A Probabilistic Numerical Method for High-Dimensional Fully Nonlinear Parabolic PDEs

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Outline

Introduction of Fully Nonlinear Parabolic PDEs

- Fully Nonlinear PDE :

$$\begin{cases} u_t + G(t, x, u, Du, D^2 u) = 0, \text{ on } [0, T) \times \mathbb{R}^d, \\ u(T, \cdot) = g(\cdot), \quad \text{on } \mathbb{R}^d, \end{cases} \tag{1}$$

  ◇ $G(t, x, y, z, \gamma) : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$;

  ◇ G is parabolic : $G_\gamma \geq 0$;

  ◇ $g : \mathbb{R}^d \to \mathbb{R}$.

- Connection with Backward SDEs :

  ◇ Semilinear PDE $\Longleftrightarrow$ BSDE

  ◇ Quasi-linear PDE $\Longleftrightarrow$ FBDSE

  ◇ Fully Nonlinear PDE $\Longleftrightarrow$ Second-order BSDE

Numerical Methods

- PDE approach : curse of dimensionality : $d \leq 3$.
- BSDES :

◇ Time discretization : J. Zhang (2004), Bouchard-Touzi (2004)

◇ Implementation : Gobet-Lemor-Warin (2005), Bender-Denk (2006), Crisan-Manolarakis (2010), ... ;

- FBSDEs : Bender-Zhang(2008),... ;
- Note : there are numerous other theoretical works, including some on non-Markovian BSDEs (path dependent PDEs). But many of them are not efficient or feasible, especially in high-dimensional case.

Numerical Methods

- Fully nonlinear PDE :
  - ◇ Convergence : viscosity solution approach by Barles-Souganidis ;
  - ◇ Rate of convergence : Krylov's "shaking the coefficients" method ;
  - ◇ A new approach by Xiaolu Tan.
- Fahim-Touzi-Warin (2010)
  - ◇ Connection with Second-order BSDEs ;
  - ◇ The proof relies on PDE arguments and Krylov's "shaking the coefficients" method.
  - ◇ bound constraint : tr $\left[(\underline{\sigma}^2)^{-1}(\overline{\sigma}^2 - \underline{\sigma}^2)\right] \leq 1$

$$\underline{\sigma}^2 I_d \leq G_\gamma \leq \overline{\sigma}^2 I_d \quad => \quad \frac{\overline{\sigma}^2}{\underline{\sigma}^2} \leq 1 + \frac{2}{d} \quad (\rightarrow 1 \text{ as } d \rightarrow \infty).$$

  - ◇ Note : When d is large, $\underline{\sigma} \approx \overline{\sigma}$, and thus the PDE is essentially semilinear.

Outline of Talk

- Our Algorithm
  - ◇ generalizes the assumption imposed in FTW (2010) ;
  - ◇ can be implemented with trinomial tree to solve low-dimensional problems efficiently
  - ◇ uses Monte-Carlo Simulation to solve **High-Dimensional** problems (12-dimensional numerical examples will be provided)
  - ◇ works for equations with G-generator(10-dimensional example will be provided)

Outline

Algorithm

- Inspiration : assuming $u(t,x)$ is a smooth Solution of PDE(1), and X is a symmetric random variable with bounded moments.

$$\begin{aligned} u(t,x) &= \mathbb{E}[u(t+h, x+\sqrt{h}X)] - h(u_t(t,x) + \frac{\mathbb{E}[X^2]}{2} D^2 u) + O(h^2) \\ &\approx \mathbb{E}[u(t+h, x+\sqrt{h}X)] + h(G(t,x,u,Du,D^2u) - \frac{\mathbb{E}[X^2]}{2} D^2 u) \end{aligned}$$

- Scheme : Partition $0 = t_0 < \cdots < t_N = T$, $h \triangleq t_i - t_{i-1}$,

$$u_h(t_N, x) := g(x), \quad u_h(t_i, x) = \mathbb{T}_h[u_h](t_i, x), \tag{2}$$

where

$$\begin{aligned} \mathbb{T}_h[u_h](t_i, x) \triangleq\ & \mathbb{E}[u_h(t_{i+1}, x+\sqrt{h}\sigma_0 X)] \\ & + hF\left(t_i, x, \mathcal{D}^0 u_h(t_i,x), \mathcal{D}^1 u_h(t_i,x), \mathcal{D}^2 u_h(t_i,x)\right) \end{aligned}$$

and $F(t,x,y,z,\gamma) \triangleq G(t,x,y,z,\gamma) - \frac{\text{tr}[\sigma_0^2\gamma]}{2}$

