Probabilistic Interpretation for Systems of Parabolic Partial Differential Equations Combined with Algebra Equations

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Contents

1 Introduction

2 Elementary properties of solutions for FBSDEs

3 Classical solution to PDE system

4 Viscosity solution to PDE system
1 Introduction

2 Elementary properties of solutions for FBSDEs

3 Classical solution to PDE system

4 Viscosity solution to PDE system
In this talk, we try to investigate the connection between the solutions of fully coupled FBSDEs and the solutions of corresponding quasilinear parabolic PDEs combined with algebra equations.

In the special case, i.e. when the diffusion coefficient $\sigma$ is independent of $Z$, the relationship between FBSDEs and PDEs is well understood. In particular, the process $Z$ admits an explicit expression:

$$\nabla u(t, x)\sigma(t, x, u(t, x)).$$
Literature

Decoupled FBSDEs:

Barles-Buckdahn-Pardoux [1997], Darling-Pardoux [1997],
Pardoux-Pradeilles-Rao [1997],
El Karoui-Kapoudjian-Pardoux-Peng-Quenez [1997], Pardoux [1998],
Kobylanski [2000], Buckdahn-Li [2008], Wu-Yu [2008]

Coupled FBSDEs:

Pardoux-Tang [PTRF1999]
In the general case, the relationship becomes implicit since the diffusion coefficient $\sigma$ depends on $Z$ itself. In fact, the implicit relationship is expressed by an algebra equation:

$$v(t, x) = \nabla u(t, x) \sigma(t, x, u(t, x), v(t, x)).$$
When $\sigma$ depends on $Z$, coupled FBSDEs are connected with a new kind of quasilinear parabolic PDE systems combined with algebra equations:

\[
\begin{align*}
\partial_t u(t, x) + (Lu)(t, x, u(t, x), v(t, x)) + g(t, x, u(t, x), v(t, x)) &= 0, \\
v(t, x) &= \nabla u(t, x)\sigma(t, x, u(t, x), v(t, x)), \\
u(T, x) &= \Phi(x),
\end{align*}
\]

(1)

where $L = (Lu_1, Lu_2, \cdots, Lu_m)^\top$ and $L$ is defined by

\[
(L\phi)(t, x, y, z) = \frac{1}{2} \sum_{i,j=1}^{n} (\sigma\sigma^\top)_{ij}(t, x, y, z) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(t, x) + \sum_{i=1}^{n} b_i(t, x, y, z) \frac{\partial \phi}{\partial x_i}(t, x).
\]
When the coefficients $b, \sigma, g$ and $\Phi$ are smooth enough, we will construct the unique classical regular solution

$$u(t, x) = Y_{t, x}^t, \quad v(t, x) = Z_{t, x}^t$$

to the PDE system (1) by introducing a family of FBSDEs parameterized by $(t, x) \in [0, T] \times \mathbb{R}^n$:

$$dX_{s, x}^t = b(s, X_{s, x}^t, Y_{s, x}^t, Z_{s, x}^t)ds + \sigma(s, X_{s, x}^t, Y_{s, x}^t, Z_{s, x}^t)dW_s,$$

$$-dY_{s, x}^t = g(s, X_{s, x}^t, Y_{s, x}^t, Z_{s, x}^t)ds - Z_{s, x}^tdW_s, \quad X_{t, x}^t = x, \quad Y_{T, x}^t = \Phi(X_{T, x}^t).$$
Without the smooth condition of the coefficients, we continue to study the weak solution in the viscosity sense with dimension $m = 1$.

- We successfully prove that $u$ defined by FBSDEs is a viscosity solution.
- We also investigate the uniqueness of viscosity solution in some special cases.
1 Introduction

2 Elementary properties of solutions for FBSDEs

3 Classical solution to PDE system

4 Viscosity solution to PDE system
Given an $m \times n$ full rank matrix $G$, we use the notations

$$h = (x, y, z), \quad A(t, h) = (-G^\top g, Gb, G\sigma)(t, h).$$

And we use the following assumptions:

(H1) $\Phi$ and $A$ are deterministic functions;

(H2) $\Phi$ and $A$ are uniformly Lipschitz continuous with respect to $x$, and $h$ respectively;

