Weierstrass Institute for
Applied Analysis and Stochastics

Dual representations for general multiple stopping problems

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Young Researchers Meeting on BSDEs, Numerics and Finance

jww C. Bender (Universität des Saarlandes) and J. Schoenmakers (WIAS Berlin)

Outline

I Recap of multiple stopping problems in discrete time

- Swing options
- Dual representations for multiple stopping
- Volume constraints & refraction periods

II Generalized multiple stopping problems

- Modeling devices
- Examples
- Dual representations
- Numerical experiments

Part I:

Recap of multiple stopping problems in discrete time

Setup

- discrete time (tenor) structure $\mathbb{T} = \{t_0 < t_1 < \ldots < t_T\}$, here $t_i := i$
- filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_i)_{i=0,\ldots,T}, \mathbb{P}\right)$ with $\mathbb{P}$ pricing measure
- adapted process $(S_i)_{i=0,\ldots,T}$ with values in $\mathbb{R}_+^D$ modeling asset prices
- cashflow $(Z_i)_{i=0,\ldots,T} \in \sigma(S)$ such that $\max_{i=0,\ldots,T} |Z_i| \in L^1$

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Plain vanilla swing contract

- trading days $j \in \mathbb{T} = \{0, \ldots, T\}$ on the electricity market
- trade only allowed on $L \leq T$ days up to maturity
- **at most one** package can be traded, waiting time is **one day**
- possible payoffs: $Z_j = (S_j - K)^+$ or $Z_j = (K - S_j)^+$ etc.

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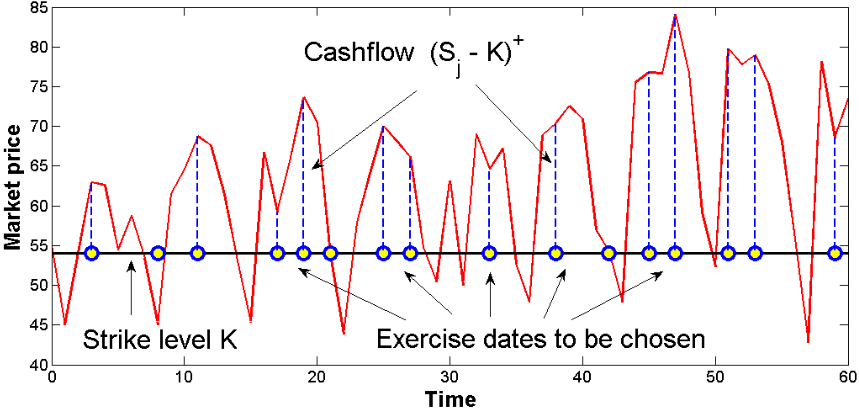
Swing option: holder of the option has the right to buy/sell L times electricity at $0 \leq j_1 < \ldots < j_L \leq T$ which have to be chosen


Cashflow $(S_j - K)^+$
Strike level K
Exercise dates to be chosen
Market price
Time
85
80
75
70
65
60
55
50
45
40
0
10
20
30
40
50
60

Multiple stopping problem (MSP): find **time zero price** by solving

$$Y_0^{*,L} := \sup_{0 \le \tau^1 < \ldots < \tau^L \le T} \mathbb{E} \sum_{k=1}^{L} Z_{\tau_j}$$

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Snell envelope of Z under L exercise rights

$Y_i^{*L} := \sup_{i \le \tau^1 < \ldots < \tau^L \le T} \mathbb{E}_i \sum_{k=1}^{L} Z_{\tau^k}$

- Bellman principle $\rightsquigarrow Y_i^{*k} = \max\left\{Z_i + \mathbb{E}_i Y_{i+1}^{*k-1}, \mathbb{E}_i Y_{i+1}^{*k}\right\}$
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$$\tau_i^{p,L} := \inf \left\{ \tau_i^{p-1,L} < j \le T : Z_j + \mathbb{E}_j Y_{j+1}^{*L-p} \ge \mathbb{E}_j Y_{j+1}^{*L-p+1} \right\}$$

