A General Comparison Theorem for Backward Stochastic Differential Equations

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Abstract

A useful result when dealing with Backward Stochastic Differential Equations is the comparison theorem of Peng (1992). When the equations are not based on Brownian motion, the comparison theorem no longer holds in general. We here present a condition for a comparison theorem to hold for Backward Stochastic Differential Equations based on arbitrary Martingales. This theorem applies to both vector and scalar situations. Applications to the theory of nonlinear expectations are then explored.

1 Introduction

The theory of Backward Stochastic Differential Equations, (BSDEs), is an active area of research in both Mathematical Finance and Stochastic Control. Typically, one begins by defining processes (Y, Z) through an equation of the form

$$Y_t - \int_{]t,T]} F(\omega, u, Y_{u-}, Z_u)du + \int_{]t,T]} Z_u dM_u = Q. \tag{1}$$

Here Q is a square-integrable terminal condition, F a progressively measurable 'driver' function, and M an N-dimensional Brownian Motion, all defined on a probability space with filtration generated by M. Recent work has also allowed the presence of jumps and the use of other underlying processes. However, these typically require the addition of another martingale process, as a martingale representation theorem may not hold. See [9] for some general results. In [4], we considered the situation where M is the compensated jump martingale

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generated by a continuous-time, finite state Markov Chain and showed that solutions existed for equations of this type.

A fundamental result, first obtained by Peng, [16], is the 'Comparison Theorem' for BSDEs. This result is connected to the Pontryagin Maximum Principle in optimal control, and, as is explored by [20, Ch. 8] for the linear case, to the theory of no-Arbitrage in a financial market. In [16], the comparison theorem was established for scalar BSDEs based on N-dimensional Brownian motion. Various other works have extended this result, primarily through the addition of jump terms. [2] gives a comparison theorem for BSDEs based on N-dimensional Brownian motion and an independent Poisson random measure. Other particular cases, with Poisson random measures and Lévy processes, can be seen in [20], [19], [15], [10] and [21]. [2] also includes a counterexample, which shows that the conditions of the comparison theorem for simple Brownian motion are insufficient when jumps are present. A forthcoming paper, [5], gives a comparison theorem for BSDEs based on continuous time Markov chains. To obtain these theorems, added conditions on the driver F, in particular on its relation to the jump component of the underlying process, are needed. This paper generalises these conditions, by expressing them in the language of equivalent measures.

In the vector valued context, [14] considers vector-valued BSDEs based on N-dimensional Brownian motion, and, under some further technical conditions, (which degenerate into the classical requirements in one dimension), presents a comparison theorem in this context. These conditions are considerably less intuitive than those in one dimension, and do not as easily lead to applications in optimal control, or to a theory of multidimensional nonlinear evaluations and expectations (as in [7], [19] and others). The conditions we present in this paper are different, and are a more natural componentwise extension of the scalar case. Various comparison theorems for vector-valued BSDEs based on continuous time Markov chains are given in [5], however under restrictive conditions on the driver F.

The above results are all in contexts where the simple BSDE (1), or its variants including random measures, has a solution. In these cases, an orthogonal martingale, the dL term in (2), is not required. This paper does not assume this, and the comparison theorem also applies to the generalised BSDE (2) considered in [9].

In this paper, we generalise these results, by making clearer the relationship between the comparison theorem and the existence of equivalent (super-)martingale measures. This highlights the relationship between the comparison theorem and no-Arbitrage in a financial market, by reference to the Fundamental Theorem of Asset Pricing (see [8]). By expressing the conditions for the comparison theorem in this way, the proofs become considerably simpler, and also extend naturally to BSDEs based on any martingale process, as they do not depend on a Lévy characterisation or on the Markov property. They are also the natural extension of the conditions derived in [6], for discrete time BSDEs. (Note that in some cases, ([6, Thm 7]), in this discrete time framework, one can show that the requirements of the comparison theorem are *necessary* for the monotonicity of the BSDE solutions.)

We demonstrate the usefulness of this result in defining dynamically consistent nonlinear expectations.

