Representation of Dynamic Time-Consistent Convex Risk Measures with Jumps

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Outlines

- Risk measures (VaR,...); Dynamic time-consistent convex risk measures
- Backward Stochastic Differential Equations and g-expectation with jumps
- Relation between g-expectation and dynamic time-consistent convex risk measure
- Integral representation of the minimal penalty term
- An example

Risk Measures in the Literature

$(\Omega, \mathscr{F}, P)$: Probability space

X: R.V.

- Standard Deviation: $R(X) := E[(X - E[X])^2]$
  e.g. Markowitz, Portfolio Selection, 1952;
- Value at Risk (J.P. Morgan): $\forall \alpha \in (0,1)$

$$VaR(\alpha) := \inf\{x : \ P(X - X_0 \leq x) \geq \alpha\},$$

$$CVaR(\alpha) := E[X - X_0 \mid X - X_0 \leq VaR(\alpha)];$$

- Stone Family of risk measures (1970s)

$$R(k, \bar{X}, X^*) := \left(E\left[|X - X^*|^k \mathbb{I}_{X \leq \bar{X}}\right]\right)^{\frac{1}{k}};$$

- ......

Coherent risk measures

Artzner, Delbaen, Eber and Heath, Math. Finance, 1999

$$\rho(\cdot) : \mathbb{L}^\infty(\Omega, \mathscr{F}, P) \to \mathbb{R},$$

satisfies four axioms:

- $\rho(X) \leq 0$, $\forall X \geq 0$,
- Subadditivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$,
- Translation invariance: $\rho(X+c) = \rho(X) - c$,
- Positive homogeneity: $\rho(\lambda X) = \lambda \rho(X)$, $\forall \lambda \geq 0$,

If ρ satisfies Fatou Property: $\rho(X) \leq \lim_n \rho(X_n)$, $\forall X_n \to X$,

$$\rho(X) = \sup_{Q \in \mathcal{P}_0} \{E_Q[-X]\},$$

$\mathcal{P}_0$ is a closed and convex set of probabilities.

Convex risk measures

Föllmer and Schied, Finance and Stochastics, 2002

- Replace "positive homogenity" by the convexity:

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y), \ \forall \lambda \in [0,1].$$

Then

$$\rho(X) = \sup_{Q\in\mathcal{P}}\{E_Q[-X] - C(Q)\},$$

$\mathcal{P}$ contains all the probabilities, and

$$C(Q) := \sup_{X\in\mathbb{L}^\infty}\{E_Q[-X] - \rho(X)\}$$

is called the minimal penalty term.

Dynamic Time-Consistent Convex Risk Measures

Detlefsen and Scandolo, Finance and Stochastics, 2005

$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, P)$: filtered probability space.

A family of mappings

$$\rho_{t,s}(\cdot): \mathbb{L}^2(\mathscr{F}_s) \to \mathbb{L}^2(\mathscr{F}_t), \quad 0 \le t \le s \le T$$

(A1) Monotonicity: $\forall X, Y \in \mathbb{L}^2(\mathscr{F}_s),\ X \ge Y,\ \rho_{t,s}(X) \le \rho_{t,s}(Y)$;

(A2) Translation invariance: $\forall Z \in \mathbb{L}^2(\mathscr{F}_t)$,

$$\rho_{t,s}(X+Z) = \rho_{t,s}(X) - Z;$$

(A3) Convexity: for all $\beta \in [0,1],\ X, Y \in \mathbb{L}^2(\mathscr{F}_s)$,

$$\rho_{t,s}(\beta X + (1-\beta)Y) \le \beta\, \rho_{t,s}(X) + (1-\beta)\rho_{t,s}(Y);$$

Dynamic Time-Consistent Convex Risk Measures

(A4) Normalization: $\rho_{t,s}(0) = 0$.

(A5) Time consistency: $\rho_{t,s}(X) = \rho_{t,r}(-\rho_{r,s}(X)), \qquad \forall r \in [t,s]$.

Definition

$(\rho_{t,s}(\cdot))_{0\le t\le s\le T}$ satisfying (A1)-(A5) is called a dynamic time-consistent convex risk measure (DTC risk measure).

