BSDEs and Strict Local Martingales

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Goal

Given parameters $g : \mathbb{R}_+^d \to \mathbb{R}$ and $f : [0, T] \times \mathbb{R}_+^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$,

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dB_s, \quad 0 \le t \le T, \qquad \text{(BSDE)}$$

where each component of X is a nonnegative local martingale.

Question: Can we find multiple solutions?

Main results (roughly)

- g has linear growth,
- f is bounded in z + some additional assumptions,
- X is a strict local martingale,

Then there exist two (sometimes infinite many) solutions in $(\mathcal{S}^p, \mathcal{M}^p)$, $0 < p < 1$,

- in one solution $(\overline{Y}, \overline{Z})$, $\overline{Y}$ is of class D,
- in another solution (Y, Z), Y is not of class D,
- $Y_0 > \overline{Y}_0$.

When X is a diffusion,

- multiple viscosity solutions to quasi-linear PDE,
- sufficient condition for uniqueness (comparison).

Thank you very much!

A motivational example

$$dX_t = -X_t^2 dB_t, \quad X_0 = x > 0.$$

X is the reciprocal 3-dim Bessel process.

X is a strict local martingale with $\mathbb{E}[X_T^2] < \infty$.

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Consider the following BSDE:

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One solution: $\overline{Y}_\cdot = \mathbb{E}[X_T | \mathcal{F}_\cdot]$ and its associated integrand $\overline{Z}$.

$$\mathbb{E}[\sup_{0\leq t\leq T} \overline{Y}_t^2] < \infty \text{ and } \mathbb{E}[\int_0^T \overline{Z}_s^2 ds] < \infty.$$

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Another solution: $(Y, Z) = (X, -X^2)$.

- $\mathbb{E}[\int_0^T Z_s^2 ds] = \mathbb{E}[\int_0^T (X_s^2)^2 ds] = \infty$.
- $Y = X$ is not of class D and $\mathbb{E}[\sup_{0\le t\le T} Y_t] = \infty$.
- $Y_0 = X_0 > \mathbb{E}[X_T] = \overline{Y}_0$.

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- $Y = X$ is not of class D and $\mathbb{E}[\sup_{0\leq t\leq T} Y_t] = \infty$.
- $Y_0 = X_0 > \mathbb{E}[X_T] = \overline{Y}_0$.

However, both solutions are $\mathbb{L}^p$ integrable with $p \in (0, 1)$.

There are at least two solutions in the same class of processes!

Bubble

Let X be the price process of a risky asset under a risk neutral measure.

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European option price $\mathbb{E}[g(X_T) \mid \mathcal{F}_t]$ solves (BSDE) when $f \equiv 0$.

[Loewenstein-Willard], [Cox-Hobson], [Jarrow-Protter] ...

Other applications:

- Stochastic Portfolio Theory [Fernholz-Karatzas et al.]
- Benchmark Approach [Platen et al.]

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Parameters	Results
$\mathbb{L}^2$	[Pardoux & Peng 90] existence and uniqueness of $\mathbb{L}^2-$ solution
$\mathbb{L}^p \, (p \in (1, 2))$	[El Karoui et al. 97] existence of $\mathbb{L}^p-$ solution
$\mathbb{L}^p \, (p \in (1, 2))$	existence and uniqueness in [Briand et al. 03]
$\mathbb{L}^1$	[Peng 97] a special type of BSDE
$\mathbb{L}^1$	f has sublinear growth in z, [Briand et al. 03] existence and uniqueness in class D .

g-local martingales

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We regard solutions to (BSDE) as g-local martingales.

The non-class D solution can be viewed as g-strict local martingale.

Assumptions on g

Denote

$$\underline{X} = \sum_{i=1}^{d} X^i$$

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The terminal function g is continuous, nonnegative, and

$$K := \sup \left\{ \frac{g(x)}{1+\underline{x}} : x \in \mathbb{R}_+^d \right\} < \infty.$$

Therefore, $0 \leq g(x) \leq K(1+\underline{x})$ and $g(X_T) \in \mathbb{L}^1$.

We do not a priori assume $g(X_T) \in \mathbb{L}^p$ for some $p > 1$.

Assumptions on f

f is jointly continuous in all its variables.

$$\begin{aligned}
&|f(t,x,y,z) - f(t,x,y,z')| \le \nu|z - z'|, \\
&(y - y')(f(t,x,y,z) - f(t,x,y',z)) \le \mu(y - y')^2, \\
&f(t,x,y,z) \ge 0, \\
&f(t,x,0,z) \le H(t,\underline{x}).
\end{aligned}$$

This implies

$$f(t,x,y,z) \le \mu y + H(t,\underline{x}), \qquad \text{for any } y \ge 0 \text{ and } z.$$

Here $H : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

- H is locally bounded on $[0,T] \times \mathbb{R}_+$.
- $\mathbb{E}[\int_0^T H(t,\underline{X}_t)dt] < \infty$.
- $r \mapsto H(t,r)$ is nondecreasing and concave.