2 BSDEs with arbitrary Martingales

Let M be an arbitrary càdlàg martingale on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ satisfying the usual assumptions of completeness and right-continuity. Let $\|\cdot\|$ denote the Euclidean norm, and $L^0(\mathcal{F}_t)$ the set of $\mathcal{F}_t$ measurable random variables Q with $\mathbb{P}(\|Q\| = +\infty) = 0$.

For a fixed deterministic $T \leq +\infty$ and for a given integrable, progressively measurable function $F : \Omega \times [0,T] \times \mathbb{R}^K \times \mathbb{R}^{K\times N} \to \mathbb{R}^K$, we shall consider equations of the form

$$Y_r - \int_{]r,t]} F(\omega, u, Y_{u-}, Z_u)du + \int_{]r,t]} Z_u dM_u + \int_{]r,t]} dL_u = Q \tag{2}$$

where $0 \leq r \leq t \leq T$ and Q is an $\mathbb{R}^K$ valued $\mathcal{F}_t$ measurable terminal condition. A solution to (2) is a triple (Y, Z, L), with Y a càdlàg, adapted, $\mathbb{R}^K$ valued process, Z a predictable $\mathbb{R}^{K\times N}$ valued process and L a $\mathbb{R}^K$ valued càdlàg martingale, with $L_0 = 0$, which is orthogonal to M, that is, the $K \times N$ dimensional matrix process

$$\langle L, M\rangle = (\langle e_i^* L, e_j^* M\rangle)_{i=1,\dots,K}^{j=1,\dots,N} \equiv 0.$$

We here restrict our attention to deterministic T, however, through appropriate modification of the driver F, it is easy to see that this extends to the case when T is a stopping time (see [5]). We know from [9] that, under certain assumptions about M, F and Q, this equation will always have a solution (Y, Z, L). If we consider a space such that a martingale representation theorem holds for M then, without loss of generality, $L = 0$. We assume here that the driver F is integrated with respect to time (du), however the general case, considered in [9], where du is replaced with dC_u, for C a continuous, adapted, increasing process, is a straightforward modification of the results given here. (The only slight difficulty is in obtaining an appropriate version of Grönwall's inequality.)

As the focus of this paper is on comparison results, rather than proving the existence of BSDE solutions, we shall refrain from explicitly making the assumptions of [9], and in general, denote by $\mathbf{Q}^F_{s,t} \subseteq L^0(\mathcal{F}_t)$ the set of values Q such that, for all $r \in [s,t]$, (2) has a unique solution, up to indistinguishability on $[s,t] \times \Omega$ for the triple $(Y_r, \int_{]s,r]} Z_u dM_u, L_r)$.

This solution may be constrained to satisfy certain conditions, for example, in [4] and [9] it is (implicitly) required that $\sup_{r\in[s,t]} E[\|Y_r\|^2|\mathcal{F}_s] < +\infty$. In this case, the solution may only be unique among those processes satisfying these constraints. We shall assume that, for $r \in [s,t]$, this solution satisfies the integrability assumption

$$E[\|Y_r\||\mathcal{F}_s] < +\infty \ \mathbb{P}\text{-a.s.}$$

and take any other assumptions necessary for uniqueness as implicit.

Lemma 1. *The following properties of $\mathbf{Q}^F_{s,t}$ are immediately apparent:*

1. *For all $s \leq r \leq t$, $\mathbf{Q}^F_{s,t} \subseteq \mathbf{Q}^F_{r,t}$.*
2. *For all $t < T$, without loss of generality, $\mathbf{Q}^F_{t,t} = L^0(\mathcal{F}_t)$.*
3. *For $s \leq t$, let $Q \in \mathbf{Q}^F_{s,t}$ and let Y_r satisfy (2). Then, for all $r \in [s,t]$, $Y_r \in \mathbf{Q}^F_{s,r}$.*

4. $\mathbf{Q}^F_{s,t} \subseteq \{Q \in L^0(\mathcal{F}_t) : E[\|Q\| | \mathcal{F}_s] < +\infty\}$.