(A6) Continuity from below: $X_n \uparrow X$, P-a.s.

$$\lim_{n\to\infty} \rho_{t,s}(X_n) = \rho_{t,s}(X),\ P\text{-a.s.};$$

(A7) $C_{t,s}(P) = 0$, where

$$C_{t,s}(Q) := \operatorname*{ess\,sup}_{X\in\mathbb{L}^\infty(\mathscr{F}_s)} \{E_Q[-X|\mathscr{F}_t] - \rho_{t,s}(X)\},\ \ \forall Q \ll P$$

is the minimal penalty term of $\rho_{t,s}$.

Representation of DTC Risk Measures

Proposition (Klöpple and Schweizer (2007), Bion-Nadal (2009))

$$\rho_{t,s}(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{P}_t} \{E_Q[-X|\mathscr{F}_t] - C_{t,s}(Q)\},$$

where $\mathcal{P}_t = \{Q \sim P \mid Q = P \text{ on } \mathscr{F}_t\}$.
Time-consistency is equivalent to

$$C_{t,s}(Q) = C_{t,r}(Q) + E_Q[C_{r,s}(Q)|\mathscr{F}_t], \quad 0 \le t \le r \le s \le T.$$

Under Brownian Filtration

- Rosazza Gianin (2006), Risk measure via g-expectation, Insurance Mathematics and Economics
- Delbaen, Peng and Rosazza Gianin (2010), Representation of the penalty term of dynamic concave utilities, Finance and Stochastics

Backward Stochastic Differential Equation with Jumps

$$\begin{cases} dY_t = -\, g(t, Y_t, Z_t, H_t)\, dt + Z_t\, dW_t + \int_{\mathbb{E}} H_t(e)\, \widetilde{\mu}(dedt); \\ Y_T = \xi. \end{cases} \tag{1}$$

Denote (Y, Z, H) as the solution, $\mathcal{E}_g[\xi|\mathscr{F}_t] := Y(t)$ as the g-expectation.

$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, P)$, usual conditions

- d-dimensional Brownian motion $\{W_t\}_{t\in[0,T]}$
- Poisson random measure μ on $[0, T] \times \mathbb{E}$, $\mathbb{E} := \mathbb{R}\backslash\{0\}$,

$$\widetilde{\mu}(dtde) := \mu(dtde) - dt\lambda(de),$$

$$\int_{\mathbb{E}} (1 \wedge |e|^2)\lambda(de) < +\infty.$$

Assumptions on g

(H1) $|g(t,y,z,h)-g(t,\hat{y},\hat{z},\hat{h})| \le L(|y-\hat{y}|+|z-\hat{z}|+\|h-\hat{h}\|)$;

(H2) $E\left[\int_0^T |g_0(t)|^2\,dt\right] < +\infty$, $g_0(t) := g(t,0,0,0)$;

(H3) $\exists \kappa_1 \ge 0$, $\kappa_2 \in (-1,0]$, such that

$$g(t,y,z,h)-g(t,y,z,\hat{h}) \le \int_{\mathbb{E}} (h(e)-\hat{h}(e))\gamma_t^{y,z,h,\hat{h}}(e)\lambda(de),$$

where

$$\kappa_2(1\wedge|e|) \le \gamma_t^{y,z,h,\hat{h}}(e) \le \kappa_1(1\wedge|e|),$$

(H4) $g(t,y,0,0)=0,\ a.e., a.s.$;

(H5) g is independent of y.

Representation of the generator

Proposition

Fixed $x, p, y \in \mathbb{R}$, $\forall \varepsilon > 0, t + \varepsilon \leq T$. *Consider the following FBSDE*

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(X_u^{t,x})du + \int_t^s \sigma(X_u^{t,x})dW_u \\ \qquad + \int_t^s \int_{\mathbb{E}} \eta(e, X_{u-}^{t,x})\widetilde{\mu}(dedu), \quad s \in [t, t+\varepsilon], \\ Y_s^{t,x,p,y} = y + p(X_{t+\varepsilon}^{t,x} - x) + \int_s^{t+\varepsilon} g(u, Y_u^{t,x,p,y}, Z_u^{t,x,p,y}, H_u^{t,x,p,y})du \\ \qquad - \int_s^{t+\varepsilon} Z_u^{t,x,p,y} dW_u - \int_s^{t+\varepsilon} \int_{\mathbb{E}} H^{t,x,p,y}(u,e)\widetilde{\mu}(dedu), \\ \qquad s \in [t, t+\varepsilon], \end{cases}$$