- consequently $Y_i^{*L} = \mathbb{E}_i \sum_{p=1}^{L} Z_{\tau_i^{p,L}}$

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Theorem: Rogers (2002), Haugh & Kogan (2002)

Let $\mathcal{M}$ be the set of all martingales M satisfying $M_0 = 0$. Then

$$Y_i^* = \inf_{M \in \mathcal{M}} \mathbb{E}_i \max_{i \le j \le T} \left(Z_j - M_j + M_i \right) = \mathbb{E}_i \max_{i \le j \le T} \left(Z_j - M_j^* + M_i^* \right)$$

where the infimum is attained for the Doob martingale M^* of the Snell envelope $Y^* = Y_0^* + M^* - A^*$, where $A_i^* \in \mathcal{F}_{i-1}$, $A_0^* = 0$ and increasing.

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Consequence

- M any martingale $\rightsquigarrow$ $Y_0^* \leq Y_0^{up}(M) := \mathbb{E}\left[\max_{0 \leq j \leq T} \left(Z_j - M_j \right)\right]$
- $Y_0^* = Y_0^{up}(M^*)$

Dual by Meinshausen & Hambly (2004)

$Y_i^{*L} = \sum_{k=1}^{L} \left(Y_i^{*k} - Y_i^{*k-1} \right)$ with $Y^{*0} := 0$ and

$$Y_i^{*k} - Y_i^{*k-1} =$$

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Salient points

- characterize marginal value as infimum of stopping time **and** martingale
- important device of the proof: $\tau^k = \inf\{j : Z_j + \mathbb{E}_j Y_{j+1}^{*k-1} \ge \mathbb{E}_j Y_{j+1}^{*k}\}$

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Question: **Pure** martingale dual (as in standard stopping) available?

Dual by Schoenmakers (2010)

For $0 \leq i \leq T$ and $L \geq 1$,

$$Y_i^{*L} = \inf_{M^1,\ldots,M^L \in \mathcal{M}} \mathbb{E}_i \Big[\max_{i \leq j_1 < \ldots < j_L \leq T} \sum_{k=1}^{L} \big(Z_{j_k} - M_{j_k}^{L-k+1} + M_{j_{k-1}}^{L-k+1} \big) \Big]$$

$$= \max_{i \leq j_1 < \ldots < j_L \leq T} \sum_{k=1}^{L} \big(Z_{j_k} - M_{j_k}^{*L-k+1} + M_{j_{k-1}}^{*L-k+1} \big)$$

where M^{*L-k+1} is the Doob martingale of Y^{*L-k+1}.

- exploit Bellman principle $Y_i^{*L} = \max \left\{ Z_i + \mathbb{E}_i Y_{i+1}^{*L-1}, \mathbb{E}_i Y_{i+1}^{*L} \right\}$
- interpret Y_i^{*L} as Snell envelope of cashflow $Z_i + \mathbb{E}_i Y_{i+1}^{*L-1}$
- proceed by induction

Modify plain vanilla swing option with L exercise rights by adding following constraints:

- **volume constraint process** $v_t \in \{1, \ldots, L\}$ representing maximal number of exercise rights at $t \in \{0, \ldots, T\}$
- **refraction period** $\rho^{(i)} \in \{i + 1, \ldots, T\}$ is stopping time specifying minimal waiting time between two exercises

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Existing dual representations

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- $v_t \in \{1, \dots, L\}$ and $\rho^{(i)} \equiv i+1$: **Aleksandrov & Hambly (2010)** and **Bender (2010a)** (**with stopping times and martingales**)
- $v_t \equiv 1$ and $\rho^{(i)} \equiv i+\delta$ with $\delta \geq 1$: **Bender (2010b)** (**in the spirit of pure martingale representation**)

Part II:

Generalized multiple stopping problems

Under $L \geq 1$ exercise rights, interpret cashflow X as a mapping

- $X : \{0, \ldots, T\}^L \times \Omega \to \mathbb{R}$
- $X_{i_1,\ldots,i_L} \in \mathcal{F}_{i_L}$ and $X_{i_1,\ldots,i_L} \in L^1$