Probability Space

- $X = (X_1, \cdots, X_d)^T$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. $X_1, \cdots, X_d$ - Independent

$$X_i = \begin{cases} 1/\sqrt{p}, & p/2 \\ 0, & 1-p \\ -1/\sqrt{p}, & p/2 \end{cases} .$$

- Note : $\mathbb{E}X_i = \mathbb{E}X_i^3 = 0,\ \mathbb{E}X_i^2 = 1,\ \mathbb{E}X_i^4 = 1/p.$

- Denote

$$\widehat{X}^2 \triangleq \begin{bmatrix} X_1^2 & 0 & \cdots & 0 \\ 0 & X_2^2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & X_d^2 \end{bmatrix}.$$

Approximation of Derivatives

- $\mathcal{D}^0\phi(t_i,x) \triangleq \mathbb{E}[\phi(t_{i+1}, x+\sigma_0\sqrt{h}X)]$
- First derivative approximation :

$$\mathcal{D}^1\phi(t,x) \triangleq \mathbb{E}\left[\phi(t+h, x+\sigma_0\sqrt{h}X)\frac{\left(\sigma_0^T\right)^{-1}X}{\sqrt{h}}\right].$$

◇ Note : assuming ϕ is smooth, then :

$$\mathcal{D}^1\phi(t,x) \quad = \quad D\phi + h\sigma_0^2 D^3\phi/p + O(h^2) = D\phi + O(h)$$

- Second derivative approximation :

$$\mathcal{D}^2\phi(t,x) \triangleq \mathbb{E}\Big[\phi(t+h, x+\sigma_0\sqrt{h}X)\times \left(\frac{(1-p)XX^T+(3p-1)\hat{X}^2-2pI_d}{\sigma_0^2 h(1-p)}\right)\Big].$$

General Convergence Results

- Convergence of Scheme : G. Barles, P.E. Souganidis (1991)

◇ **Monotonicity** : $\varphi_1, \ \varphi_2 \in C([0, T] \times \mathbb{R}^d)$

$$\varphi_1 \leq \varphi_2 \Rightarrow \mathbb{T}_h[\varphi_1] \leq \mathbb{T}_h[\varphi_2].$$

◇ **Stability** : $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |u_h(t,x)| \leq \mathbb{C}$ (independent of h).

◇ **Consistency** : ϕ : smooth & with bounded derivatives

$$\lim_{\substack{(t',x') \to (t,x) \\ (h,c) \to (0,0) \\ t'+h \leq T}} \frac{[c+\phi](t',x') - \mathbb{T}_h[c+\phi](t',x')}{h}$$
$$= -\left(\phi_t + G(t,x,\phi,D\phi,D^2\phi)\right)(t,x).$$

◇ **Camparison Principle** for viscosity solutions holds

Standing Assumptions

- $\|G(t, x, 0, 0, 0)\|_\infty < \infty$;
- G : Lipschtiz-continuous with respect to (y, z, γ) uniformly in t;
- $g : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous;
- Note :
  - ◇ We may weaken slightly the Lipschitz continuity of G.

Additional Assumptions and Remarks(1)

- **Additional Key Assumptions** : there exists $\theta \geq 0$ such that
  - $\diamond$ $G_\gamma = G_\gamma^0(\text{diagonal}) + G_\gamma^1, \; \theta G_\gamma + G_\gamma^1 \geq 0$;
  - $\diamond$ $\exists \underline{\sigma}, \; \overline{\sigma} > 0, \; 0 < \underline{\sigma}^2 I_d \leq G_\gamma \leq \overline{\sigma}^2 I_d$,

$$\frac{\overline{\sigma}^2}{\underline{\sigma}^2} \leq 1 + \frac{2}{d} + \frac{(d+2)^2}{8d\theta} \cdot \left[\left(\frac{2}{d+2} - \theta \right)^+ \right]^2$$

- Remarks(1) :
  - $\diamond$ When G_γ is diagonal, then $\theta = 0$ and thus the bound constraint is not needed
  - $\diamond$ $1 + \frac{2}{d}$ is exactly the bound in FTW (2010), so our result covers theirs.
  - $\diamond$ $\underline{\sigma}$, and $\overline{\sigma}$ can be generalized to matrices easily.

