Valuation and Optimal Decision for Perpetual American Employee Stock Options under a Constrained Viscosity Solution Framework

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Outline

Introduction

- Employee Stock Options (ESOs)
  - Call options issued by a company
  - Based on the company's common stock
  - Granting the holder the right to buy a certain number of stocks at a predetermined price (Strike Price) during a specified period
- Extensive Use
  - Granted as compensation or reward to employees by companies
  - Employees profit from exercising ESOs

Features of ESOs

- American-style: can be exercised at any time before expiration
- Long maturity: from 5-10 years
- Job termination risk: get fired or leave the company voluntarily
- A vesting period: during which exercising is forbidden
- Transfer and hedging restrictions: transfer of ESOs and short selling the underlying stock are forbidden

Valuation and Optimal Decision for ESOs

- Companies (Valuation)
  - estimate the cost of ESOs
  - report to the Financial Accounting Standards Board(FASB)
- Employees (Optimal decision)
  - how to exercise a block of ESOs over time
  - maximize the returns

Difficulties

ESOs operate in an incomplete market.

Standard valuation methods fail.

Related Literature

- Exercise at a single date: treat every option equally
  - J.Carpenter, *The exercise and valuation of executive stock options*, Journal of Financial Economics(1998)48, 127–158.
  - B.J.Hall, K.J.Murphy, *Stock option for undiversified executives*, Journal of Accounting and Economics(2002)33, 3–42.
- Multi-period Model: Utility Maximization
  - Ashish Jain, Ajay Subramanian, *The intertemporal exercise and valuation of employee options*, The Accounting Review(2004)Vol.79 No.3, 705–743.
  - L.C.G. Rogers, JoséScheinkman, *Optimal exercise of executive stock options*, Finance Stoch(2007)11, 357–372.

Our Model

- Assumptions
  - An employee holding a block of perpetual American ESOs
  - Exercise process: Continuous fluid model with a bounded exercise rate
- Method
  - Maximize the overall discounted exercise returns
  - Stochastic control approach: *works well in an incomplete market*
- Results
  - Value-based maximum defines the cost of ESOs
  - Determine the optimal exercise strategy

Problem Formulation

- Time horizon: $t \in [0, \infty)$ $\quad (t = 0$: Grant Date)
- Stock price of the company: X_t

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x$$

  - μ: expected stock return rate
  - σ: constant volatility
  - W_t: standard Browian motion (BM)

Problem Formulation

- The employee is granted N shares of perpetual American ESOs with strike price K at grant date 0
- Exercise process:
  - the accumulated number of exercised options up to t: Y_t
  - a continuous fluid model

$$dY_t = u_t dt, \quad Y_0 = y$$

  - control set $\Gamma = [0, \lambda]$
  - $\{Y_t\}_{t \geq 0}$: a non-negative non-decreasing right-continuous process not exceeding N

Problem Formulation

- State variable $(X_t, Y_t) \in \bar{S}$: $S = (0, \infty) \times (0, N)$
- Admissible Control $u(\cdot)$ with initial value $(x, y) \in \bar{S}$

Definition (Admissible)

1. $u(\cdot)$ is $\mathcal{F}_t = \sigma\{X_s : s \leq t\}$ adapted
2. $u(t) \in \Gamma$ for all $t \geq 0$
3. the corresponding state process $(X_t, Y_t) \in \bar{S}$ for all $t \geq 0$

Denote by $\mathcal{A} = \mathcal{A}(x, y)$ the set of all admissible controls.

Optimization

- Objective function

$$J(x, y; u.) = E\left[\int_0^\infty e^{-\rho t}\left(X_t - K\right)^+ u_t dt \mid X_0 = x, Y_0 = y\right]$$

  - discount rate ρ: $\rho > \mu > 0$
  - payoff function: $(X_t - K)^+ = \begin{cases} X_t - K, & X_t > K \\ 0, & X_t \leq K \end{cases}$

- Value function

$$v(x, y) = \sup_{u(\cdot) \in \mathcal{A}(x,y)} J(x, y; u(\cdot))$$

HJB Equation

- HJB equation (Stochastic analysis theory)

$$\mathcal{L}v + \max_{u\in\Gamma}(u\mathcal{B}v) = 0, \quad (x, y) \in [0, \infty) \times [0, N] \tag{1}$$

where

$$\mathcal{L}v = \mu x v_x + \frac{\sigma^2}{2} x^2 v_{xx} - \rho v,$$
$$\mathcal{B}v = v_y + (x - K)^+.$$