Representation of the generator (continue)

where g satisfies $(H1)$-$(H2)$
$b:\mathbb{R}\to\mathbb{R}$, $\sigma:\mathbb{R}\to\mathbb{R}^d$, $\eta:\mathbb{E}\times\mathbb{R}\to\mathbb{R}$, $\eta(e,0)\in L^2(\mathbb{E})$, and
$\exists\ L>0$ such that

$$|b(x_1)-b(x_2)|+|\sigma(x_1)-\sigma(x_2)|\leq L|x_1-x_2|,\quad \forall x_1,x_2\in\mathbb{R},$$

$$|\eta(e,x_1)-\eta(e,x_2)|\leq L(1\wedge|e|)|x_1-x_2|,\quad \forall x_1,x_2\in\mathbb{R},$$

then there exists $A\subset[0,T]$ with full Lebesgue measure, such that
$\forall t\in A$, $\forall q\in[1,2)$,

$$L^q\text{-}\lim_{\varepsilon\downarrow 0}\frac{Y_t^{t,x,p,y}-y}{\varepsilon}=g(t,y,\sigma(x)p,\eta(\cdot,x)p)+b(x)p.$$

Converse Comparison Theorem

Theorem

Suppose that g_1 *and* g_2 *are two generators of BSDE* (1)*, and they satisfy assumptions (H1), (H3) and (H4). If* $\mathcal{E}_{g_1}[\xi|\mathscr{F}_t] \geq \mathcal{E}_{g_2}[\xi|\mathscr{F}_t]$ *for all* $\xi \in \mathbb{L}^2(\mathscr{F}_T)$*, then there exists a subset* $\mathcal{S} \subseteq [0,T]$ *with* $v([0,T]\backslash\mathcal{S}) = 0$ *(*v *is the Lebesgue measure), such that for any* $t \in \mathcal{S}$*,*

$$g_1(t,y,z,h) \geq g_2(t,y,z,h), \quad P\text{-a.s.}$$

for all $y \in \mathbb{R},\ z \in \mathbb{R}^d,\ h \in \mathbb{L}^2(\mathbb{E})$*.*

Corollary

Corollary

Let g satisfy the assumptions (H1), (H3) and (H4). Then for all $\beta \in [0,1]$, the following are equivalent,

(1) $\forall \xi_1,\ \xi_2 \in \mathbb{L}^2(\mathscr{F}_T)$,

$$\mathcal{E}_g[\beta\xi_1 + (1-\beta)\xi_2|\mathscr{F}_t] \le \beta\mathcal{E}_g[\xi_1|\mathscr{F}_t] + (1-\beta)\mathcal{E}_g[\xi_2|\mathscr{F}_t];$$

(2) for all $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$ *and* $h_1, h_2 \in \mathbb{L}^2(\mathbb{E})$,

$$g(t, \beta y_1 + (1-\beta)y_2, \beta z_1 + (1-\beta)z_2, \beta h_1 + (1-\beta)h_2)$$
$$\le \beta g(t, y_1, z_1, h_1) + (1-\beta)g(t, y_2, z_2, h_2);$$

Relation Between DTC Risk Measure and g-expectation

Proposition

Suppose that g satisfies (H1)-(H3). Then the following are equivalent:
(1) $\mathcal{E}_g[-\cdot|\mathscr{F}_t], t \in [0,T]$ is a DTC risk measure.
(2) g satisfies (H4) and (H5), and g is jointly convex with respect to z and h.

Relation Between DTC Risk Measure and g-expectation

Proposition

If a DTC risk measure $\rho_{t,T}(\cdot)$ *is strictly monotone and* $\rho_{t,T}(-\cdot)$ *is* $\mathcal{E}_{g_{\kappa_1,\kappa_2}}$*-dominated for some* $\kappa_1 \geq 0$ *and* $\kappa_2 \in (-1,0]$*, then*
(1) $\exists g : \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{L}^2(\mathbb{E}) \to \mathbb{R}$ *such that*

$$\rho_{t,T}(\cdot) = \mathcal{E}_g[- \cdot |\mathscr{F}_t];$$

(2) g *satisfies (H1)-(H5) and is jointly convex with respect to* z *and* h*. Moreover,* κ_1 *is the Lipschtz coefficient on* z*,* $\kappa_1 - \kappa_2$ *on* h*, and* κ_1 *and* κ_2 *are the two coefficient in (H2).*