Proof. Property 1 follows from the fact that, if we have a unique solution to (2) on $[s,t]$, then we have a unique solution on $[r,t] \subseteq [s,t]$. Property 2 is because (2) degenerates into the tautology $Y_t = Q$, which clearly has a unique solution up to indistinguishability. Property 3 is simply due to a rearrangement of (2). Property 4 is due to the assumed integrability condition, evaluated for $Y_t = Q$. □

3 A Comparison Theorem

In the following, a vector inequality is assumed to hold componentwise.

Theorem 1 (Comparison Theorem). *Suppose we have two BSDEs corresponding to coefficients and terminal values (F^1, Q^1) and (F^2, Q^2), $Q^1 \in \mathbf{Q}^{F^1}_{s,t}$, $Q^2 \in \mathbf{Q}^{F^2}_{s,t}$. Let (Y^1, Z^1, L^1) and (Y^2, Z^2, L^2) be the associated solutions. We suppose the following conditions hold:*

(i) $Q^1 \geq Q^2$ *$\mathbb{P}$-a.s.*

(ii) $du \times \mathbb{P}$*-a.s. on* $[s,t] \times \Omega$,

$$F^1(\omega, u, Y^2_{u-}, Z^2_u) \geq F^2(\omega, u, Y^2_{u-}, Z^2_u).$$

(iii) For each i, there exists a measure $\tilde{\mathbb{P}}_i$ equivalent to $\mathbb{P}$ such that the ith component of X, as defined for $r \in [s,t]$ by

$$\begin{aligned} e_i^* X_r := &- \int_{]s,r]} e_i^*[F^1(\omega, u, Y^2_{u-}, Z^1_u) - F^1(\omega, u, Y^2_{u-}, Z^2_u)]du \\ &+ \int_{]s,r]} e_i^*[Z^1_u - Z^2_u]dM_u + e_i^*[L^1_r - L^2_r] \end{aligned}$$

is a $\tilde{\mathbb{P}}_i$ supermartingale on $[s,t]$.

(iv) For all $r \in [s,t]$, if

$$\begin{aligned} &e_i^* Y^1_r - E_{\tilde{\mathbb{P}}_i}\left[\int_{]r,t]} e_i^* F^1(\omega, u, Y^1_{u-}, Z^1_u)du \middle| \mathcal{F}_r\right] \\ &\quad \geq e_i^* Y^2_r - E_{\tilde{\mathbb{P}}_i}\left[\int_{]r,t]} e_i^* F^1(\omega, u, Y^2_{u-}, Z^1_u)du \middle| \mathcal{F}_r\right] \end{aligned}$$

for all i, then $Y^1_r \geq Y^2_r$ componentwise.

It is then true that $Y^1 \geq Y^2$ on $[s,t]$, except possibly on some evanescent set.

Proof. We omit the ω and u arguments of F for clarity. Then, for $r \in [s,t]$

$$\begin{aligned} &Y^1_r - Y^2_r - \int_{]r,t]} [F^1(Y^1_{u-}, Z^1_u) - F^2(Y^2_{u-}, Z^2_u)]du \\ &\quad + \int_{]r,t]} [Z^1_u - Z^2_u]dM_u + \int_{]r,t]} dL^1_u - \int_{]t,T]} dL^2_u = Q^1 - Q^2 \geq 0 \end{aligned} \tag{3}$$

which can be rearranged to give

$$
\begin{aligned}
&Y_r^1 - Y_r^2 - \int_{]r,t]} [F^1(Y_{u-}^1, Z_u^1) - F^1(Y_{u-}^2, Z_u^1)]du \\
&\geq \int_{]r,t]} [F^1(Y_{u-}^2, Z_u^2) - F^2(Y_{u-}^2, Z_u^2)]du + \int_{]r,t]} [F^1(Y_{u-}^2, Z_u^1) - F^1(Y_{u-}^2, Z_u^2)]du \\
&\quad - \int_{]r,t]} [Z_u^1 - Z_u^2]dM_u - \int_{]r,t]} dL_u^1 + \int_{]r,t]} dL_u^2.
\end{aligned}
\tag{4}
$$