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Unconstrained general MSP: $Y_i^{*L} = \sup_{i \leq \tau^1 \leq \ldots \leq \tau^L} \mathbb{E}_i X_{\tau^1,\ldots,\tau^L}$

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Definition

- $Y_i^{*L-k+1,j_1,\ldots,j_{k-1}} := \sup_{i \leq \tau^k \leq \ldots \leq \tau^L} \mathbb{E}_i X_{j_1,\ldots,j_{k-1},\tau^k,\ldots,\tau^L}$
- $Y_r^{*L,\emptyset} := Y_r^{*L}$
- $Y_r^{*0,j_1,\ldots,j_L} := X_{j_1,\ldots,j_L}$

Theorem: Bender, Schoenmakers & Z. (2011)

For any $0 \le i \le T$ and any set of martingales $\left(M_r^{L-k+1,j_1,\ldots,j_{k-1}}\right)_{r \ge j_{k-1}}$ where $1 \le k \le L$ and $i =: j_0 \le j_1 \le \ldots \le j_{k-1}$, we have

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2. $Y_i^{*L} = \max_{i \le j_1 \le \ldots \le j_L \le T} \Big(X_{j_1,\ldots,j_L} + \sum_{k=1}^{L} \big(M_{j_{k-1}}^{*L-k+1,j_1,\ldots,j_{k-1}} - M_{j_k}^{*L-k+1,j_1,\ldots,j_{k-1}} \big) \Big)$

where $M_r^{*L-k+1,j_1,\ldots,j_{k-1}}$ is the Doob martingale of $Y_r^{*L-k+1,j_1,\ldots,j_{k-1}}$.

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2. $Y_i^{*L} = \max_{i \leq j_1 \leq \ldots \leq j_L \leq T} \Big(X_{j_1,\ldots,j_L} + \sum_{k=1}^{L} \big(M_{j_{k-1}}^{*L-k+1,j_1,\ldots,j_{k-1}} - M_{j_k}^{*L-k+1,j_1,\ldots,j_{k-1}} \big) \Big)$

where $M_r^{*L-k+1,j_1,\ldots,j_{k-1}}$ is the Doob martingale of $Y_r^{*L-k+1,j_1,\ldots,j_{k-1}}$.

$\rightsquigarrow$ Gyurko, Hambly & Witte (2011)

Huge family of martingales!

- $M^{L-k+1,j_1,\ldots,j_{k-1}}$ parameterized via $i \leq j_1 \leq \ldots \leq j_{k-1} \leq T$
- if $L \geq 1$ and $T > 0$ are big there are too many martingales!

$\Longrightarrow$ need for simpler parameterized martingales

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Towards tractable martingales: structuring the cashflow

- $(U_i^k)_{0 \leq i \leq T}$ adapted integrable processes for $k = 1, \ldots, L$
- $(V_i^l)_{0 \leq i \leq T}$ adapted processes, strictly positive and uniformly bounded from above for $l = 1, \ldots, L$ **($V^l \equiv 1$ in this talk)**
- pre-cashflow defined $\tilde{X}_{j_1,\ldots,j_L} := \sum_{k=1}^{L} U_{j_k}^k \prod_{l=1}^{k-1} V_{j_l}^l = \sum_{k=1}^{L} U_{j_k}^k$

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- pre-cashflow defined $\tilde{X}_{j_1,\dots,j_L} := \sum_{k=1}^L U_{j_k}^k \prod_{l=1}^{k-1} V_{j_l}^l = \sum_{k=1}^L U_{j_k}^k$
- $\mathcal{E}_p(j_1, \dots, j_p) := \#\{r : 1 \leq r \leq p, j_r = j_p\}$ counts number of rights exercised at time j_p

Modeling volume constraints and refractions periods

$$\mathcal{C}_p(j_1,\ldots,j_p) := \begin{cases} 1, & \forall 1 \le l \le p : \mathcal{E}_l(j_1,\ldots,j_l) \le v_{j_l} \text{ and } \forall 1 \le l \le p : j_l > j_{l-1} \Longrightarrow j_l \ge \rho^{j_{l-1}} \\ 0, & \text{else} \end{cases}$$