$\mathcal{E}_{g_{\kappa_1,\kappa_2}}$-Domination

Definition

$$\phi[\cdot|\mathscr{F}_t] : \mathbb{L}^2(\mathscr{F}_T) \to \mathbb{L}^2(\mathscr{F}_t), \quad \forall t \leq T,$$

If for all $\xi_1, \xi_2 \in \mathbb{L}^2(\mathscr{F}_T)$,

$$\phi[\xi_1 + \xi_2] - \phi[\xi_2] \leq \mathcal{E}_{g_{\kappa_1,\kappa_2}}[\xi_1],$$

where

$$\begin{aligned} g_{\kappa_1,\kappa_2}(t,z,h) :=& \kappa_1|z| + |\kappa_1| \int_{\mathbb{E}} (1 \wedge |e|) h^+(e) \lambda(de) \\ & - \kappa_2 \int_{\mathbb{E}} (1 \wedge |e|) h^-(e) \lambda(de). \end{aligned}$$

Then ϕ is $\mathcal{E}_{g_{\kappa_1,\kappa_2}}$-dominated.

Truncated DTC Risk Measure

- $\rho_{t,s}$: strictly monotone, $\mathcal{E}_{g_{\kappa_1,\kappa_2}}$-dominated, BSDE
- what about a general $\rho_{t,s}$?

Define

$$\rho^n_{t,s}(X) := \operatorname*{ess\,sup}_{Q\in\mathcal{P}^n_t}\left\{E_Q[-X|\mathscr{F}_t] - C_{t,s}(Q)\right\},$$

where

$$\mathcal{P}^n_t := \Big\{ Q \in \mathcal{P}_t \;\Big|\; |\theta(u,\omega)| \le n, \\ -(1-\frac{1}{n})(1\wedge|e|) \le \zeta(u,e,\omega) \le n(1\wedge|e|),\ \forall u\in[t,T]\Big\}$$

with

$$\frac{dQ}{dP} = \mathscr{E}xp\left\{\int_0^T \theta_s dW_s + \int_0^T\int_{\mathbb{E}} \zeta(e,s)\widetilde{\mu}(deds)\right\}.$$

Let

$$C^n_{t,s}(Q) := \begin{cases} C_{t,s}(Q), & Q \in \mathcal{P}^n_t; \\ +\infty, & else, \end{cases}$$

then

$$\rho^n_{t,s}(X) := \operatorname*{ess\,sup}_{Q \in \mathcal{P}_t} \left\{ E_Q[-X|\mathscr{F}_t] - C^n_{t,s}(Q) \right\},$$

$\rho^n_{t,s}(\cdot)$ is $\mathcal{E}_{g_{n,-\frac{1}{n}}}$-dominated.

Proposition

We have the following two assertions for $\rho^n_{t,s}$:
(1) $\rho^n_{t,s}$ is also a DTC risk measure satisfying (A1)-(A7) with $C^n_{t,s}$ being its minimal penalty term;
(2) $\exists g_n : [0,T] \times \Omega \times \mathbb{R}^d \times \mathbb{L}^2(\mathbb{E}) \to \mathbb{R}$ satisfying (H1)-(H5) and jointly convex with respect to z and h, such that

$$
\begin{aligned}
\rho^n_{t,T}(X) = & - X + \int_t^T g_n(s, Z_s, H_s)\, ds - \int_t^T Z_s \, dW_s \\
& - \int_t^T \int_{\mathbb{E}} H_s(e)\, \widetilde{\mu}(deds), \quad t \in [0,T].
\end{aligned}
$$

Integral Representation of $C^n_{t,s}$

Proposition

Define

$$f_n(t,\omega,a,b) := \sup_{(z,h)\in\mathbb{R}^d\times\mathbb{L}^2(\mathbb{E})} \{\langle a,z\rangle + \langle b,h\rangle - g_n(t,\omega,z,h)\}$$

for all $(a,b)\in\mathbb{R}^n\times\mathbb{L}^2(\mathbb{E})$. *Here* f_n *can take the value* $+\infty$ *and the integration here is defined to be extended. Then*

$$C^n_{t,s}(Q) = E_Q\left[\int_t^s f_n(r,\theta_r,\zeta_r)dr \,\middle|\, \mathscr{F}_t\right], \quad \forall Q\sim P,$$

and

$$\rho^n_{t,s}(X) = \operatorname*{ess\,sup}_{Q\in\mathcal{P}_t} E_Q\left[-X - \int_t^s f_n(r,\theta_r,\zeta_r)dr \,\middle|\, \mathscr{F}_t\right].$$