Additional Assumptions and Remarks(2)

- **Additional Key Assumptions** : there exists $\theta \geq 0$ such that
  - $G_\gamma = G_\gamma^0(\text{diagonal}) + G_\gamma^1$, $\theta G_\gamma + G_\gamma^1 \geq 0$;
  - $\exists \underline{\sigma}$, $\overline{\sigma} > 0$, $0 < \underline{\sigma}^2 I_d \leq G_\gamma \leq \overline{\sigma}^2 I_d$,

$$\frac{\overline{\sigma}^2}{\underline{\sigma}^2} \leq 1 + \frac{2}{d} + \frac{(d+2)^2}{8d\theta} \cdot \left[\left(\frac{2}{d+2} - \theta \right)^+ \right]^2$$

- Remarks(2) :
  - When $\underline{\sigma}$ is 0, we can truncate it to be positive definite.
  - Examples that don't follow this assumption but can be solved by our scheme will be provided.
  - The general G-generator doesn't satisfy this assumption, but our scheme works on it as well.

Rate of Convergence

Theorem (Smooth Solution case)

Assume $u \in C_b^{1,3}$ is the solution of PDE, and u_h is the numerical solution, then

$$|u - u_h| \leq Ch.$$

Theorem (Viscosity Solution case : Barles-Jakobsen (2007))

Assuming that PDE (1) is of Hamilton-Jacobi-Bellman type, and with some slightly stronger conditions to HJB coefficients, we have

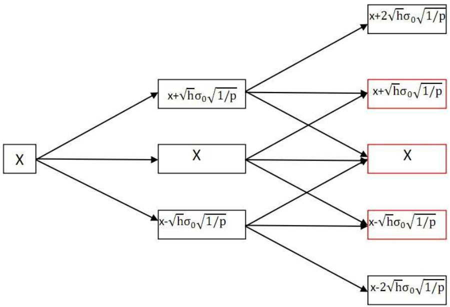
$$-Ch^{1/10} \leq u - u_h \leq Ch^{1/4}.$$

Weighted Average(Trinomial Tree)

Computing $\mathcal{D}^m u_h(t_i, X_{t_i}) := \mathbb{E}\left[\phi^m(X_{t_{i+1}}) \middle| X_{t_i}\right],\ m = 0, 1, 2.$

- Fast, stable, best choice for low-dimensional problem.
- Number of Nodes : $(2N+1)^d$ (N : Number of time steps)

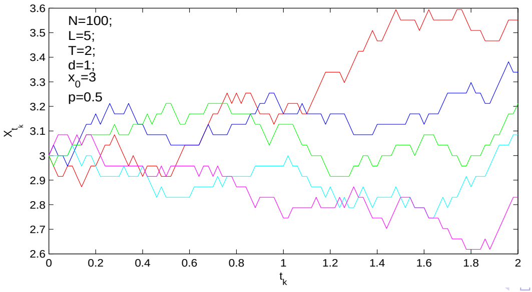
$$\mathbb{E}\left[\phi^m(X_{t_{i+1}}) \middle| X_{t_i} = x\right] = \sum^{3^d} \phi^m(X_{t_{i+1}}^{t_i,x}) P(X_{t_{i+1}}^{t_i,x})$$



Least Square Regression

Gobet-Lemor-Warin (2005) ; Bender-Denk (2006).

- It can handle high-dimensional problems (up to 12 in my Laptop). e.g. 1.3×10^7 paths is enough to discretize a 12 dimensional PDE into around 160 time steps by LSR, but it can only discretize the same PDE into 2 time steps if we use finite difference method.
- The variance of the result is small if a large amount of paths are sampled.