We have that $\int_{]r,t]}[F^1(Y_{u-}^2, Z_u^2) - F^2(Y_{u-}^2, Z_u^2)]du \geq 0$ by assumption (ii). As $e_i^* X_r$ is a $\tilde{\mathbb{P}}_i$ supermartingale, we know that the process given by

$$
\begin{aligned}
e_i^* \tilde{X}_r &:= e_i^* X_r - E_{\tilde{\mathbb{P}}_i}[e_i^* X_t | \mathcal{F}_r] \\
&= E_{\tilde{\mathbb{P}}_i}\Big[\int_{]r,t]} e_i^*[F^1(Y_{u-}^2, Z_u^1) - F^1(Y_{u-}^2, Z_u^2)]du \\
&\qquad - \int_{]r,t]} e_i^*[Z_u^1 - Z_u^2]dM_u - \int_{]r,t]} e_i^* dL_u^1 + \int_{]t,T]} e_i^* dL_u^2 \Big| \mathcal{F}_r \Big]
\end{aligned}
\tag{5}
$$

is also a $\tilde{\mathbb{P}}_i$-supermartingale, with $e_i^* \tilde{X}_t = 0$ $\tilde{\mathbb{P}}_i$-a.s. Hence $e_i^* \tilde{X}_r \geq 0$.

For each i, taking a $\tilde{\mathbb{P}}_i|\mathcal{F}_r$ conditional expectation throughout (4) and premultiplying by e_i^* gives

$$
e_i^* Y_r^1 - e_i^* Y_r^2 - E_{\tilde{\mathbb{P}}_i}\left[\int_{]r,t]} e_i^*[F^1(Y_{u-}^1, Z_u^1) - F^1(Y_{u-}^2, Z_u^1)]du \middle| \mathcal{F}_r \right] \geq 0.
$$

By Assumption (iv), this then proves $Y_r^1 \geq Y_r^2$ componentwise $\mathbb{P}$-a.s. for each $r \in [s,t]$. As $Y^1 - Y^2$ is càdlàg, we have that $Y^1 - Y^2$ is indistinguishable from a nonnegative process and, therefore, the inequality holds up to evanescence. □

Corollary 1. *Theorem 1 remains true if Assumptions (iii) and (iv) are replaced by*

(iii') For each i, there exists a measure $\tilde{\mathbb{P}}_i$ equivalent to $\mathbb{P}$ such that the ith component of X, as defined for $r \in [s,t]$

$$
\begin{aligned}
e_i^* X_r := &- \int_{]s,r]} e_i^*[F^1(\omega, u, Y_{u-}^1, Z_u^1) - F^1(\omega, u, Y_{u-}^1, Z_u^2)]du \\
&+ \int_{]s,r]} e_i^*[Z_u^1 - Z_u^2]dM_u + e_i^*[L_t^1 - L_t^2]
\end{aligned}
$$

is a $\tilde{\mathbb{P}}_i$ supermartingale on $[s,t]$.

(iv') For all r, if

$$
\begin{aligned}
&e_i^* Y_r^1 - E_{\tilde{\mathbb{P}}_i}\left[\int_{]r,t]} e_i^* F^1(\omega, u, Y_{u-}^1, Z_u^2)du \middle| \mathcal{F}_t \right] \\
&\qquad \geq e_i^* Y_r^2 - E_{\tilde{\mathbb{P}}_i}\left[\int_{]r,t]} e_i^* F^1(\omega, u, Y_{u-}^2, Z_u^2)du \middle| \mathcal{F}_t \right]
\end{aligned}
$$

for all i, then $Y_r^1 \geq Y_r^2$ componentwise.

Proof. In this case, the decomposition in (4) and the definition of $\tilde{X}$ in (5) are correspondingly changed. The rest of the proof remains valid. $\square$

Remark 1. In the scalar case, $(K = 1)$, when M is a Brownian motion generating $\{\mathcal{F}\}$ and $F^1(\omega, u, Y_{u-}, \cdot)$ is uniformly Lipschitz continuous in Z_u, and in many other cases, we can use the standard results of Girsanov's theorem to show that Assumption (iii) holds.

Remark 2. A significant special case, particularly in the context of Dynamic Risk Measures, is when F is assumed not to depend on Y. (See, for example, [3], [18], [19], [5] and [6].) In this case, Assumption (iv) is trivial.