The class $\mathcal{C}$

Look for (BSDE) solution inside the following class:

$$\mathcal{C} := \left\{ Y : 0 \leq Y \leq C \left(K(1 + \underline{X}_t) + \mathbb{E}\left[\int_t^T H(s, \underline{X}_s) ds \middle| \mathcal{F}_t \right] \right) \right\}.$$

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Proposition

For a solution (Y, Z) to (BSDE) such that $Y \in \mathcal{C}$,

$$\mathbb{E}\left[\sup_{t \in [0,T]} |Y_t|^p \right] < \infty \quad \text{and} \quad \mathbb{E}\left[\left(\int_0^T |Z_s|^2 ds \right)^{p/2} \right] < \infty,$$

for any $p \in (0,1)$, i.e., $(Y, Z) \in (\mathcal{S}^p, \mathcal{M}^p)$.

Main results

Theorem

(i) $\exists$ *a solution* $(\overline{Y}, \overline{Z})$ *such that* $\overline{Y} \in \mathcal{C}$ *and* $\overline{Y}$ *is of class D.*

(ii) *For any other solution* $(\widetilde{Y}, \widetilde{Z})$ *such that* $\widetilde{Y} \in \mathcal{C}$, $\widetilde{Y}_t \geq \overline{Y}_t$.

Define $\overline{g}(x) := K(1 + \underline{x}) - g(x)$. *Assume that*

$$\begin{aligned} &\overline{g}(X_\cdot) \text{ is a supermartingale on } [0, T], \\ &\exists \text{ a nondecreasing univariate } \overline{G} : \mathbb{R}_+ \to \mathbb{R}_+, \\ &\overline{g}(x) \leq \overline{G}(\underline{x}) \quad \text{and} \quad \lim_{r \to \infty} \overline{G}(r)/r = 0. \end{aligned}$$

(iii) *Then when* X *is a strict local mart,* $\exists$ *another solution* (Y, Z) *such that* $Y \in \mathcal{C}$, *but* Y *is not of class D, moreover,* $Y_0 > \overline{Y}_0$.

Remarks and examples

Multiple solutions $\implies$ comparison fails in $\mathcal{C}$.

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Example (Zero generator)

When $f \equiv 0$,

$$\overline{Y}_\cdot = \mathbb{E}[g(X_T)|\mathcal{F}_\cdot] \quad \text{and} \quad Y_\cdot = K(\underline{X}_\cdot - \mathbb{E}[\underline{X}_T|\mathcal{F}_\cdot]) + \mathbb{E}[g(X_T)|\mathcal{F}_\cdot].$$

BSDE with quadratic growth in z

Consider

$$P_t = \log \underline{X}_T + \int_t^T \left(\alpha + \frac{1}{2} |Q_s|^2 \right) ds - \int_t^T Q_s \, dB_s. \tag{1}$$

Define $(Y, Z) := (e^P, e^P Q)$. It satisfies

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[Delbaen & Hu & Richou 11]: uniqueness of solution to (1) holds

$$\mathbb{E}\left[e^{\gamma \sup_{0\le t\le T} P_t^+} + e^{\epsilon \sup_{0\le t\le T} P_t^-}\right] < \infty, \qquad \text{for some } \gamma > 1 \text{ and } \epsilon > 0.$$

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The additional solution (P, Q) is outside the previous class.

Construction of multiple solutions

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$$Y_t^n = \xi_n + \int_t^T \mathbb{I}_{\{s \leq \tau_n\}} f(s, X_s, Y_s^n, Z_s^n) ds - \int_t^T Z_s^n dB_s, \qquad \text{for each } n \geq 0.$$

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We have

$$\mathbb{P} - \lim_{n\to\infty} \xi_n = g(X_T) \quad \text{and} \quad \mathbb{P} - \lim_{n\to\infty} \overline{\xi}_n = g(X_T).$$

But the convergence may not be in $\mathbb{L}^1$.

This allows $\{Y_n\}_{n\geq 0}$ and $\{\overline{Y}_n\}_{n\geq 0}$ converge to two different solutions.

Two remarks

f is bounded in z + assumptions on $H \implies$

$$Y_t^n \leq C \left(K(1 + \underline{X}_t) + \int_t^T H(s, \underline{X}_t) ds \right), \quad t \in [0, T].$$

Then use the localization technique in [Briand & Hu 06].

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f non-neg. + g linear growth $\Longrightarrow$

$$Y_t = \lim_{n\to\infty} Y_t^n \ge K(\underline{X}_t - \mathbb{E}[\underline{X}_T \,|\, \mathcal{F}_t]) + \mathbb{E}[g(X_T) \,|\, \mathcal{F}_t].$$

Then X strict local martingale $\Longrightarrow$ Y is not of class D.

The Markovian case

Given $\sigma : (0,\infty)^d \to \mathbb{R}^{d\times d}$ which is locally Lipschitz,

$$dX_s^{x,i} = \sum_{j=1}^{d} \sigma_{ij}(X_s^x)dB_s^j, \quad X_0^x = x \in (0,\infty)^d, \quad i = 1,\cdots,d.$$

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We assume X does not hit the boundary of $(0,\infty)^d$ in finite time.

