Probabilistic Interpretation for Systems of Parabolic Partial Differential Equations Combined with Algebra Equations

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Contents

Introduction

In this talk, we try to investigate the connection between the solutions of fully coupled FBSDEs and the solutions of corresponding quasilinear parabolic PDEs combined with algebra equations.

- Peng, *Open problems on BSDEs*, 1999.

Special case

In the special case, i.e. when the diffusion coefficient σ is independent of Z, the relationship between FBSDEs and PDEs is well understood. In particular, the process Z admits an explicit expression:

$$\nabla u(t,x)\sigma(t,x,u(t,x)).$$

Literature

Decoupled FBSDEs:

Peng [1991, 1992], Pardoux-Peng [1993], Peng [1997],
Barles-Buckdahn-Pardoux [1997], Darling-Pardoux [1997],
Pardoux-Pradeilles-Rao [1997],
El Karoui-Kapoudjian-Pardoux-Peng-Quenez [1997], Pardoux [1998],
Kobylanski [2000], Buckdahn-Li [2008], Wu-Yu [2008]

Coupled FBSDEs:

Pardoux-Tang [PTRF1999]

General case

In the general case, the relationship becomes implicit since the diffusion coefficient σ depends on Z itself. In fact, the implicit relationship is expressed by an algebra equation:

$$v(t,x) = \nabla u(t,x)\sigma(t,x,u(t,x),v(t,x)).$$

Main result (1/3)

When σ depends on Z, coupled FBSDEs are connected with a new kind of quasilinear parabolic PDE systems combined with algebra equations:

$$\begin{cases} \partial_t u(t,x) + (\mathcal{L}u)(t,x,u(t,x),v(t,x)) + g(t,x,u(t,x),v(t,x)) = 0, \\ v(t,x) = \nabla u(t,x)\sigma(t,x,u(t,x),v(t,x)), \\ u(T,x) = \Phi(x), \end{cases} \tag{1}$$

where $\mathcal{L} = (Lu_1, Lu_2, \cdots, Lu_m)^\top$ and L is defined by

$$(\mathcal{L}\phi)(t,x,y,z) = \frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^\tau)_{ij}(t,x,y,z)\frac{\partial^2\phi}{\partial x_i \partial x_j}(t,x) + \sum_{i=1}^{n} b_i(t,x,y,z)\frac{\partial \phi}{\partial x_i}(t,x).$$

Main result (2/3)

When the coefficients b, σ, g and Φ are smooth enough, we will construct the unique classical regular solution

$$u(t,x) = Y_t^{t,x}, \quad v(t,x) = Z_t^{t,x}$$

to the PDE system (1) by introducing a family of FBSDEs parameterized by $(t,x) \in [0,T] \times \mathbb{R}^n$:

$$\begin{cases} dX_s^{t,x} = b(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})ds + \sigma(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})dW_s, \\ -dY_s^{t,x} = g(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})ds - Z_s^{t,x} dW_s, \\ X_t^{t,x} = x, \quad Y_T^{t,x} = \Phi(X_T^{t,x}). \end{cases} \tag{2}$$

Main result (3/3)

Without the smooth condition of the coefficients, we continue to study the weak solution in the viscosity sense with dimension $m = 1$.

- We successfully prove that u defined by FBSDEs is a viscosity solution.
- We also investigate the uniqueness of viscosity solution in some special cases.

Standard assumptions (1/2)

Given an $m \times n$ full rank matrix G, we use the notations

$$h = (x, y, z), \quad A(t, h) = (-G^\top g, Gb, G\sigma)(t, h).$$

And we use the following assumptions:

(H1) Φ and A are deterministic functions;

(H2) Φ and A are uniformly Lipschitz continuous with respect to x, and h respectively;

(H3) A is continuous with respect to t;

Standard assumptions (2/2)