Constrained MSP: $Y_i^{*l} = \sup_{\substack{i \le \tau^1 \le \ldots \le \tau^l \le T \\ \mathcal{C}_l(\tau^1,\ldots,\tau^l)=1}} \mathbb{E}_i \sum_{k=1}^{l} U_{j_k}^k$

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$\rightsquigarrow Y_i^{*L} = \sup_{i \le \tau^1 \le \ldots \le \tau^L} \mathbb{E}_i X_{\tau^1,\ldots,\tau^L}$ recasts into above by

$$X_{j_1,\ldots,j_L} = \begin{cases} \sum_{k=1}^{L} U_{j_k}^k, & \text{if } \mathcal{C}_L(j_1,\ldots,j_L) = 1 \\ -N, & \text{else} \end{cases}$$

straightforward to check that Y_i^{*L} satisfies the following **dynamic program**

Lemma: Bender, Schoenmakers & Z. (2011)

For $r \geq 0$ and $1 \leq k \leq L$, we have

$$Y_r^{*L-k+1} = \mathbb{E}_r Y_{r+1}^{*L-k+1} \quad \vee \quad \max_{1 \leq n \leq v_r \wedge L-k+1} \left(\sum_{p=k}^{k+n-1} U_r^p + \mathbb{E}_r Y_{\rho^r}^{*L-k-n-1} \right)$$

Swing option: for general volume constraints v and refraction periods ρ^i (stopping times), put $U^p \equiv Z$ and obtain

Multiplicative framework Tractable Theorem

$$X_{j_1,\ldots,j_L} = \begin{cases} \sum_{k=1}^{L} Z_{j_k}, & \text{if } \mathcal{C}_L(j_1,\ldots,j_L) = 1 \\ -N, & \text{else} \end{cases}$$

Theorem: Bender, Schoenmakers & Z. (2011)

For any set of martingales $\left(M_r^{L-k+1}\right)_{r\geq 0}$ and any set of adapted integrable processes $\left(A_r^{L-k+1}\right)_{r\geq 0}$, it holds

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2 $$Y_i^{*L} = \max_{\substack{i\leq j_1\leq \ldots \leq j_L \leq T \\ \mathcal{C}_L(j_1\ldots,j_L)=1}} \left(\sum_{k=1}^L U_{j_k}^k + \sum_{k=1}^L \left(M_{j_{k-1}}^{*L-k+1} - M_{j_k}^{*L-k+1} + \mathbb{E}_{j_k} A_{\rho^{j_{k-1}}}^{*L-k+1} - \mathbb{E}_{j_{k-1}} A_{\rho^{j_{k-1}}}^{*L-k+1} \right) \right)$$

where the pair (M^{*L-k+1}, A^{*L-k+1}) arises from the Doob decomposition of Y_r^{*L-k+1}.

Note: if $\rho^{j_{k-1}} = j_{k-1} + 1 \Longrightarrow \mathbb{E}_{j_k} A^{*L-k+1}_{\rho^{j_{k-1}}} - \mathbb{E}_{j_{k-1}} A^{*L-k+1}_{\rho^{j_{k-1}}} = 0$

Key ingredient of proof: for $r \geq j_{k-1}$ we have to work a bit to find

$$M_r^{*L-k+1,j_1,\dots,j_{k-1}} - M_{j_k}^{*L-k+1,j_1,\dots,j_{k-1}} = \left(M_r^{*L-k+1} - M_{j_{k-1}}^{*L-k+1} + \mathbb{E}_{j_{k-1}} A^{*L-k+1}_{\rho^{j_{k-1}}} - \mathbb{E}_r A^{*L-k+1}_{\rho^{j_{k-1}}}\right)$$

However, maximum runs over huge set! $\max_{\substack{i \leq j_1 \leq \dots \leq j_L \leq T \\ \mathcal{C}_L(j_1 \dots, j_L) = 1}} \left(\dots \right)$