- Boundary Condition

$$v(0, y) = 0, \quad 0 \le y \le N.$$

Properties of Value Function

1. For each $x \in [0, \infty)$, $v(x, y)$ is non-increasing in y
2. For each $y \in [0, N]$, $v(x, y)$ is non-decreasing in x
3. $v(x, y)$ is Lipschitz continuous in (x, y), that is

$$\begin{aligned} |v(x_1, y_1) - v(x_2, y_2)| &\leq \min\left\{\frac{\lambda}{\rho - \mu}, 2(N - y_1)\right\} |x_1 - x_2| \\ &+ (2x_2 + 1)|y_1 - y_2| \end{aligned}$$

for any $(x_i, y_i) \in [0, \infty) \times [0, N]$ $(i = 1, 2)$.

Definition for Constrained Viscosity Solution

- Semicontinuous functions in $\bar{S}$

$$\begin{aligned} USC(\bar{S}) &= \{v : \bar{S} \to \mathbb{R} \cup \{-\infty\} \mid v \text{ is upper semicontinuous}\}, \\ LSC(\bar{S}) &= \{v : \bar{S} \to \mathbb{R} \cup \{+\infty\} \mid v \text{ is lower semicontinuous}\}. \end{aligned}$$

- Viscosity Supersolution

  $w(x, y) \in LSC(\bar{S})$ is a supersolution of (1) in $\bar{S}$ if and only if

$$\mathcal{L}\varphi + \max_{u \in \Gamma}(u\mathcal{B}\varphi)\Big|_{(x_0,y_0)} \leq 0 \tag{2}$$

  whenever $\varphi(x, y) \in C^{2,1}$ and $w(x, y) - \varphi(x, y)$ has a local minimum at $(x_0, y_0) \in \bar{S}$ with $w(x_0, y_0) = \varphi(x_0, y_0)$.

Definition for Constrained Viscosity Solution

- Viscosity Subsolution

  $w(x,y) \in USC(S)$ is a subsolution of (1) in S if and only if

$$\mathcal{L}\varphi + \max_{u\in\Gamma}(u\mathcal{B}\varphi)\big|_{(x_0,y_0)} \geq 0 \tag{3}$$

  whenever $\varphi(x,y) \in C^{2,1}$ and $w(x,y) - \varphi(x,y)$ has a local maximum at $(x_0,y_0) \in S$ with $w(x_0,y_0) = \varphi(x_0,y_0)$.

Definition (Constrained Viscosity Solution)

A continuous function w is a constrained viscosity solution of (1) if it is both a viscosity supersolution of (1) in $\bar{S}$ and a viscosity subsolution of (1) in S.

Comparison Principle

Theorem

Let $\underline{v} \in USC(S)$ *is a subsolution of (1) in* S *and* $\bar{v} \in LSC(\bar{S})$ *is a supersolution of (1) in* $\bar{S}$ *. Furthermore,* $\underline{v}, -\bar{v}$ *grow linearly in* X*, i.e. there exists a constant* $C > 0$ *such that,*

$$\underline{v}(X), \ -\bar{v}(X) \leq C|X| \text{ for } X \in \bar{S},$$

and

$$\underline{v}(0) = \bar{v}(0) = 0.$$

Then if $\rho > 2\mu + \sigma^2$*, we have* $\underline{v} \leq \bar{v}$ *in* $\bar{S}$*.*

Proof by contradiction.

Value Function as a Constrained Viscosity Solution

Theorem

The continuous value function $v(x, y)$ growing linearly in (x, y) is the unique constrained viscosity solution of (1) in $\bar{S}$.