(H4) there exist three nonnegative constants β_1, β_2 and μ_1 satisfying $\beta_1 + \beta_2 > 0$, $\mu_1 + \beta_2 > 0$, moreover $\beta_1 > 0$, $\mu_1 > 0$ (resp. $\beta_2 > 0$) in the case of $m > n$ (resp. $n > m$), such that, for each $h = (x, y, z)^\top$, $\overline{h} = (\overline{x}, \overline{y}, \overline{z})^\top$,

$$\langle \Phi(x) - \Phi(\overline{x}),\ G(x - \overline{x}) \rangle \geq \mu_1 \left| G(x - \overline{x}) \right|^2,$$

$$\begin{aligned}\langle A(t,h) - A(t,\overline{h}),\ h - \overline{h} \rangle \leq\ & -\beta_1 \left| G(x - \overline{x}) \right|^2 \\ & -\beta_2 \left(\left| G^\top (y - \overline{y}) \right|^2 + \left| G^\top (z - \overline{z}) \right|^2 \right).\end{aligned}$$

Existence and uniqueness of FBSDEs

Comparing with the previous section, we consider a generalized family of FBSDEs:

$$\begin{cases} dX_s^{\tau,\zeta} = b(s, X_s^{\tau,\zeta}, Y_s^{\tau,\zeta}, Z_s^{\tau,\zeta})ds + \sigma(s, X_s^{\tau,\zeta}, Y_s^{\tau,\zeta}, Z_s^{\tau,\zeta})dW_s, \\ -dY_s^{\tau,\zeta} = g(s, X_s^{\tau,\zeta}, Y_s^{\tau,\zeta}, Z_s^{\tau,\zeta})ds - Z_s^{\tau,\zeta} dW_s, \\ X_\tau^{\tau,\zeta} = \zeta, \quad Y_T^{\tau,\zeta} = \Phi(X_T^{\tau,\zeta}), \quad s \in [\tau, T]. \end{cases} \tag{3}$$

Under Assumptions (H1)-(H4), FBSDE (3) admits a unique triple of solutions

$$(X_s^{\tau,\zeta}, Y_s^{\tau,\zeta}, Z_s^{\tau,\zeta})_{s\in[\tau,T]} \in \mathcal{S}^2(\tau, T; \mathbb{R}^n) \times \mathcal{S}^2(\tau, T; \mathbb{R}^m) \times L^2_{\mathcal{F}}(\tau, T; \mathbb{R}^{m\times d}).$$

(Hu-Peng [PTRF1995], Peng-Wu [SICON1999])

Remarks on monotonicity framework

Due to the fully coupled nature of the equations, only with the uniform Lipschitz condition (H2) for coefficients, FBSDE (3) does not necessary have an adapted solution (a counterexample is given by Antonelli [AAP1993]). So we add the monotonicity conditions (H4). Some advantages of the monotonicity framework are as follows:

- Easy to verify.
- Many existing examples of FBSDEs in optimal control and Hamiltonian systems.
- Need not to impose the non-degenerate condition on σ.

Methods to coupled FBSDEs

- Method of Continuation. Hu-Peng [PTRF1995], Peng-Wu [SICON], Yong [PTRF1997, TMS2010].
- Method of Contraction Mapping. Antonelli [AAP1993], Pardoux-Tang [PTRF1999].
- Method of Four-Step Scheme. Ma-Protter-Yong [PTRF1994].
- A Unified Scheme. Ma-Wu-Zhang-Zhang [Working paper 2010].

L^2-estimates

Proposition (Yong [PTRF1997], Wu [1998]). Let Assumptions (H1) (H2) and (H4) hold. We have the following estimates:

$$\mathbb{E}\left[\sup_{t\le s\le T}|X_s^{t,\zeta}|^2+\sup_{t\le s\le T}|Y_s^{t,\zeta}|^2+\int_t^T|Z_s^{t,\zeta}|^2ds\right]\le C\mathbb{E}\left\{1+|\zeta|^2\right\},$$

$$\mathbb{E}\left[\sup_{t\le s\le T}|X_s^{t,\zeta}-X_s^{t,\zeta'}|^2+\sup_{t\le s\le T}|Y_s^{t,\zeta}-Y_s^{t,\zeta'}|^2+\int_t^T|Z_s^{t,\zeta}-Z_s^{t,\zeta'}|^2ds\right]$$
$$\le C\mathbb{E}\left\{|\zeta-\zeta'|^2\right\},$$

where C is a universal constant.