$\rightsquigarrow$ Remedy by calculating maximum using a ***recursion***

▸ Recursion details ▸ Numerical recipe ▸ Numerical results

Off-peak swing option: allows buying at most one package on weekdays and two packages on weekends, i.e.

$$\text{for } j = 0, \ldots, T = 50: \quad v_j := \begin{cases} 1, & \text{if } j \text{ is a weekday} \\ 2, & \text{if } j \text{ is a weekend day} \end{cases}$$

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δ	L	$Y_0^{up,L}$	upper bound using Bender (2010b)
1	1	1.86485 (0.0019)	1.8638 (0.0019)
1	2	3.40832 (0.003)	3.4078 (0.003)
1	4	5.90956 (0.0043)	5.9170 (0.0045)
1	6	7.92669 (0.005)	7.9470 (0.0058)
1	8	9.61643 (0.0058)	9.6493 (0.0069)
1	10	11.0553 (0.00064)	11.1035 (0.0079)

Tractable maximum

δ	L	$Y_0^{low,L}$	$Y_0^{up,L}$	95% confidence interval	L	$Y_0^{low,L}$	$Y_0^{up,L}$	95% confidence interval
1	4	5.89744	5.91312	[5.89078, 5.91464]	6	7.91351	7.93364	[7.90538, 7.93527]
2	4	5.73736	5.76003	[5.73078, 5.76192]	6	7.55394	7.58547	[7.54595, 7.58773]
3	4	5.62688	5.65097	[5.62027, 5.65305]	6	7.28486	7.32349	[7.27684, 7.32611]
4	4	5.51763	5.5468	[5.51105, 5.54915]	6	7.01995	7.06275	[7.01198, 7.06577]
5	4	5.40154	5.43295	[5.39496, 5.43546]	6	6.7418	6.78724	[6.73383,6.79042]
6	4	5.27976	5.30967	[5.27316, 5.31227]	6	6.45197	6.49774	[6.44401, 6.50102]
8	4	5.06733	5.10055	[5.06078, 5.10335]	6	5.9401	5.98546	[5.93238, 5.989]
10	4	4.85039	4.88637	[4.84386, 4.88953]	6	5.46672	5.50782	[5.45916, 5.51116]
12	4	4.63227	4.66583	[4.62575, 4.66884]	6	5.08473	5.11779	[5.07729, 5.12079]
14	4	4.43104	4.45997	[4.42453, 4.46279]	6	4.76734	4.79957	[4.76006, 4.80269]
16	4	4.25079	4.27955	[4.24441, 4.28237]	6	4.39537	4.42311	[4.38864, 4.42584]
18	4	4.07804	4.10338	[4.07164, 4.10605]	6	4.18139	4.20744	[4.17473, 4.21004]
20	4	3.94562	3.96789	[3.93923, 3.97034]	6	4.02465	4.04716	[4.01803, 4.04968]
1	8	9.60253	9.62348	[9.59318, 9.62507]	10	11.0436	11.0661	[11.0332, 11.0677]
2	8	8.97188	9.01806	[8.96279, 9.02078]	10	10.0822	10.1411	[10.0721, 10.1443]
3	8	8.48335	8.53629	[8.47425, 8.53952]	10	9.31393	9.3793	[9.30393, 9.38283]
4	8	8.00789	8.06203	[7.99887, 8.06551]	10	8.58082	8.63832	[8.57102, 8.64178]
5	8	7.5251	7.57926	[7.5161, 7.58278]	10	7.9058	7.961	[7.89611, 7.96454]
6	8	7.06562	7.11754	[7.05669, 7.12102]	10	7.33533	7.38481	[7.32577, 7.38835]
8	8	6.18418	6.23051	[6.17596, 6.23403]	10	6.20274	6.24781	[6.19445, 6.2513]
10	**8**	**5.54885**	**5.58774**	**[5.54107, 5.59089]**	**10**	**5.54885**	**5.58774**	**[5.54107, 5.59089]**

Thank you for your attention!