Simulation

- Choose a sequence of basis functions $e_1(t_i, x), \cdots, e_\lambda(t_i, x)$ to project the conditional expectation.
- Basic idea : $\mathbb{E}\left[\phi(X_{t_{i+1}}) \middle| X_{t_i}\right] \approx \sum_{j=1}^{\lambda} \alpha_j e_j(t_i, X_{t_i})$ (with projection error), where $\{\alpha_j\}$ are $\sigma(X_{t_i})-$measurable r.v. such that

$$\begin{aligned}\{\alpha_j\}_{j=1}^{\lambda} &= \arg\min_{\alpha_1,\dots,\alpha_\lambda} \mathbb{E}\left[\left|\sum_{j=1}^{\lambda} \alpha_j e_j(t_i, X_{t_i}) - \phi(X_{t_{i+1}})\right|^2 \middle| X_{t_i}\right] \\ &\approx \arg\min_{\alpha_1,\dots,\alpha_\lambda} \frac{1}{L}\sum_{l=1}^{L}\left|\sum_{j=1}^{\lambda} \alpha_j e_j(t_i, X^l_{t_i}) - \phi(X^l_{t_{i+1}})\right|^2\end{aligned}$$

with simulation error where $\left\{\{X^l_{t_i}\}_{i=0}^{N}\right\}_{l=1}^{L}$ are L paths sampled.

Errors

- $Error_{total} = Error_{discretization} + Error_{projection} + Error_{simulation}$
- In most of the numerical examples below we know the true solution, so we may choose "perfect" basis functions and focus on $Error_{discretization}$ and $Error_{simulation}$ only.
- $Error_{projection}$ depends on the choice of basis functions. How to find good basis functions is still unknown.
- The typical candidates of basis functions in the literature are : Monomials, Hermite polynomials, the terminal condition $g(\cdot)$ and its derivatives.

Outline

A 3-dimensional PDE

- **Example 1**

$$
\begin{aligned}
&u_t + \tfrac{1}{2}\sup_{\underline{\sigma}\le\sigma\le\overline{\sigma}}\left(\sigma^2\mathrm{tr}\left[D^2u\right]\right) - f(u,Du) = 0,\ \ 0\le t\le T\\
&u(T,x) = \sin(T + x_1 + \ldots + x_d), \quad \text{on } \mathbb{R}^d,
\end{aligned} \tag{3}
$$

and $f(u,Du) = \frac{1}{d}\left(\sum_{i=1}^{d}\frac{\partial u}{\partial x_i}\right) - \frac{d}{2}\inf_{\underline{\sigma}\le\sigma\le\overline{\sigma}}\left(\sigma^2 u\right),\ \ d = 3.$

- True solution : $u(t,x) = \sin(t + x_1 + \ldots + x_d)$.
- Numerical Scheme :

$$
\begin{aligned}
&u_h(t_i,x) = \mathbb{E}[u_h(t_{i+1}, x + \sigma_0\sqrt{h}X)]\\
&+h\Big\{\frac{1}{2}\sup_{\underline{\sigma}\le\sigma\le\overline{\sigma}}\left(\sigma^2\mathrm{tr}[\mathcal{D}^2u]\right) - \frac{1}{2}\mathrm{tr}[\sigma_0^2\mathcal{D}^2u] + f(\mathcal{D}^0u,\mathcal{D}u)\Big\}.
\end{aligned}
$$

- How to choose the parameters ?

Choice of discretization parameters

Maintaining **Monotonicity**.

- Assuming that $0 < \sigma_1^2 \leq G_\gamma \leq \sigma_2^2$, if G_γ is diagonal, then we choose $\sigma_0 = 2\sigma_1$ and $p = \min\{\frac{1}{3}, \frac{1}{1+\sup[\mathrm{tr}((\sigma_1^2)^{-1}\sigma_2^2)]-d}\}$
- Assuming that $\theta G_\gamma + G_\gamma^1 \geq 0$, then we will choose $\frac{1}{p} = \max\{3, \frac{1}{\theta} + 2 - \frac{d}{2}\}$ and $\sigma_0 = \sqrt{\frac{2p+(3p-1)\theta}{p}}\sigma_1$.
- If $\sigma_1^2 = 0$, we can truncate G_γ from below with a positive definite matrix $\varepsilon I_d > 0$ by substituting $G + ((\varepsilon I_d - G_\gamma) \bigvee 0) \cdot \gamma$ for G.
- Large σ_0 and small p generally lead to convergence, though the monotonicity may not hold.