The following backwards version of Grönwall's inequality will be useful.

Lemma 2. *Suppose $\phi : [s,t] \to \mathbb{R}$ is such that, for constants $\alpha \geq 0$, $\beta \geq 0$,*

$$\phi_r \leq \alpha + \beta \int_{]r,t]} \phi_u du$$

for all $r \in [s,t]$. Then $\phi_r \leq \alpha e^{-\beta(t-r)}$.

Proof. Write $\eta_r = \alpha + \beta \int_{]r,t]} \phi_u du$. Then

$$\frac{d\eta_r}{dr} = -\beta\phi_r \geq -\beta\eta_r.$$

Hence if $\nu_r = e^{-\beta r}\eta_r$,

$$\frac{d\nu_r}{dr} = -\beta e^{-\beta r}\eta_r + e^{-\beta r}\frac{d\eta_r}{dr} \geq 0.$$

This implies ν is nondecreasing, and so $\nu_r \leq \nu_t$, which by rearrangement gives

$$\eta_r \leq \alpha e^{-\beta(t-r)}.$$

Finally, as $\phi_r \leq \eta_r$, we have the result. $\square$

Theorem 2. *Consider the scalar,$(K = 1)$, case, where, for all $u \in [s,t]$, $F^1(\omega, u, \cdot, Z_u^1)$ is uniformly Lipschitz continuous with respect to Y, that is, there exists $c \geq 0$ such that, for any Y^1, Y^2,*

$$|F^1(\omega, u, Y_{u-}^1, Z_u^1) - F^1(\omega, u, Y_{u-}^2, Z_u^1)| \leq c|Y_{u-}^1 - Y_{u-}^2|, \ du \times \mathbb{P}\text{-a.s.}$$

Then Assumption (iv) of Theorem 1 is trivial, (and hence can be omitted).

Proof. As we are in the scalar case, we can omit the e_i from the statement of the assumption. Hence, we wish to show that, given for all $r \in [s,t]$

$$\begin{aligned} Y_r^1 - E_{\tilde{\mathbb{P}}}&\left[\int_{]r,t]} F^1(\omega, u, Y_{u-}^1, Z_u^1) du \middle| \mathcal{F}_r\right] \\ &\geq Y_r^2 - E_{\tilde{\mathbb{P}}}\left[\int_{]r,t]} F^1(\omega, u, Y_{u-}^2, Z_u^1) du \middle| \mathcal{F}_r\right] \end{aligned}$$

we must have $Y_r^1 \geq Y_r^2$.

Simple rearrangement gives

$$Y_r^1 - Y_r^2 \geq E_{\tilde{\mathbb{P}}}\left[\int_{]r,t]} [F^1(\omega, u, Y_{u-}^1, Z_u^1) - F^1(\omega, u, Y_{u-}^2, Z_u^1)]du \middle| \mathcal{F}_r\right].$$

By Lipschitz continuity, we know that

$$Y_r^1 - Y_r^2 \geq -cE_{\tilde{\mathbb{P}}}\left[\int_{]r,t]} |Y_{u-}^1 - Y_{u-}^2|du \middle| \mathcal{F}_r\right].$$

Let $A = \{\omega : Y_r^1 - Y_r^2 < 0\} \in \mathcal{F}_r$. Then we have

$$I_A|Y_r^1 - Y_r^2| \leq c\int_{]r,t]} E_{\tilde{\mathbb{P}}}\left[I_A|Y_u^1 - Y_u^2||\mathcal{F}_r\right] du,$$

where the left limits within the integration can be omitted as $Y^1 - Y^2$ is càdlàg.

Taking a $\tilde{\mathbb{P}}$ expectation,

$$E_{\tilde{\mathbb{P}}}[I_A|Y_t^1 - Y_t^2|] \leq c\int_{]r,t]} E_{\tilde{\mathbb{P}}}[I_A|Y_u^1 - Y_u^2|]du,$$

and application of Lemma 2 then implies

$$E_{\tilde{\mathbb{P}}}[I_A|Y_r^1 - Y_r^2|] = 0$$

and so $I_A(Y_r^1 - Y_r^2) = 0$ $\tilde{\mathbb{P}}$-a.s. As $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent, this proves

$$Y_r^1 - Y_r^2 \geq 0$$

$\mathbb{P}$-a.s., as desired. □

Remark 3. By Theorem 2 and Remark 1, we can see that the classical inequality of Peng [16] is simply a special case of Theorem 1. Similarly the scalar comparison in [5, Thm 4.2] also follows as a special case. However, as shown in the counterexamples presented in [5, Example 5.1], for the vector case, Assumption (iv) remains nontrivial, (cf. Remark 5).