Problem. Without additional assumptions, the corresponding L^p-estimates $(p>2)$ do not exist in the literature.

Function u

$Y_t^{t,x}$ is deterministic. We define a function u from $[0,T]\times\mathbb{R}^n$ to $\mathbb{R}^m$:

$$u(t,x) := Y_t^{t,x}. \tag{4}$$

Remark. $Z_t^{t,x}$ is deterministic also, but it is valueless to define another function v from $[0,T]\times\mathbb{R}^{m\times d}$ to $\mathbb{R}^{m\times d}$ in the same way:

$$v(t,x) := Z_t^{t,x}. \tag{5}$$

The main reason is that the trajectories of the process $(Z_s^{t,x})_{s\in[t,T]}$ are not continuous. In fact, under the standard assumptions (H1)-(H4), we only know that Z belongs to the space $L^2_{\mathcal{F}}(t,T;\mathbb{R}^{m\times d})$, which allow us to arbitrarily change the values of the process Z in any $\mathcal{P}$-null set. In particular, $\Omega\times\{t\}$ is a $\mathcal{P}$-null set, which means that $Z_t^{t,x}$ can be any $\mathbb{R}^{m\times d}$ matrix and Definition (5) says nothing.

Properties of u — Monotonicity

Proposition. Let Assumption (H1)-(H4) hold. u is monotonic in the following sense: there exists a nonnegative number $\nu_1 \geq 0$ such that, for each $x, \bar{x} \in \mathbb{R}^n$,

$$\langle u(t,x) - u(t,\bar{x}),\ G(x-\bar{x})\rangle \geq \nu_1 |x-\bar{x}|^2.$$

Moreover, when $\beta_1 > 0, \mu_1 > 0, \beta_2 \geq 0$ and $n \leq m$, the constant $\nu_1 > 0$.

Remark. A function satisfies the above inequality with $\nu_1 \geq 0$, we call it G-monotonic; With $\nu_1 > 0$, we all it strong G-monotonic.

Properties of u — Continuity and Markovian

Proposition. Let Assumption (H1)-(H4) hold.

- u is continuous with respect to (t, x). In particular, u is Lipschitz continuous in x;
- for each $\mathbb{F}$-stopping time $\tau \leq T$, each $\zeta \in L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n)$, $(X_s^{\tau,\zeta}, Y_s^{\tau,\zeta}, Z_s^{\tau,\zeta})_{s\in[\tau,T]}$ denotes the unique solution of FBSDE (3), then

$$u(\tau, \zeta) = Y_\tau^{\tau,\zeta}.$$

Remark. For each deterministic $(t, x) \in [0, T] \times \mathbb{R}^n$, if we select $\zeta = X_\tau^{t,x}$, from the above Markovian property and the uniqueness of FBSDEs,

$$u(\tau, X_\tau^{t,x}) = Y_\tau^{\tau, X_\tau^{t,x}} = Y_\tau^{t,x}.$$

A uniqueness result

Theorem

We assume (H1)-(H4), if $u^1 \in C^{1,2}([0,T] \times \mathbb{R}^n)$ *and* (u^1, v^1) *is a solution of PDE system (1),* (u^1, v^1) *is uniformly Lipschitz continuous with respect to* x*, then for each* $(t,x) \in [0,T] \times \mathbb{R}^n$*,* $u^1(t,x)$ *is uniquely determined by* $Y_t^{t,x}$*.*

From Itô's formula and the uniqueness of FBSDEs, we proved it.

The theorem unveils a fascinating fact that there exists some connection between the family of coupled FBSDEs (2) and PDE system (1).

Smoothness Assumption

(A1)

$$
\begin{aligned}
& b \in C^3_{l,b}(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}; \mathbb{R}^n), \\
& \sigma \in C^3_{l,b}(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}; \mathbb{R}^{n\times d}), \\
& g \in C^3_{l,b}(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}; \mathbb{R}^m), \\
& \Phi \in C^3_{l,b}(\mathbb{R}^n; \mathbb{R}^m).
\end{aligned}
$$

Regularity of u

First, we strengthen the continuity of u:

Proposition. u is Lipschitz continuous in x and $1/2$-Hölder continuous in t.