Numerical Results

Taking $x_0 = (5, 6, 7)$, T=0.5 we have the following results :

N	Approx. $\overline{\sigma}^2 = 2$
20	-0.72984
40	-0.74028
60	-0.74382
80	-0.74667
100	-0.74560
120	-0.74738
140	-0.74790
160	-0.74829
Ans.	-0.75099

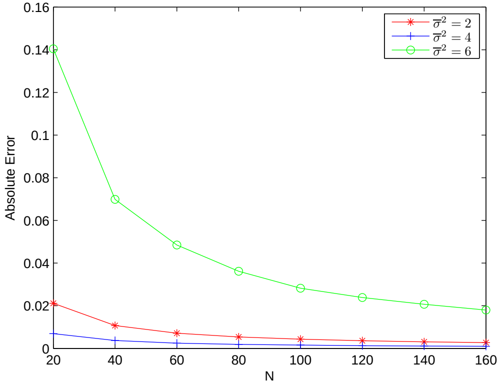


Figure: Results and the corresponding errors when $d = 3$, $\underline{\sigma} = 1$.

Comparison with finite difference

N	Ours	F.D.
20	-0.72984	-0.76420
40	-0.74028	-0.75785
60	-0.74382	-0.75562
80	-0.74667	-0.75447
100	-0.74560	-0.75379
120	-0.74738	-0.75332
140	-0.74790	-0.75300
160	-0.74829	-0.75274
Ans	-0.75099	-0.75099

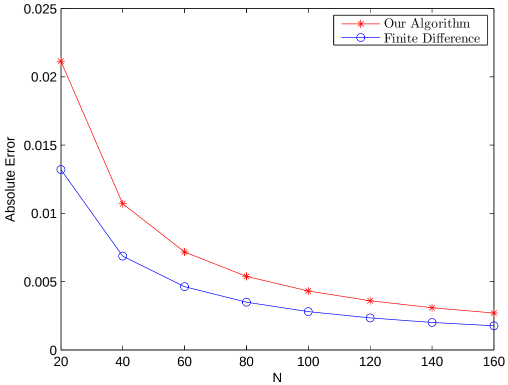
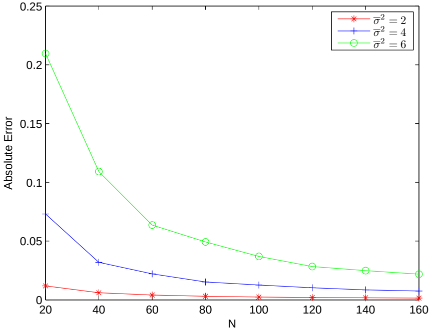


Figure: Comparing our scheme and finite difference with $\underline{\sigma} = 1$, $\overline{\sigma} = 2$.

σ Truncation

If $\sigma = 0$ in Example 1, we will approximate Equation (3) by

$$u_t + \frac{1}{2}\sup_{\varepsilon \le \sigma \le \overline{\sigma}} \left(\sigma^2 \mathrm{tr}\left[D^2 u\right]\right) - f(u, Du) = 0, \; 0 \le t \le T,$$
$$u(T, x) = \sin(T + x_1 + ... + x_d), \quad \text{on } \mathbb{R}^d, \; \varepsilon = 0.01$$



A 12-dimensional PDE with known solution

Example 2. We try to solve by LSR a PDE with the same setting as Example 1 except that $d = 12$.

- Choice of basis functions : $\{1, x, g(x), g'(x)\}$
- Choice of simulation parameters :

  We don't know how to balance the variance, error and the cost by choosing L, the amount of paths sampled, and K, the number of tests we should conduct before taking the average.

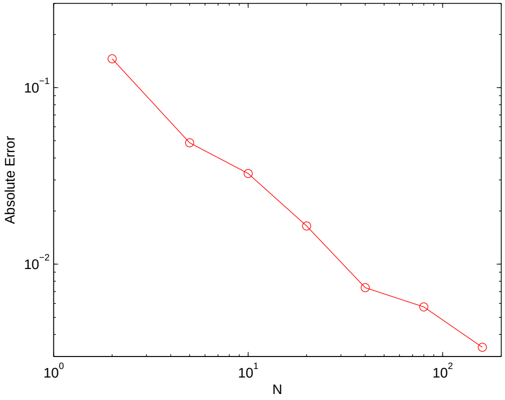
A 12-dimensional PDE with known solution

Fixing $d = 12, \underline{\sigma}^2 = 1, \ \overline{\sigma}^2 = 2, \ T = 0.2$, and $x_0 = (1, 2, ..., 12)$, we test our algorithm by the LSQ method to get :

N	L	K	Avg(Ans.)	Var(Avg.)	cost (in seconds)
2	2083	160	0.659639	3.53×10^{-6}	4.48×10^{-2}
5	13021	64	0.562635	1.99×10^{-6}	1.46×10^{-1}
10	52083	32	0.546598	8.41×10^{-7}	1.17×10^{0}
20	208333	16	0.530432	8.04×10^{-7}	1.08×10^{1}
40	833333	8	0.521343	2.25×10^{-7}	9.11×10^{1}
80	3333333	4	0.519701	1.21×10^{-7}	7.28×10^{2}
160	13333333	2	0.517363	6.17×10^{-8}	5.86×10^{3}

True solution : $\sin(\sum x_0) = 0.513978$.