Limit function

Lemma

Define

$$f(t,\omega,a,b) = \inf_n f_n(t,\omega,a,b),$$

then for any (t,ω,a,b), *the following two are alternative:*
(i) $\exists n$, *such that* $f_n(t,\omega,a,b) < +\infty$, *then,* $\forall m \geq n$,
$f_m(t,\omega,a,b) = f_n(t,\omega,a,b) = f(t,\omega,a,b)$;
(ii) $\forall n$, $f_n(t,\omega,a,b) = +\infty$, *then we define* $f(t,\omega,a,b) = +\infty$.

$$\widehat{\mathcal{P}} := \Big\{ Q \sim P \mid \zeta(u,e) > -(1 \wedge |e|) \Big\}.$$

Theorem

Let $\rho_{t,s}(\cdot)$ be a DTC risk measure satisfying assumption $(A1)$-$(A7)$. Then, for any $Q \in \widehat{\mathcal{P}}$, we have

$$C_{t,s}(Q) \leq E\left[\int_t^s f(r, \theta_r, \zeta_r) dr \,\middle|\, \mathscr{F}_t \right]$$

with the equality "$=$" holding for $Q \in \cup_{n=1}^{\infty} \mathcal{P}_t^n$.

Let $\mathbb{E} := \{1\}$, then $\mu(dtde) := N(dt)$ is a Poisson process.

Theorem

$$C_{t,s}(Q) = E_Q\left[\int_t^s f(u, \theta_u, \zeta_u)du \,\middle|\, \mathscr{F}_t\right], \ Q \sim P,$$

and

$$\rho_{t,s}(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{P}_t} E_Q\left[-X - \int_t^s f(r, \theta_r, \zeta_r)dr \,\middle|\, \mathscr{F}_t\right].$$

- If $\mathbb{E}$ is a finite set, similar equalities hold too.

Example: Loss Function

Loss function $l: \mathbb{R} \to \mathbb{R}$, nondecreasing, convex

$$\mathscr{A}_{t,T} = \left\{X \in L^\infty(\mathscr{F}_T) \mid E_P[l(-X)|\mathscr{F}_t] \leq x_0\right\}$$

$$\rho^{\mathscr{A}}_{t,T}(X) := \operatorname{ess\,inf}\{\ \xi \in L^\infty \mid \xi + X \in \mathscr{A}\}$$

time-consistent $\Leftrightarrow$ l is a linear or exponential function

$$l(x) := exp\{x\}, \qquad x_0 := 1.$$

$$\rho^{\mathscr{A}}_{t,T}(X) = \log\left(E_P[exp\{-X\}|\mathscr{F}_t]\right) = \operatorname*{ess\,sup}_{Q \in \mathcal{P}_t}\left\{E_Q[-X|\mathscr{F}_t] - C_{t,T}(Q)\right\},$$

$$C_{t,T}(Q) := E_Q\left[\log \frac{dQ}{dP} \,\middle|\, \mathscr{F}_t\right]$$

$$= E_Q\left[\int_t^T \left[\frac{|\theta_s|^2}{2} + \int_{\mathbb{E}}\Big(\zeta(s,e)\log(1+\zeta(s,e))\right.\right.$$

Example (continue)

$$f(t,a,b) = \frac{|a|^2}{2} + \int_{\mathbb{E}} \Big(b(e)\log(1+b(e)) + \log(1+b(e)) - b(e) \Big) \nu(de).$$

Define

$$g(t,z,h) = \frac{|z|^2}{2} + \int_{\mathbb{E}} \left[-h(e) + exp\{h(e)\} - 1 \right] \lambda(de),$$

then, f is the conjugate function of g, and vice versa.

$$\rho^{\mathscr{A}}_{t,T}(X) = -X + \int_t^T g(t,Z,H)\, ds - \int_t^T Z(s)\, dW_s + \int_t^T \int_{\mathbb{E}} H(s,e)\, \widetilde{\mu}(de\, ds).$$

Thank you!