Optimal Exercise Decision

- HJB equation:

$$\mathcal{L}v + \max_{0 \le u \le \lambda} (u\mathcal{B}v) = 0, \quad (x, y) \in [0, \infty) \times [0, N]$$

- No-exercise Region(NR) and Exercise Region(ER)

$$NR := \{(x, y) : \ \mathcal{B}v(x, y) \le 0\}; \quad ER := \{(x, y) : \ \mathcal{B}v(x, y) > 0\}$$

- Optimal Exercise Rate

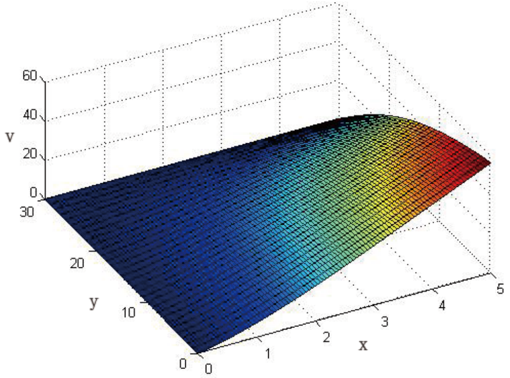
$$u^*(x, y) = \begin{cases} 0, & \text{if } (x, y) \in NR, \\ \lambda, & \text{if } (x, y) \in ER. \end{cases}$$

(the feedback control depending on v)

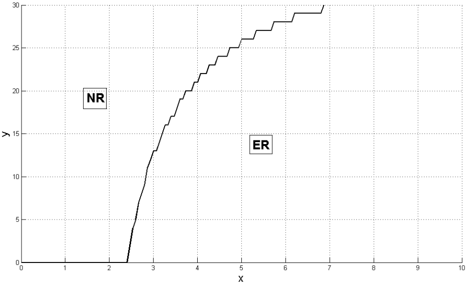
- Model Parameters:

  $\mu = 0.1,\ \sigma = 0.3,\ \rho = 0.15,\ \lambda = 1,\ K = 2,\ M = 5,\ N = 30.$

- Value Function $v(x, y)$:

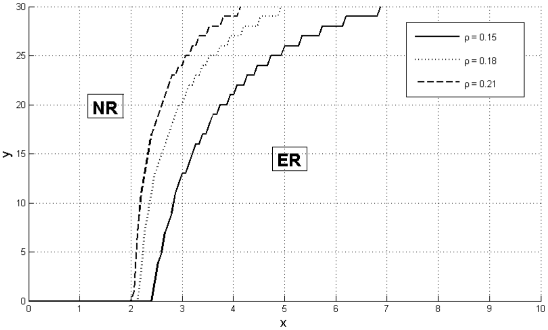


NR,ER and the Threshold Boundary



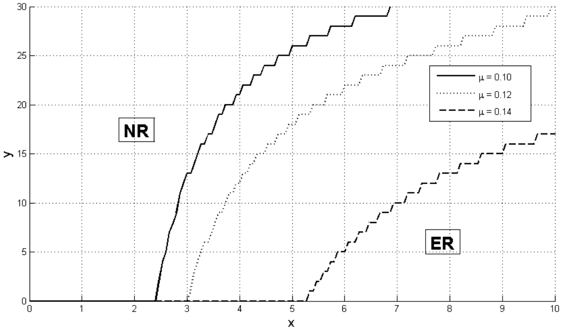
- NR ($u^* = 0$): hold and wait
- ER ($u^* = \lambda$): exercise immediately
- Threshold-style Strategy: easy to implement in practice

Impact of Discount Rate ρ



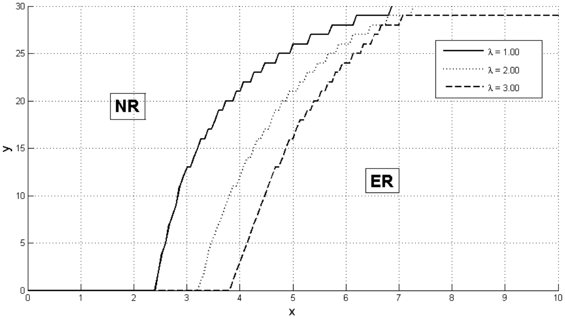
- $\rho = 0.15,\ 0.18,\ 0.21$
- ρ: time scale
- Larger ρ encourages earlier exercise action.

Impact of Expected Stock Return Rate μ



- $\mu = 0.15,\ 0.1,\ 0.12,\ 0.14$
- μ: increasing capacity of the stock price
- Larger μ encourages more patient waiting.

Impact of Exercise Rate Restriction λ



- $\lambda = 1,\ 2,\ 3$
- λ: upper bound of exercise rate
- Larger λ encourages more patient waiting.

Thank You!