Second, we study the continuous differentiability:

Proposition. Let Assumptions (H1)-(H4) and (A1) hold. The function u belongs to $C^2_{l,b}(\mathbb{R}^n;\mathbb{R}^m)$.

Remark. For the decoupled FBSDEs, by virtue of Kolmogorov's lemma and L^p-estimates (for all $p \geq 2$) of SDEs and BSDEs, Pardoux and Peng (1992) obtained the same smoothness result. For the general coupled case, we improve Pardoux and Peng's method and drew the conclusion through Lebesgue's dominated convergence theorem and only L^2-estimates.

Differentiability in Malliavin's sense (1/2)

Combining the method of El Karoui-Peng-Quenez [1997] and the *method of continuation*, we establish the differentiability in Malliavin's sense.

Proposition. Under Assumption (H1)-(H4) and (A1), the solution $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})_{t\le s\le T}$ of FBSDE (2) is Malliavin differentiable, and a version of $(D_\theta X_s^{t,x}, D_\theta Y_s^{t,x}, D_\theta Z_s^{t,x})_{\theta\le s\le T}$ is given by

$$\begin{cases} D_\theta X_s^{t,x} = \sigma(\Theta_\theta^{t,x}) + \int_\theta^s \nabla b(\Theta_r^{t,x}) \cdot D_\theta \Theta_r^{t,x} dr + \int_\theta^s \nabla\sigma(\Theta_r^{t,x}) \cdot D_\theta \Theta_r^{t,x} dW_r, \\ D_\theta Y_s^{t,x} = \Phi'(X_T^{t,x}) D_\theta X_T^{t,x} + \int_s^T \nabla g(\Theta_r^{t,x}) \cdot D_\theta \Theta_r^{t,x} dr - \int_s^T D_\theta Z_r^{t,x} dW_r, \end{cases}$$

where $\Theta_r^{t,x} = (X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})$. Moreover $(D_s Y_s^{t,x})_{t\le s\le T}$ provides a version of $(Z_s^{t,x})_{t\le s\le T}$.

Differentiability in Malliavin's sense (2/2)

From the above proposition and the Markov property of u, we know that $Z_s^{t,x}$ satisfies the algebra equation:

$$\begin{aligned} Z_s^{t,x} &= D_s Y_s^{t,x} \\ &= D_s u(s, X_s^{t,x}) \\ &= \nabla u(s, X_s^{t,x}) D_s X_s^{t,x} \\ &= \nabla u(s, X_s^{t,x}) \sigma(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}). \end{aligned}$$

Algebra equation (1/2)

Lemma. Let Assumptions (H1)-(H4) hold. We assume that $u^1 \in C^1_{l,b}(\mathbb{R}^n; \mathbb{R}^m)$ is G-monotonic. When $\beta_1 > 0, \mu_1 > 0, \beta_2 \geq 0$ and $n \leq m$, we further assume that u^1 is strong G-monotonic. Then

(a) the algebra equation

$$v^1(t,x) = \nabla u^1(t,x)\sigma(t,x,u^1(t,x),v^1(t,x))$$

admits at most one solution for any (t,x);

(b) an estimate:

$$|v^1(t,x)| \leq C|\sigma(t,x,u^1(t,x),0)|;$$

(c) $v^1(t,x)$ is continuous.

Algebra equation (2/2)

From the above lemma, $Z_s^{t,x}$ is continuous in $s \in [t, T]$. So we can define

$$v(t,x) := Z_t^{t,x},$$

and $v(t,x)$ is the unique solution of the algebra equation

$$v(t,x) = \nabla u(t,x)\sigma(t,x,u(t,x),v(t,x)).$$

Classical solution

Theorem. Under Assumption (H1)-(H4) and (A1), u is of class $C^{1,2}_{l,b}([0,T]\times\mathbb{R}^n)$, v is of class $C^{0,0}([0,T]\times\mathbb{R}^n)$, and (u,v) solves PDE system (1).

