Probabilistic numerical approximation for stochastic control problems

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Outline

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1. Introduction
2. Probabilistic approximation
   - A non-Markovian control problem
   - Convergence results
3. A probabilistic interpretation

Control problem and numerical methods

- A stochastic control problem

$$\sup_{\nu} \mathbb{E}\, \Phi(X_T^{\nu}), \qquad X_t^{\nu} := x_0 + \int_0^t \sigma(\nu_s)\, dW_s.$$

- Dynamic programming equation

$$\partial_t v(t,x) + \sup_u \Big(\frac{1}{2}\sigma^2(u) D^2 v(t,x) \Big) = 0, \qquad v(T,x) = \Phi(x).$$

- Finite difference method :

$$v_k^n = v_k^{n+1} + \Delta t \sup_u \Big(\frac{1}{2}\sigma^2(u) \frac{v_{k+1}^{n+1} - 2v_k^{n+1} + v_{k-1}^{n+1}}{\Delta x^2} \Big).$$

- Numerical analysis
  - Monotone convergence of viscosity solution (Barles and Souganidis),
  - Controlled Markov chain (Kushner).

Control problem and numerical methods

- A stochastic control problem

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The probabilistic scheme of Fahim, Touzi and Warin

- Terminal condition :

$$v_h(t_n, x) = \Phi(x).$$

- Let σ_0 be a constant and $X_h^{t,x} = x + \sigma_0 W_h$,

$$\begin{aligned} v_h(t_k, x) &= \mathbb{E}\big[v_h(t_{k+1}, X_h^{t_k,x})\big] \\ &\quad + h \sup_u \Big(\frac{1}{2}\big(\sigma^2(u) - \sigma_0^2\big)\ \mathbb{E}D^2 v_h\big(t_{k+1}, X_h^{t_k,x}\big)\Big) \\ &= \mathbb{E}\big[v_h(t_{k+1}, X_h^{t_k,x})\big] \\ &\quad + h \sup_u \Big(\frac{1}{2}\big(\sigma^2(u) - \sigma_0^2\big)\ \mathbb{E} v_h\big(t_{k+1}, X_h^{t_k,x}\big)\sigma_0^{-2}\frac{W_h^2 - h}{h^2}\Big). \end{aligned}$$

- Monotone convergence holds true for PDEs having a comparison.
- Numerical experiences : Fahim et al., Guyon et al, etc.

Backward SDEs and numerical scheme

- Semi-linear PDE

$$\partial_t v \; + \; \mu \cdot Dv \; + \; \frac{1}{2}\sigma\sigma^T \cdot D^2 v \; + \; f(t, x, v, \sigma Dv) \;\; = \;\; 0.$$

- Forward Backward SDEs :

$$X_t \;=\; x + \int_0^t \mu(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s,$$

$$Y_t \;=\; \Phi(X_\cdot) + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s.$$

- The numerical scheme

$$Y_k = \mathbb{E}_k\left[Y_{k+1}\right] + f(t_k, X_k, Y_k, Z_k)h, \quad Z_k = \mathbb{E}_k\left[Y_{k+1}\Delta W_{k+1}\right]h^{-1}.$$

- Convergence : Bouchard and Touzi(2004), Zhang(2004).

Outline

A non-Markovian stochastic control problem

• Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be filtrated probability space, E compact Polish space, $\mathcal{U}$ class of all $E-$valued $\mathbb{F}-$progressive processes. Denote $\Omega^d := C([0,T], \mathbb{R}^d)$, $Q_T := [0,T] \times \Omega^d \times E$, functions $(\mu, \sigma) : Q_T \to \mathbb{R}^d \times S_d$ bounded, uniformly continuous in $(t, \mathbf{x}, u)$ and Lipschitz in $\mathbf{x}$, for $\nu \in \mathcal{U}$,

$$X_t^\nu \quad = \quad x_0 \; + \; \int_0^t \mu(s, X_\cdot^\nu, \nu_s) ds \; + \; \int_0^t \sigma(s, X_\cdot^\nu, \nu_s) dW_s.$$