No boundary condition is needed. [Bao-Delbaen-Hu 10]

Consider the quasi-linear PDE

$$\begin{aligned} &-\partial_t u - \tfrac{1}{2}Tr\left(\sigma\sigma'\nabla^2 u\right) - f(t,x,u,\nabla u\,\sigma) = 0, && (t,x) \in [0,T) \times (0,\infty)^d,\\ &u(T,x) = g(x), && x \in (0,\infty)^d. \end{aligned} \tag{PDE}$$

[Pardoux & Peng 92], [Barles & Buckdahn & Pardoux 97] ...

Existence theorem

Theorem

There are two different viscosity solutions u and $\overline{u}$ to (PDE). Both of them are nonnegative and have at most linear growth. But

$$u(t,x) > \overline{u}(t,x) \quad \textit{for} \quad (t,x) \in [0,T) \times (0,\infty)^d.$$

When f vanishes, g has linear growth, X a strict local mart., multiple solution to (PDE) has been observed in

- stock price bubble [Heston et al. 07].
- stochastic portfolio theory [Fernholz & Karatzas 08].

Comparison (uniqueness) theorem

Assume

$$|f(t,x,y,z) - f(t,x,y,z')| \leq b(x)|z - z'|, \qquad \text{for some bdd. cont. } b.$$

Theorem (Comparison)

Suppose that there exist a positive function Ψ and a positive constant λ:

$$\begin{aligned}
&\mathcal{L}\Psi(x) \leq \lambda(1 + \Psi(x)) \text{ on } (0,\infty)^d, \\
&\lim_{x \to \mathcal{O}} \Psi(x) = \infty, \\
&\forall M > 0, \exists R \text{ s.t. } \Psi(x)/\underline{x} \geq M \text{ for all } \underline{x} \geq R, \\
&c\Psi(x) \geq b(x)|\nabla\Psi(x)\sigma(x)|, \qquad \text{on } (0,\infty)^d.
\end{aligned}$$

Then for any nonneg. subsolution u and supersolution v of at most linear growth,

$$u(t,x) \leq v(t,x), \qquad \text{for } (t,x) \in [0,T] \times (0,\infty)^d.$$

Three examples: more restrictions on $\sigma \implies$ wider dependence of f on z.

Examples: σ has at most linear growth

When $|\sigma(x)| \leq C(1+|x|)$,

$\Psi(x)$ can be chosen as $1+|x|^2$, add another function s.t. $\lim_{x\to\mathcal{O}} \Psi(x) = \infty$.

b can be any bounded function.

Actually, the comparison holds in the class of functions

$$\lim_{|x|\to\infty} |u(t,x)|e^{-A[\log|x|]^2} = 0.$$

[Barles & Buckdahn & Pardoux 97]

Example: No growth constraint on σ

f does not depend on z ($b \equiv 0$).

Assumptions in the comparison theorem is sharp in 1−dimension:

If X is a 1-dim positive martingale, then Ψ exists: $\Psi = \Psi_1 + \Psi_2$,

$$\Psi_1(x) = 2\int_c^x dy \int_c^y \frac{dz}{\sigma^2(z)} \quad \text{and} \quad \Psi_2(x) = x + \int_c^x dy \int_c^y \frac{z}{\sigma^2(z)} dz.$$

- $\lim_{x\downarrow 0} \Psi_1(x) = \infty \iff X$ does not hit 0 (Feller's test).
- $\lim_{x\to\infty} \frac{\Psi_2(x)}{x} = \infty \iff \int_c^\infty \frac{x}{\sigma^2(x)} dx = \infty \iff X$ is a martingale.

  [Delbaen & Shirakawa 02], [Mijatovic & Urusov 10]

σ has super-linear growth

Consider a 1-dim SDE

$$dX_t = \sigma(X_t)dB_t, \quad \text{where } \sigma(x) = \begin{cases} x & \text{if } x \le e \\ x\sqrt{\log x} & \text{if } x > e \end{cases}.$$

X is a martingale.

Consider

$$b(x) = \begin{cases} 1 & \text{if } x \le e \\ \frac{e}{x\sqrt{\log x}} & \text{if } x > e \end{cases}.$$

Then

$$\Psi(x) = \frac{1}{x} + x + \int_e^x dy \int_e^y \frac{z}{\sigma^2(z)} dz$$

satisfies all assumptions.

Conclusion

We study a BSDE whose terminal condition is a linear growth function of a local nonnegative martingale.

- obtain multiple solutions explicitly.
- other than a class D solution, there exists a non-class D solution, which can be viewed as g-strict local martingale.
- derive a necessary/sufficient condition for uniqueness of associated quasi-linear PDE.

"On backward stochastic differential equations and strict local martingales", Stochastic Processes and their Applications, 122 (2012) 2265-2291.

Thanks for your attention!