A 12-dimensional PDE with known solution



The absolute error is slightly greater than O(h) because of the simulation error.

A 12-Dimensional Isaacs Equation with viscosity solution

Example 3. Consider the following PDE :

$$\begin{cases} u_t + G(D^2u) = 0, \text{ on } [0,T) \times \mathbb{R}^d, \\ u(T,\cdot) = sin(T + x_1 + ... + x_d), \quad \text{on } \mathbb{R}^d, \end{cases}$$

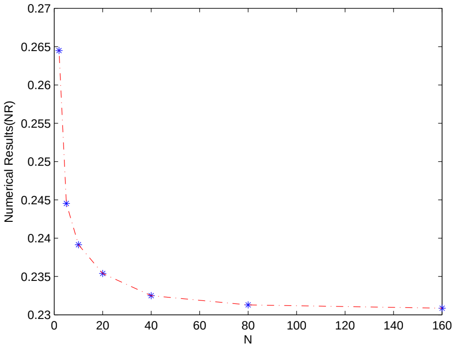
where

$$\begin{aligned} G(\gamma) &\triangleq \sum_{i=1}^{d} \sup_{0\le u\le 1} \inf_{0\le u\le 1} \left[\frac{1}{2}\sigma^2(u,v)\gamma_{ii} + f(u,v)\right] \\ &= \sum_{i=1}^{d} \inf_{0\le u\le 1} \sup_{0\le u\le 1} \left[\frac{1}{2}\sigma^2(u,v)\gamma_{ii} + f(u,v)\right], \end{aligned}$$

$\sigma^2(u,v) = (1+u+v), \ f(u,v) = -\frac{u^2}{4} + \frac{v^2}{4}$

- It can be shown that this PDE has a unique viscosity solution.

A 12-Dimensional Isaacs Equation with viscosity solution



$NR_5 - NR_2$	-0.019953
$NR_{10} - NR_5$	-0.005390
$NR_{20} - NR_{10}$	-0.003752
$NR_{40} - NR_{20}$	-0.002893
$NR_{80} - NR_{40}$	-0.001213
$NR_{160} - NR_{80}$	-0.000426

Figure: Numerical results and their differences

A 12-Dimensional Coupled FBSDE

- Coupled FBSDE : W, X, Z are d-dimensional, Y is 1-dimensional

$$X_t = X_0 + \int_0^t b(Y_s, Z_s)ds + \int_0^t \sigma(X_s, Y_s)dW_s;$$
$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s.$$

- Quasilinear PDE : $u(T, x) = g(x)$, and

$$u_t + b(u, Du\sigma) \cdot Du + \frac{1}{2}\mathrm{tr}\left(\sigma^T\sigma(x, u)D^2 u\right) + f(t, x, u, Du\sigma(x, u)) = 0.$$

- Nonlinear Feynman-Kac formula :

$$Y_t = u(t, X_t), \qquad Z_t = [Du\sigma](t, X_t).$$

- Note : the generator of the above PDE is not Lipschitz continuous w.r.t. u or Du. We use truncations to make it Lip. continuous.

A 12-Dimensional Coupled FBSDE

- **Example 4** : σ is diagonal with

$$b_i(Y,Z) \triangleq \cos(Y + Z^i),\ \ \sigma_{ii}(X,Y) \triangleq 1 + \frac{1}{3}\sin\left(\frac{\sum_{i=1}^d X^i}{d} + Y\right),$$
$$g(X) \triangleq \sin(T + \textstyle\sum_{i=1}^d X^i), \quad f(t,x,y,z) \triangleq \cdots;$$
$$T \triangleq 0.2, \quad X_0 \triangleq (2,3,...,13)$$

- True solution : $Y_t = \sin(t + \sum_{i=1}^d X_t^i)$ and $Y_0 = 0.893997$

- Numerical approximation of Y_0 :