• Let $L : Q_T \to \mathbb{R}$ and $\Phi : \Omega^d \to \mathbb{R}$ be reward functions,

$$V \quad := \quad \sup_{\nu \in \mathcal{U}} \; \mathbb{E}\Big[\int_0^T L(t, X_\cdot^\nu, \nu_t) dt \; + \; \Phi(X_\cdot^\nu)\Big].$$

The probabilistic scheme

- Let $\sigma_0 : [0, T] \times \Omega^d \to S_d$ be bounded, non-degenerate, uniformly continuous in $(t, \mathbf{x})$ and Lipschitz in $\mathbf{x}$,

$$a_u^{t,\mathbf{x}} \ := \ \sigma\sigma^T(t, \mathbf{x}, u) \ - \ \sigma_0\sigma_0^{t,\mathbf{x}}, \qquad b_u^{t,\mathbf{x}} \ := \ \mu(t, \mathbf{x}, u).$$

- **Assumption 1.** For every $(t, \mathbf{x}, u) \in Q_T$, $a_u^{t,\mathbf{x}} \geq 0$ and $\frac{1}{2} a_u^{t,\mathbf{x}} \cdot \left(a_0^{t,\mathbf{x}}\right)^{-1} \leq 1$.

- Define a process on grid $(t_k)_{0 \leq k \leq n}$, $t_k = hk$,

$$X_0^0 \ := \ x_0, \qquad X_{k+1}^0 \ := \ X_k^0 \ + \ \sigma_0(t_k, \widehat{X^0_\cdot})\Delta W_{k+1}.$$

The probabilistic scheme

- Terminal condition

$$Y_n \;:=\; \Phi(\widehat{X^0}_\cdot).$$

- Backward iteration

$$\begin{aligned} Y_k &:= \mathbb{E}_k^W[Y_{k+1}] \;+\; h\; G(t_k, \widehat{X^0}_\cdot, \Gamma_k, Z_k), \\ \Gamma_k &:= \mathbb{E}_k^W\Big[Y_{k+1}(\sigma_{0,k}^T)^{-1}\frac{\Delta W_{k+1}\Delta W_{k+1}^T - hI_d}{h^2}\sigma_{0,k}^{-1}\Big], \\ Z_k &:= \mathbb{E}_k^W\Big[Y_{k+1}(\sigma_{0,k}^T)^{-1}\frac{\Delta W_{k+1}}{h}\Big], \end{aligned}$$

with $\sigma_{0,k} := \sigma_0(t_k, \widehat{X^0}_\cdot)$ and

$$G(t, \mathbf{x}, \gamma, z) \;:=\; \sup_{u \in E}\;\Big(\; L(t, \mathbf{x}, u) + \frac{1}{2} a_u^{t,\mathbf{x}} \cdot \gamma + b_u^{t,\mathbf{x}} \cdot z \;\Big).$$

General convergence

Theorem

Let L and Φ be uniformly continues and of exponential growth in x. Then

$$Y_0^h \longrightarrow V \quad \text{as} \quad h \rightarrow 0.$$

Proof. Weak convergence method (Dupuis and Kushner).

Rate of convergence

- **Assumption 2.** $E \subset S_d \times \mathbb{R}^d$ is compact, $\mu(t, \mathbf{x}, a, b) = b$, $\sigma(t, \mathbf{x}, a, b) = a^{1/2}$ and $L(\cdot, a, b) = \ell(\cdot) \cdot u$, ℓ and Φ Lipschitz.

Theorem

There exist constants C_ε such that for every $\varepsilon > 0$,

$$|Y_0^h - V| \leq C_\varepsilon h^{\frac{1}{8} - \varepsilon}.$$

Suppose in addition that ℓ and Φ are bounded, then there is a constant C such that

$$|Y_0^h - V| \leq C \, h^{\frac{1}{8}}.$$

Proof. Invariance principle (Sakhanenko(2000), Dolinsky(2011)).