Some literature

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Exponential utility: fits into cashflow framework by setting

$$X_{j_1,\ldots,j_L} = \begin{cases} \sum_{k=1}^{L} U_{j_k}^k \prod_{m=1}^{k-1} V_{j_m}^m, & \text{if } \mathcal{C}_L(j_1,\ldots,j_L) = 1 \\ -N, & \text{else} \end{cases}$$

with risk aversion parameter $\alpha > 0$

$$V_j^l = e^{-\alpha Z_j}, \quad U_j^k = \begin{cases} 0, & \text{if } k = 1,\ldots,L-1 \\ -e^{-\alpha Z_j}, & \text{if } k = L \end{cases}$$

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then we have Bellman principle

$$\sup_{\substack{\tau^1 \le \dots \le \tau^L \\ \mathcal{C}_L(\tau^1,\dots,\tau^L)=1}} \mathbb{E}_i \left[-\exp\left(-\alpha \sum_{k=1}^{L} Z_{\tau^k} \right) \right] = \sup_{\substack{\tau^1 \le \dots \le \tau^L \\ \mathcal{C}_L(\tau^1,\dots,\tau^L)=1}} \mathbb{E}_i X_{\tau^1,\dots,\tau^L}$$

Emulating pathwise maximum

- given any $L-$tuple of martingales $M = (M^1, \ldots, M^L)$
- and any $L-$tuple of adapted processes $A = (A^1, \ldots, A^L)$,

define for $n = 0, \ldots, L$

$$\theta_i^{n,L}(M, A) := \max_{\substack{i=j_0 \le j_1 \le \ldots \le j_{L-n} \\ \mathcal{C}_{L-n}(j_1,\ldots,j_{L-n})=1}} \sum_{k=1}^{L-n} \Bigg(U_{j_k}^{n+k} + M_{j_{k-1}}^{L-n-k+1} - M_{j_k}^{L-n-k+1} + \mathbb{E}_{j_k} A_{\rho^{j_{k-1}}}^{L-k-n+1} - \mathbb{E}_{j_{k-1}} A_{\rho^{j_{k-1}}}^{L-k-n+1} \Bigg)$$

By construction: $\theta_i^{0,L}(M^*, A^*) = Y_i^{*L}$

Theorem: Bender, Schoenmakers & Z. (2011)

For any pair (M, A) we have for $i = T, \ldots, 0$ and $n = L, \ldots, 0$

$$\theta_i^{n,L} = \theta_{i+1}^{n,L} - M_{i+1}^{L-n} + M_i^{L-n} \quad \vee$$

$$\max_{\nu=1,\ldots,v_i \wedge L-n} \sum_{k=1}^{\nu} U_i^{n+k} + \Big(\theta_{\rho^i}^{n+\nu,L} - M_{\rho^i}^{L-n-\nu}$$

$$+ M_i^{L-n-\nu} + A_{\rho^i}^{L-n-\nu} - \mathbb{E}_i A_{\rho^i}^{L-n-\nu} \Big)$$

$$\theta_T^{n,L} = \sum_{k=1}^{L-n} U_T^{n+k}$$

Unit volume constraint & unit refraction period & $L = 1$ $\rightsquigarrow$

$$\theta_i = \max_{i \le j \le T} \left(Z_j + M_i - M_j\right) = \max\left\{Z_i, \max_{j+1 \le i \le T}\left(Z_j + M_i - M_j\right)\right\}$$
$$= \max\left\{Z_i, \max_{j+1 \le i \le T}\left(Z_j + M_{i+1} - M_j\right) + M_i - M_{i+1}\right\}$$

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$$= \max\left\{Z_i, \theta_{i+1} + M_i - M_{i+1}\right\}$$

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$$= \max\left\{Z_i, \theta_{i+1} + M_i - M_{i+1}\right\}$$

$$= \theta_{i+1} - (M_{i+1} - M_i) + \max\left\{0, Z_i - \theta_{i+1} + (M_{i+1} - M_i)\right\}$$