N	L	K	Avg(Ans.)	Var(Avg.)	cost (in seconds)
2	2083	160	1.462543	3.35×10^{-5}	1.56×10^{-2}
5	13021	64	1.111675	2.30×10^{-5}	2.36×10^{-1}
10	52083	32	1.014295	2.48×10^{-5}	2.43×10^{0}
20	208333	16	0.925712	8.10×10^{-6}	2.29×10^{1}
40	833333	8	0.912373	2.46×10^{-6}	1.94×10^{2}
80	3333333	4	0.908013	2.89×10^{-7}	1.56×10^{3}
160	13333333	2	0.888747	1.62×10^{-8}	3.42×10^{4}

A 12-Dimensional Coupled FBSDE

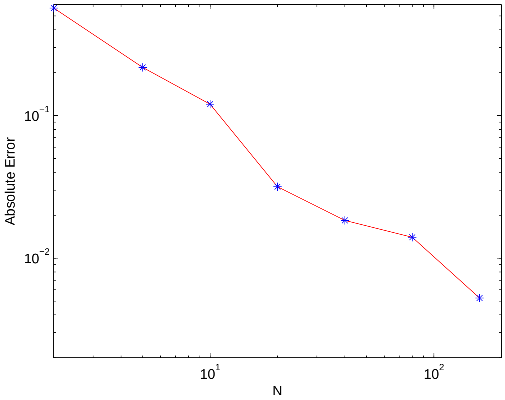


Figure: Errors for approximating the 12 dimensional FBSDE at X_0

A 10-dimensional PDE with G-generator

Example 5.

- We consider the following 10-dimensional PDE :

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{2} \sup\limits_{\underline{\sigma} \leq \sigma \leq \overline{\sigma}} \operatorname{tr}\left[\sigma^2 D^2 u\right] + f(t,x) = 0, \text{ on } [0,T) \times \mathbb{R}^d, \\ u(T,x) = \sin(T + x_1 + \frac{x_2}{2} \ldots + \frac{x_d}{d}), \quad \text{on } \mathbb{R}^d, \end{cases} \tag{4}$$

where $d = 10$, $\underline{\sigma}^2$ and $\overline{\sigma}^2$ are positive-definite matrices, and $f(t,x)$ is a function such that the true solution is

$$u(t,x) = \sin(t + x_1 + \frac{x_2}{2} + \ldots + \frac{x_{10}}{10}).$$

- ◇ This generator doesn't satisfy our key assumption.
- ◇ Our scheme works.

A 10-dimensional PDE with G-generator

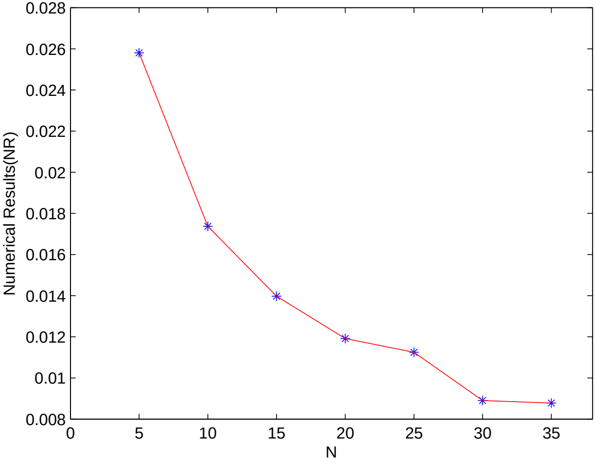
We pick x_0 and $\overline{\sigma}^2 > \underline{\sigma}^2 > 0$ arbitrarily :

$$X_0 = (2.99,\ 3.05,\ 1.54,\ 1.89,\ 2.52,\ 1.10,\ 3.21,\ 1.64,\ 1.02,\ 1.80),$$

so the true solution is 0.75805.

N	L	K	Avg(Ans.)	Var(Avg.)	cost (in seconds)
5	10000	40	0.78385	5.22×10^{-8}	13
10	10000	20	0.77542	4.28×10^{-7}	57
15	10000	13	0.77202	3.80×10^{-7}	135
20	10000	10	0.76997	4.45×10^{-7}	248
25	10000	8	0.76930	2.28×10^{-6}	395
30	10000	6	0.76696	3.25×10^{-6}	573
35	10000	5	0.76683	3.08×10^{-6}	784

A 10-dimensional PDE with G-generator



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