Theorem. Let Assumption (H1)-(H4) and (A1) hold. Let $u^1\in C^{1,2}_{l,b}([0,T]\times\mathbb{R}^n)$ is G-monotonic, $v^1\in C^{0,0}([0,T]\times\mathbb{R}^n)$ and (u^1,v^1) is a solution of PDE system. When $\beta_1>0,\mu_1>0,\beta_2\geq 0$ and $n\leq m$, we further assume that u^1 is strong G-monotonic. Then for each $(t,x)\in[0,T]\times\mathbb{R}^n$, $(u^1(t,x),v^1(t,x))$ is uniquely determined by $(Y^{t,x}_t,Z^{t,x}_t)$.

Definition of viscosity solution $(m = 1)$

Definition. Let $u \in C([0,T]\times\mathbb{R}^n)$ satisfy $u(T,x) = \Phi(x),\ x \in \mathbb{R}^n$. u is called a viscosity subsolution (resp. supersolution) of PDE system (1) if, for each $(t,x) \in [0,T)\times\mathbb{R}^n$, $\phi \in C^{2,3}([0,T]\times\mathbb{R}^n)$, $(\phi - u)$ attains a local minimum (resp. maximum) at (t,x) and $\phi(t,x) - u(t,x) = 0$, we have

$$\begin{cases} \partial_t\phi(t,x) + (\mathcal{L}\phi)(t,x,u(t,x),\psi(t,x)) + g(t,x,u(t,x),\psi(t,x)) \geq 0, \\ \psi(t,x) = \nabla\phi(t,x)\sigma(t,x,u(t,x),\psi(t,x)). \end{cases} \tag{6}$$

$$\left(\text{resp.} \begin{cases} \partial_t\phi(t,x) + (\mathcal{L}\phi)(t,x,u(t,x),\psi(t,x)) + g(t,x,u(t,x),\psi(t,x)) \leq 0, \\ \psi(t,x) = \nabla\phi(t,x)\sigma(t,x,u(t,x),\psi(t,x)). \end{cases} \right) \tag{7}$$

u is called a viscosity solution of PDE system (1) if it is both a viscosity subsolution and a supersolution.

Gradient representation lemma

Lemma

Let Assumption (H1)-(H4) hold. For each $(t,x) \in [0,T) \times \mathbb{R}^n$, $\phi \in C^{2,3}([0,T] \times \mathbb{R}^n)$, $(\phi - u)$ attains a local minimum (resp. maximum) at (t,x) and $\phi(t,x) - u(t,x) = 0$, then there exists a positive number $q(t,x) \geq 0$ such that

$$\nabla\phi(t,x) = q(t,x)G. \tag{8}$$

Remark (1/2)

The lemma says that any smooth function $f : x \in \mathbb{R}^n \mapsto f(x)$ that has the G-monotonicity property satisfies

$$\nabla f(x) = q(x) G,$$

for some function q with values in $[0, +\infty)$. Up to a change of coordinates, we may assume G to match e_1, the first vector of the canonical basis. We deduce that

$$\frac{\partial f}{\partial x_i}(x) = 0, \quad i \in \{2, \ldots, n\},$$

so that f is a function of x_1 only.

Remark (2/2)

Actually, when $m = 1$ and $G = e_1$, the problem is a one-dimensional problem only. In detail, the family of FBSDEs (2) reduces to

$$\begin{cases} dX_s^1 = b_1(s, X_s^1, Y_s, Z_s)ds + \sigma_1(s, X_s^1, Y_s, Z_s)dW_s \\ -dY_s = g(s, X_s^1, Y_s, Z_s)ds - Z_s dW_s, \\ X_t^1 = x^1, \quad Y_T = \Phi(X_T^1). \end{cases}$$

The corresponding PDE system (1) reduces to:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x_1) + \dfrac{1}{2}\left(\sigma_1 \sigma_1^\top\right)(t, x_1, u(t, x_1), v(t, x_1))\dfrac{\partial^2 u}{\partial x_1^2}(t, x_1) \\ \qquad + b_1(t, x_1, u(t, x_1), v(t, x_1))\dfrac{\partial u}{\partial x_1}(t, x_1) + g(t, x_1, u(t, x_1), v(t, x_1)) = 0, \\ v(t, x_1) = \dfrac{\partial u}{\partial x_1}(t, x_1)\sigma(t, x_1, u(t, x_1), v(t, x_1)), \\ u(T, x_1) = \Phi(x_1). \end{cases}$$