(H3) $A$ is continuous with respect to $t$;
(H4) there exist three nonnegative constants $\beta_1, \beta_2$ and $\mu_1$ satisfying $\beta_1 + \beta_2 > 0$, $\mu_1 + \beta_2 > 0$, moreover $\beta_1 > 0$, $\mu_1 > 0$ (resp. $\beta_2 > 0$) in the case of $m > n$ (resp. $n > m$), such that, for each $h = (x, y, z)^\top$, $ar{h} = (\bar{x}, \bar{y}, \bar{z})^\top$,

$$
\langle \Phi(x) - \Phi(\bar{x}), \ G(x - \bar{x}) \rangle \geq \mu_1 |G(x - \bar{x})|^2,
$$
$$
\langle A(t, h) - A(t, \bar{h}), \ h - \bar{h} \rangle \leq - \beta_1 |G(x - \bar{x})|^2
$$
$$
- \beta_2 (|G^\top (y - \bar{y})|^2 + |G^\top (z - \bar{z})|^2).
$$
Existence and uniqueness of FBSDEs

Comparing with the previous section, we consider a generalized family of FBSDEs:

$$
\begin{align*}
\begin{cases}
    dX_s^{\tau,\zeta} = b(s, X_s^{\tau,\zeta}, Y_s^{\tau,\zeta}, Z_s^{\tau,\zeta})ds + \sigma(s, X_s^{\tau,\zeta}, Y_s^{\tau,\zeta}, Z_s^{\tau,\zeta})dW_s, \\
    -dY_s^{\tau,\zeta} = g(s, X_s^{\tau,\zeta}, Y_s^{\tau,\zeta}, Z_s^{\tau,\zeta})ds - Z_s^{\tau,\zeta}dW_s, \\
    X_\tau^{\tau,\zeta} = \zeta, \quad Y_T^{\tau,\zeta} = \Phi(X_T^{\tau,\zeta}), \quad s \in [\tau, T].
\end{cases}
\end{align*}
$$

Under Assumptions (H1)-(H4), FBSDE (3) admits a unique triple of solutions

$$(X_s^{\tau,\zeta}, Y_s^{\tau,\zeta}, Z_s^{\tau,\zeta})_{s \in [\tau, T]} \in S^2(\tau, T; \mathbb{R}^n) \times S^2(\tau, T; \mathbb{R}^m) \times L^2_{\mathcal{F}}(\tau, T; \mathbb{R}^{m \times d}).$$

(Hu-Peng [PTRF1995], Peng-Wu [SICON1999])
Remarks on monotonicity framework

Due to the fully coupled nature of the equations, only with the uniform Lipschitz condition (H2) for coefficients, FBSDE (3) does not necessarily have an adapted solution (a counterexample is given by Antonelli [AAP1993]). So we add the monotonicity conditions (H4). Some advantages of the monotonicity framework are as follows:

- Easy to verify.
- Many existing examples of FBSDEs in optimal control and Hamiltonian systems.
- Need not to impose the non-degenerate condition on $\sigma$. 
Methods to coupled FBSDEs

- **Method of Continuation.** Hu-Peng [PTRF1995], Peng-Wu [SICON], Yong [PTRF1997, TMS2010].

- **Method of Contraction Mapping.** Antonelli [AAP1993], Pardoux-Tang [PTRF1999].

- **Method of Four-Step Scheme.** Ma-Protter-Yong [PTRF1994].

- **A Unified Scheme.** Ma-Wu-Zhang-Zhang [Working paper 2010].
**$L^2$-estimates**

**Proposition** (Yong [PTRF1997], Wu [1998]). Let Assumptions (H1) (H2) and (H4) hold. We have the following estimates:

$$
\mathbb{E}\left[ \sup_{t \leq s \leq T} |X_{s,t}^{t,\zeta}|^2 + \sup_{t \leq s \leq T} |Y_{s,t}^{t,\zeta}|^2 + \int_t^T |Z_{s,t}^{t,\zeta}|^2 \, ds \right] \leq C \mathbb{E}\left\{ 1 + |\zeta|^2 \right\},
$$

$$
\mathbb{E}\left[ \sup_{t \leq s \leq T} |X_{s,t}^{t,\zeta} - X_{s,t}^{t,\zeta'}|^2 + \sup_{t \leq s \leq T} |Y_{s,t}^{t,\zeta} - Y_{s,t}^{t,\zeta'}|^2 + \int_t^T |Z_{s,t}^{t,\zeta} - Z_{s,t}^{t,\zeta'}|^2 \, ds \right]
\leq C \mathbb{E}\left\{ |\zeta - \zeta'|^2 \right\},
$$

where $C$ is a universal constant.