Outline

A probabilistic interpretation

- Control problem, $d = 1$,

$$\sup_{\nu} \mathbb{E}\ \Phi(X_T^{\nu}), \qquad X_t^{\nu} \ := \ x_0 \ + \ \int_0^t \sigma(\nu_s) dW_s.$$

- PDE : $\partial_t v(t,x) \ + \ \sup_{u \in E} \left(\frac{1}{2}\sigma^2(u) D^2 v(t,x) \right) \ = \ 0.$

- Scheme of Fahim, Touzi and Warin : Let $X_h^{t,x} = x + \sigma_0 W_h$,

$$\begin{aligned} v_h(t,x) &= \mathbb{E}\big[v_h(t_{k+1}, X_h^{t_k,x}) \big] \\ &\quad + h \sup_{u \in E} \Big(\frac{1}{2} \big(\sigma^2(u) - \sigma_0^2\big) \ \mathbb{E} v_h\big(t_{k+1}, X_h^{t_k,x}\big) \sigma_0^{-2} \frac{W_h^2 - h}{h^2} \Big) \\ &= \sup_{u \in E} \mathbb{E}\Big[v_h(t_{k+1}, x + \sigma_0 W_h) \Big(1 - \frac{1}{2} a_u \sigma_0^{-2} + \frac{1}{2} a_u \sigma_0^{-2} \frac{W_h^2}{h} \Big) \Big]. \end{aligned}$$

A probabilistic interpretation

- For every function φ of exponential growth,

$$\mathbb{E}\Big[\varphi(\sigma_0 W_h)\Big(1-\frac{1}{2}a_u\sigma_0^{-2}+\frac{1}{2}a_u\sigma_0^{-2}\frac{W_h^2}{h}\Big)\Big]$$
$$=\int_{\mathbb{R}}\varphi(x)\frac{1}{\sqrt{2\pi h}\sigma_0}e^{-\frac{x^2}{2\sigma_0^2 h}}\Big(1-\frac{1}{2}a_u\sigma_0^{-2}+\frac{1}{2}a_u\sigma_0^{-4}\frac{x^2}{h}\Big)\,dx.$$

- Let $F_h(u,x)$ be cumulative distribution function of $f_h(u,x)$,

$$f_h(u,x) \;:=\; \frac{1}{\sqrt{2\pi h}\sigma_0}e^{-\frac{x^2}{2\sigma_0^2 h}}\Big(1-\frac{1}{2}a_u\sigma_0^{-2}+\frac{1}{2}a_u\sigma_0^{-4}\frac{x^2}{h}\Big).$$

- Hence the scheme turns to be

$$v_h(t_k,x) \;=\; \sup_{u\in E}\mathbb{E}\Big[v_h(t_{k+1},x+F_h^{-1}(u,U_{k+1}))\Big].$$

A probabilistic interpretation

• Let $U_1, \cdots, U_n$ be i.i.d., $\mathcal{A}_h$ be set of strategies $\phi = (\phi_0, \cdot, \phi_{n-1})$ taking value in E,

$$X^{\phi}_{i+1} \;:=\; X^{\phi}_i \;+\; F_h^{-1}\big(\phi_i(X^{\phi}_0, \cdots, X^{\phi}_i),\; U_{i+1}\big),$$

• Let

$$V^h_0 \;:=\; \sup_{\phi \in \mathcal{A}_h} \mathbb{E}\left[\Phi(X^{\phi}_n)\right].$$

Theorem

The numerical solution is equivalent to the above optimization problem, i.e.

$$v_h(0, x_0) \;=\; V^h_0, \qquad \text{or} \;\; Y^h_0 \;=\; V^h_0.$$