Unit volume constraint & unit refraction period & $L = 1 \rightsquigarrow$

$$\begin{aligned}
\theta_i &= \max_{i \le j \le T} \left(Z_j + M_i - M_j \right) = \max \left\{ Z_i, \max_{j+1 \le i \le T} \left(Z_j + M_i - M_j \right) \right\} \\
&= \max \left\{ Z_i, \max_{j+1 \le i \le T} \left(Z_j + M_{i+1} - M_j \right) + M_i - M_{i+1} \right\} \\
&= \max \left\{ Z_i, \theta_{i+1} + M_i - M_{i+1} \right\} \\
&= \theta_{i+1} - (M_{i+1} - M_i) + \max \left\{ 0, Z_i - \theta_{i+1} + (M_{i+1} - M_i) \right\} \\
&= \theta_{i+1} - (M_{i+1} - M_i) + K_{i+1} - K_i
\end{aligned}$$

Unit volume constraint & unit refraction period & $L = 1$ $\rightsquigarrow$

$$
\begin{aligned}
\theta_i &= \max_{i \le j \le T} \left(Z_j + M_i - M_j\right) = \max\left\{Z_i, \max_{j+1 \le i \le T} \left(Z_j + M_i - M_j\right)\right\} \\
&= \max\left\{Z_i, \max_{j+1 \le i \le T} \left(Z_j + M_{i+1} - M_j\right) + M_i - M_{i+1}\right\} \\
&= \max\left\{Z_i, \theta_{i+1} + M_i - M_{i+1}\right\} \\
&= \theta_{i+1} - (M_{i+1} - M_i) + \max\left\{0, Z_i - \theta_{i+1} + (M_{i+1} - M_i)\right\} \\
&= \theta_{i+1} - (M_{i+1} - M_i) + K_{i+1} - K_i \\
&\asymp \theta_{i+1} - \int_i^{i+1} Z_s \mathrm{d}W_s + \int_i^{i+1} \mathrm{d}K_s, \; \theta_j \ge Z_j
\end{aligned}
$$

Unit volume constraint & unit refraction period & $L = 1$ $\rightsquigarrow$

$$\begin{aligned}
\theta_i &= \max_{i \le j \le T} \left(Z_j + M_i - M_j\right) = \max\left\{Z_i, \max_{j+1 \le i \le T} \left(Z_j + M_i - M_j\right)\right\} \\
&= \max\left\{Z_i, \max_{j+1 \le i \le T} \left(Z_j + M_{i+1} - M_j\right) + M_i - M_{i+1}\right\} \\
&= \max\left\{Z_i, \theta_{i+1} + M_i - M_{i+1}\right\} \\
&= \theta_{i+1} - (M_{i+1} - M_i) + \max\left\{0, Z_i - \theta_{i+1} + (M_{i+1} - M_i)\right\} \\
&= \theta_{i+1} - (M_{i+1} - M_i) + K_{i+1} - K_i \\
&\asymp \theta_{i+1} - \int_i^{i+1} Z_s \mathrm{d}W_s + \int_i^{i+1} \mathrm{d}K_s,\ \theta_j \ge Z_j
\end{aligned}$$

$\rightsquigarrow$ Schoenmakers, Huang & Z. (2011)

$\rightsquigarrow$ Bernhart, Pham, Tankov & Warin (2011)

▸ Tractable maximum

Hands-on numerical recipe for calculating lower and upper bounds

1. $Y^{low,L-k+1}$: approximate lower bound of $Y^{*L-k+1}, k = 1, \ldots, L$ (e.g. Longstaff & Schwartz (2001))
2. $\hat{Y}^{L-k+1}$: approximation of Y^{*L-k+1} via dynamic program
3. numerical Doob decomposition of $\hat{Y}^{L-k+1}$ (e.g. Andersen & Broadie (2004) using one layer of nested MC)
4. $Y^{up,L-k+1}$: use dual representation, i.e. plug in numerical Doob decomposition, replace expectation by sample mean, recursion for the pathwise maximum
5. $\left[Y_0^{low,L} - \beta \cdot \sigma^{low,L}, Y_0^{up,L} + \beta \cdot \sigma^{up,L}\right]$ confidence interval

▸ Tractable maximum