**Problem.** Without additional assumptions, the corresponding $L^p$-estimates ($p > 2$) do not exist in the literature.
Function $u$

$Y_{t,x}^t$ is deterministic. We define a function $u$ from $[0, T] \times \mathbb{R}^n$ to $\mathbb{R}^m$:

$$u(t, x) := Y_{t,x}^t. \quad (4)$$

**Remark.** $Z_{t,x}^t$ is deterministic also, but it is valueless to define another function $v$ from $[0, T] \times \mathbb{R}^{m \times d}$ to $\mathbb{R}^{m \times d}$ in the same way:

$$v(t, x) := Z_{t,x}^t. \quad (5)$$

The main reason is that the trajectories of the process $(Z_{t,x}^s)_{s \in [t,T]}$ are not continuous. In fact, under the standard assumptions (H1)-(H4), we only know that $Z$ belongs to the space $L^2_{\mathcal{F}}(t, T; \mathbb{R}^{m \times d})$, which allow us to arbitrarily change the values of the process $Z$ in any $\mathcal{P}$-null set. In particular, $\Omega \times \{t\}$ is a $\mathcal{P}$-null set, which means that $Z_{t,x}^t$ can be any $\mathbb{R}^{m \times d}$ matrix and Definition (5) says nothing.
Properties of $u$ — Monotonicity

**Proposition.** Let Assumption (H1)-(H4) hold. $u$ is monotonic in the following sense: there exists a nonnegative number $\nu_1 \geq 0$ such that, for each $x, \bar{x} \in \mathbb{R}^n$,

$$\langle u(t,x) - u(t,\bar{x}), G(x - \bar{x}) \rangle \geq \nu_1 |x - \bar{x}|^2.$$ 

Moreover, when $\beta_1 > 0, \mu_1 > 0, \beta_2 \geq 0$ and $n \leq m$, the constant $\nu_1 > 0$.

**Remark.** A function satisfies the above inequality with $\nu_1 \geq 0$, we call it $G$-monotonic; With $\nu_1 > 0$, we call it strong $G$-monotonic.
**Proposition.** Let Assumption (H1)-(H4) hold.

- $u$ is continuous with respect to $(t, x)$. In particular, $u$ is Lipschitz continuous in $x$;
- for each $\mathbb{F}$-stopping time $\tau \leq T$, each $\zeta \in L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n)$, 
  $\left( X_{s}^{\tau, \zeta}, Y_{s}^{\tau, \zeta}, Z_{s}^{\tau, \zeta} \right)_{s \in [\tau, T]}$ denotes the unique solution of FBSDE (3), then

  $$u(\tau, \zeta) = Y_{\tau}^{\tau, \zeta}.$$

**Remark.** For each deterministic $(t, x) \in [0, T] \times \mathbb{R}^n$, if we select $\zeta = X_{t}^{t, x}$, from the above Markovian property and the uniqueness of FBSDEs,

$$u(\tau, X_{t}^{t, x}) = Y_{\tau}^{\tau, X_{t}^{t, x}} = Y_{\tau}^{t, x}.$$
1. Introduction
2. Elementary properties of solutions for FBSDEs
3. Classical solution to PDE system
4. Viscosity solution to PDE system
A uniqueness result

Theorem

We assume (H1)-(H4), if $u^1 \in C^{1,2}([0, T] \times \mathbb{R}^n)$ and $(u^1, v^1)$ is a solution of PDE system (1), $(u^1, v^1)$ is uniformly Lipschitz continuous with respect to $x$, then for each $(t, x) \in [0, T] \times \mathbb{R}^n$, $u^1(t, x)$ is uniquely determined by $Y^t_x$.

From Itô’s formula and the uniqueness of FBSDEs, we proved it.

The theorem unveils a fascinating fact that there exists some connection between the family of coupled FBSDEs (2) and PDE system (1).
Smoothness Assumption

(A1)

\[ b \in \mathcal{C}^3_{l,b}(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}; \mathbb{R}^n), \]
\[ \sigma \in \mathcal{C}^3_{l,b}(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}; \mathbb{R}^{n \times d}), \]
\[ g \in \mathcal{C}^3_{l,b}(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}; \mathbb{R}^m), \]
\[ \Phi \in \mathcal{C}^3_{l,b}(\mathbb{R}^n; \mathbb{R}^m). \]
Regularity of $u$

First, we strengthen the continuity of $u$:

**Proposition.** $u$ is Lipschitz continuous in $x$ and 1/2-Hölder continuous in $t$.

Second, we study the continuous differentiability:

**Proposition.** Let Assumptions (H1)-(H4) and (A1) hold. The function $u$ belongs to $C^2_{l,b}(\mathbb{R}^n; \mathbb{R}^m)$.

