f(t,\omega,y,z) = \sum_{i=1}^N f(t,\omega,y,z)h_n^i$. Choose y_n^i, z_n^i such that $h_n^i(y_n^i, z_n^i) > 0$. Set $f_n(t,\omega,y,z) = \sum_{i=1}^N f(t,\omega,y_n^i,z_n^i)h_n^i$.

Sketch of the Proof to Theorem 9 (continued)

Step 5. f is bounded, Lipschitz.

For any $n \in \mathbb{N}$, choose $h^n \in C_0^\infty(\mathbb{R}^2)$ such that $I_{B(n)} \leq h^n \leq I_{B(n+1)}$ and $\{h^n\}$ are uniformly Lipschitz w.r.t. n. Set $f_n = fh^n$, which are uniformly Lipschitz.

Step 6. For the general f.

Set $f_n = [f \vee (-n)] \wedge n$, which are uniformly Lipschitz.

Thank you!