Treatment of the algebra equation (1/2)

Lemma

Let Assumptions (H1)-(H4) hold. For each $(t,x) \in [0,T) \times \mathbb{R}^n$, $\phi \in C^{2,3}([0,T] \times \mathbb{R}^n)$, $(\phi - u)$ attains a local minimum (resp. maximum) at (t,x) and $\phi(t,x) - u(t,x) = 0$, then there exists a domain

$$D^{t,x}(\delta_1) = \{(s,y) \in [0,T] \times \mathbb{R}^n \,\big|\, t - \delta_1 < s < t + \delta_1,\ |y - x| < \delta_1\}, \tag{9}$$

where $\delta_1 > 0$ is a constant depending on t, x and ϕ, such that

Treatment of the algebra equation (2/2)

(a) **(Existence and Uniqueness)** the algebra equation

$$\psi(s,y) = \nabla\phi(s,y)\sigma(s,y,u(s,y),\psi(s,y)) \tag{10}$$

has a unique solution for each $(s,y) \in D^{t,x}(\delta_1)$;

(b) **(An Estimate)** for each $(s,y) \in D^{t,x}(\delta_1)$,

$$|\psi(s,y)| \le 2\,|\nabla\phi(s,y)|\,|\sigma(s,y,u(s,y),0)|\,;$$

(c) **(Continuity)** ψ is continuous in $D^{t,x}(\delta_1)$.

Existence theorem

Theorem

Let Assumption (H1)-(H4) hold. The function u defined by (4) is continuous and it is a viscosity solution of PDE system (1).

The proof is a combination of the method of Pardoux-Tang [PTRF1999] and the techniques about algebra equation. The proof framework is reduction to absurdity.

Uniqueness: Special case 1 $\sigma = \sigma(t,x)$

$$\begin{cases} \partial_t u(t,x) + \frac{1}{2} Tr\Big((\sigma\sigma^\top)(t,x) D^2 u(t,x)\Big) \\ \qquad\qquad + \langle b(t,x,u(t,x),\nabla u(t,x)\sigma(t,x)),\ Du(t,x)\rangle \\ \qquad\qquad + g(t,x,u(t,x),\nabla u(t,x)\sigma(t,x)) = 0, \\ u(t,x) = \Phi(x), \end{cases} \tag{11}$$

Theorem

Let Assumptions (H1)-(H3) hold. Then there exists at most one viscosity solution of (11) in the class of continuous functions which are Lipschitz continuous in spatial variable x.

Uniqueness: Special case 2 $\sigma(t, x, y, 0) = 0$

In this case, FBSDEs (2) become a family of FBODEs without randomness. (Wu [Ph.D. Thesis 1997])

Lemma

Let Assumption (H1)-(H5) hold. If $u \in C([0, T] \times \mathbb{R}^n)$ is Lipschitz continuous and monotonic in x, and u is a viscosity solution of PDE system (1), then u is a viscosity solution of the following PDE system:

$$\begin{cases} \partial_t u(t,x) + \langle b(t,x,u(t,x),0),\ Du(t,x)\rangle + g(t,x,u(t,x),0) = 0, \\ u(T,x) = \Phi(x). \end{cases} \tag{12}$$

Uniqueness: Special case 2 $\sigma(t,x,y,0)=0$

Theorem

Let Assumption (H1)-(H5) hold. Then there exists at most one viscosity solution of PDE system (1) in the class of continuous functions which are Lipschitz continuous and monotonic in spatial variable x.

Proof. We notice that PDE (12) is the special case of PDE (11) when $\sigma \equiv 0$, so Theorem in Case 1 gives the uniqueness of viscosity solution of PDE (12). Lemma in Case 2 shows that the uniqueness of viscosity solution of PDE (12) implies the uniqueness of PDE (1).

Thank you for your attention!