**Remark.** For the decoupled FBSDEs, by virtue of Kolmogorov’s lemma and $L^p$-estimates (for all $p \geq 2$) of SDEs and BSDEs, Pardoux and Peng (1992) obtained the same smoothness result. For the general coupled case, we improve Pardoux and Peng’s method and drew the conclusion through Lebesgue’s dominated convergence theorem and only $L^2$-estimates.
Combining the method of El Karoui-Peng-Quenez [1997] and the method of continuation, we establish the differentiability in Malliavin’s sense.

**Proposition.** Under Assumption (H1)-(H4) and (A1), the solution $(X_{s}^{t,x}, Y_{s}^{t,x}, Z_{s}^{t,x})_{t \leq s \leq T}$ of FBSDE (2) is Malliavin differentiable, and a version of $(D_{\theta}X_{s}^{t,x}, D_{\theta}Y_{s}^{t,x}, D_{\theta}Z_{s}^{t,x})_{\theta \leq s \leq T}$ is given by

\[
\begin{align*}
D_{\theta}X_{s}^{t,x} &= \sigma(\Theta_{\theta}^{t,x}) + \int_{\theta}^{s} \nabla b(\Theta_{r}^{t,x}) \cdot D_{\theta} \Theta_{r}^{t,x} \, dr + \int_{\theta}^{s} \nabla \sigma(\Theta_{r}^{t,x}) \cdot D_{\theta} \Theta_{r}^{t,x} \, dW_{r}, \\
D_{\theta}Y_{s}^{t,x} &= \Phi'(X_{T}^{t,x})D_{\theta}X_{T}^{t,x} + \int_{s}^{T} \nabla g(\Theta_{r}^{t,x}) \cdot D_{\theta} \Theta_{r}^{t,x} \, dr - \int_{s}^{T} D_{\theta}Z_{r}^{t,x} \, dW_{r},
\end{align*}
\]

where $\Theta_{r}^{t,x} = (X_{r}^{t,x}, Y_{r}^{t,x}, Z_{r}^{t,x})$. Moreover, $(D_{s}Y_{s}^{t,x})_{t \leq s \leq T}$ provides a version of $(Z_{s}^{t,x})_{t \leq s \leq T}$. 
From the above proposition and the Markov property of $u$, we know that $Z_{s}^{t,x}$ satisfies the algebra equation:

$$Z_{s}^{t,x} = D_s Y_{s}^{t,x}$$

$$= D_s u(s, X_{s}^{t,x})$$

$$= \nabla u(s, X_{s}^{t,x}) D_s X_{s}^{t,x}$$

$$= \nabla u(s, X_{s}^{t,x}) \sigma(s, X_{s}^{t,x}, Y_{s}^{t,x}, Z_{s}^{t,x}).$$
Lemma. Let Assumptions (H1)-(H4) hold. We assume that $u^1 \in C^1_{l,b}(\mathbb{R}^n; \mathbb{R}^m)$ is $G$-monotonic. When $\beta_1 > 0, \mu_1 > 0, \beta_2 \geq 0$ and $n \leq m$, we further assume that $u^1$ is strong $G$-monotonic. Then

(a) the algebra equation

$$v^1(t, x) = \nabla u^1(t, x) \sigma(t, x, u^1(t, x), v^1(t, x))$$

admits at most one solution for any $(t, x)$;

(b) an estimate:

$$|v^1(t, x)| \leq C|\sigma(t, x, u^1(t, x), 0)|;$$

(c) $v^1(t, x)$ is continuous.
From the above lemma, $Z_{s,x}^t$ is continuous in $s \in [t, T]$. So we can define

$$v(t, x) := Z_{t,x}^t,$$

and $v(t, x)$ is the unique solution of the algebra equation

$$v(t, x) = \nabla u(t, x) \sigma(t, x, u(t, x), v(t, x)).$$
Theorem. Under Assumption (H1)-(H4) and (A1), \( u \) is of class \( C^{1,2}([0, T] \times \mathbb{R}^n) \), \( v \) is of class \( C^{0,0}([0, T] \times \mathbb{R}^n) \), and \((u, v)\) solves PDE system (1).

Theorem. Let Assumption (H1)-(H4) and (A1) hold. Let \( u^1 \in C^{1,2}([0, T] \times \mathbb{R}^n) \) is \( G \)-monotonic, \( v^1 \in C^{0,0}([0, T] \times \mathbb{R}^n) \) and \((u^1, v^1)\) is a solution of PDE system. When \( \beta_1 > 0, \mu_1 > 0, \beta_2 \geq 0 \) and \( n \leq m \), we further assume that \( u^1 \) is strong \( G \)-monotonic. Then for each \((t, x) \in [0, T] \times \mathbb{R}^n\), \((u^1(t, x), v^1(t, x))\) is uniquely determined by \((Y^{t,x}_T, Z^{t,x}_T)\).
1 Introduction

2 Elementary properties of solutions for FBSDEs

3 Classical solution to PDE system

4 Viscosity solution to PDE system
**Definition of viscosity solution**  \((m = 1)\)

**Definition.** Let \(u \in C([0, T] \times \mathbb{R}^n)\) satisfy \(u(T, x) = \Phi(x), \ x \in \mathbb{R}^n\). \(u\) is called a viscosity subsolution (resp. supersolution) of PDE system (1) if, for each \((t, x) \in [0, T) \times \mathbb{R}^n\), \(\phi \in C^{2,3}([0, T] \times \mathbb{R}^n)\), \((\phi - u)\) attains a local minimum (resp. maximum) at \((t, x)\) and \(\phi(t, x) - u(t, x) = 0\), we have

\[
\begin{cases}
\partial_t \phi(t, x) + (\mathcal{L}\phi)(t, x, u(t, x), \psi(t, x)) + g(t, x, u(t, x), \psi(t, x)) \geq 0, \\
\psi(t, x) = \nabla \phi(t, x)\sigma(t, x, u(t, x), \psi(t, x)).
\end{cases}
\]

\[
\begin{cases}
\partial_t \phi(t, x) + (\mathcal{L}\phi)(t, x, u(t, x), \psi(t, x)) + g(t, x, u(t, x), \psi(t, x)) \leq 0, \\
\psi(t, x) = \nabla \phi(t, x)\sigma(t, x, u(t, x), \psi(t, x)).
\end{cases}
\]

\(u\) is called a viscosity solution of PDE system (1) if it is both a viscosity subsolution and a supersolution.
Let Assumption (H1)-(H4) hold. For each \((t, x) \in [0, T) \times \mathbb{R}^n\),
\(\phi \in C^{2,3}([0, T] \times \mathbb{R}^n)\), \((\phi - u)\) attains a local minimum (resp. maximum) at \((t, x)\) and \(\phi(t, x) - u(t, x) = 0\), then there exists a positive number \(q(t, x) \geq 0\) such that

\[
\nabla \phi(t, x) = q(t, x)G. \tag{8}
\]
Remark (1/2)

The lemma says that any smooth function $f : x \in \mathbb{R}^n \mapsto f(x)$ that has the $G$-monotonicity property satisfies

$$\nabla f(x) = q(x)G,$$

for some function $q$ with values in $[0, +\infty)$. Up to a change of coordinates, we may assume $G$ to match $e_1$, the first vector of the canonical basis. We deduce that

$$\frac{\partial f}{\partial x_i}(x) = 0, \quad i \in \{2, \ldots, n\},$$

so that $f$ is a function of $x_1$ only.
Remark (2/2)

Actually, when $m = 1$ and $G = e_1$, the problem is a one-dimensional problem only. In detail, the family of FBSDEs (2) reduces to

$$
\begin{align*}
    dX_s^1 &= b_1(s, X_s^1, Y_s, Z_s)ds + \sigma_1(s, X_s^1, Y_s, Z_s)dW_s \\
    -dY_s &= g(s, X_s^1, Y_s, Z_s)ds - Z_sdW_s, \\
    X_t^1 &= x^1, \quad Y_T = \Phi(X_T^1).
\end{align*}
$$

The corresponding PDE system (1) reduces to:

$$
\begin{align*}
    \frac{\partial u}{\partial t}(t, x_1) + \frac{1}{2} \left( \sigma_1 \sigma_1^\top \right)(t, x_1, u(t, x_1), v(t, x_1)) \frac{\partial^2 u}{\partial x_1^2}(t, x_1) \\
    &\quad + b_1(t, x_1, u(t, x_1), v(t, x_1)) \frac{\partial u}{\partial x_1}(t, x_1) + g(t, x_1, u(t, x_1), v(t, x_1)) = 0, \\
    v(t, x_1) &= \frac{\partial u}{\partial x_1}(t, x_1) \sigma(t, x_1, u(t, x_1), v(t, x_1)), \\
    u(T, x_1) &= \Phi(x_1).
\end{align*}
$$
Lemma

Let Assumptions (H1)-(H4) hold. For each \((t, x)\) \(\in [0, T) \times \mathbb{R}^n\), \(\phi \in C^{2,3}([0, T] \times \mathbb{R}^n)\), \((\phi - u)\) attains a local minimum (resp. maximum) at \((t, x)\) and \(\phi(t, x) - u(t, x) = 0\), then there exists a domain

\[
D^{t,x}(\delta_1) = \{(s, y) \in [0, T] \times \mathbb{R}^n \mid t - \delta_1 < s < t + \delta_1, |y - x| < \delta_1\}, \tag{9}
\]

where \(\delta_1 > 0\) is a constant depending on \(t, x\) and \(\phi\), such that
(a) **(Existence and Uniqueness)** the algebra equation

\[ \psi(s, y) = \nabla \phi(s, y) \sigma(s, y, u(s, y), \psi(s, y)) \]  \hspace{1cm} (10)

has a unique solution for each \((s, y) \in D_{t,x}^{t,x}(\delta_1)\);

(b) **(An Estimate)** for each \((s, y) \in D_{t,x}^{t,x}(\delta_1)\),

\[ |\psi(s, y)| \leq 2 |\nabla \phi(s, y)| |\sigma(s, y, u(s, y), 0)| ; \]

(c) **(Continuity)** \(\psi\) is continuous in \(D_{t,x}^{t,x}(\delta_1)\).
Existence theorem

**Theorem**

Let Assumption (H1)-(H4) hold. The function $u$ defined by (4) is continuous and it is a viscosity solution of PDE system (1).

The proof is a combination of the method of Pardoux-Tang [PTRF1999] and the techniques about algebra equation. The proof framework is reduction to absurdity.
Uniqueness: Special case 1  \( \sigma = \sigma(t, x) \)

\[
\begin{aligned}
\partial_t u(t, x) + \frac{1}{2} \text{Tr} \left( (\sigma \sigma^\top)(t, x) D^2 u(t, x) \right) \\
+ \langle b(t, x, u(t, x), \nabla u(t, x) \sigma(t, x)), Du(t, x) \rangle \\
+ g(t, x, u(t, x), \nabla u(t, x) \sigma(t, x)) = 0, \\
u(t, x) = \Phi(x),
\end{aligned}
\] (11)

Theorem

Let Assumptions (H1)-(H3) hold. Then there exists at most one viscosity solution of (11) in the class of continuous functions which are \textit{Lipschitz continuous} in spatial variable \( x \).
In this case, FBSDEs (2) become a family of FBODEs without randomness. (Wu [Ph.D. Thesis 1997])

Lemma

Let Assumption (H1)-(H5) hold. If \( u \in C(\mathbb{R}^n) \) is Lipschitz continuous and monotonic in \( x \), and \( u \) is a viscosity solution of PDE system (1), then \( u \) is a viscosity solution of the following PDE system:

\[
\begin{align*}
\partial_t u(t, x) + \langle b(t, x, u(t, x), 0), Du(t, x) \rangle + g(t, x, u(t, x), 0) &= 0, \\
ur(t, x) &= \Phi(x).
\end{align*}
\]

(12)
Theorem

Let Assumption (H1)-(H5) hold. Then there exists at most one viscosity solution of PDE system (1) in the class of continuous functions which are Lipschitz continuous and monotonic in spatial variable $x$.

Proof. We notice that PDE (12) is the special case of PDE (11) when $\sigma \equiv 0$, so Theorem in Case 1 gives the uniqueness of viscosity solution of PDE (12). Lemma in Case 2 shows that the uniqueness of viscosity solution of PDE (12) implies the uniqueness of PDE (1).
Thank you for your attention!