Definition 1. *The comparison between Y^1 and Y^2 will be called **strict** on $[s,t]$ if the conditions of Theorem 1 hold, and, for any $A \in \mathcal{F}_s$ such that $Y_s^1 = Y_s^2$ $\mathbb{P}$-a.s. on A, we have $Y_u^1 = Y_u^2$ on $[s,t] \times A$, up to evanescence.*

Lemma 3. *If the comparison is strict on $[s,t]$, then for any $A \in \mathcal{F}_s$ such that $Y_s^1 = Y_s^2$ $\mathbb{P}$-a.s. on A, it follows that*

- *$Q^1 = Q^1$ $\mathbb{P}$-a.s. on A,*
- *$F^1(\omega, u, Y_{u-}^2 Z_u^2) = F^2(\omega, u, Y_{u-}^2, Z_u^2)$ $du \times \mathbb{P}$-a.s. on $[s,t] \times A$, and*
- *for $r \in [s,t]$, up to indistinguishability, on A,*

$$\int_{]s,r]} Z_u^1 dM_u = \int_{]s,r]} Z_u^2 dM_u$$

and

$$L_r^1 = L_r^2.$$

Proof. We omit the ω and t arguments of F^1 and F^2 for clarity. Let $\tilde{X}$ be as in (5), and let S be the process defined by

$$\begin{aligned} e_i^* S_r := e_i^* E_{\tilde{\mathbb{P}}_i} \left[Q^1 - Q^2 \middle| \mathcal{F}_r \right] \\ + e_i^* E_{\tilde{\mathbb{P}}_i} \left[\int_{]r,t]} [F^1(Y^2_{u-}, Z^2_u) - F^2(Y^2_{u-}, Z^2_u)] du \middle| \mathcal{F}_r \right] + \tilde{e}_i^* X_r. \end{aligned} \tag{6}$$

Then $e_i^* S$ is a $\tilde{\mathbb{P}}_i$-supermartingale, as the first term is a $\tilde{\mathbb{P}}_i$-martingale, the second is nonincreasing in r by Assumption (ii) of Theorem 1, and the third is a $\tilde{\mathbb{P}}_i$-supermartingale by Assumption (iii) of Theorem 1. Furthermore, each of these terms is nonnegative.

Taking a $\tilde{\mathbb{P}}|\mathcal{F}_r$ conditional expectation through (2), we have that, for all $r \in [s, t]$,

$$Y^1_r - Y^2_r = S_r + E_{\tilde{\mathbb{P}}} \left[\int_{]r,t]} [F^1(Y^1_{u-}, Z^1_u) - F^1(Y^2_{u-}, Z^1_u)] du \middle| \mathcal{F}_r \right]. \tag{7}$$

If $Y^1_r = Y^2_r$ on $[s, t] \times A$ up to evanescence, then it is clear from (7) that $S_r = 0$ $\mathbb{P}$-a.s. on $[s, t] \times A$. Hence, by nonnegativity, each of the terms on the right hand side of (6) must be zero. The first two points of the lemma immediately follow.

Consider the BSDE (2) satisfied by Y^2. As $F^1(Y^2_{u-}, Z^2_u) = F^2(Y^2_{u-}, Z^2_u)$ $du \times \mathbb{P}$-a.s. on $[s, t] \times A$ and $Q^1 = Q^2$ $\mathbb{P}$-a.s. on A, we know that

$$Y^2_r - \int_{]r,t]} F^2(Y^2_{u-}, Z^2_u) du + \int_{]r,t]} Z^2_u dM_u + \int_{]r,t]} dL^2_u = Q^2$$

is $\mathbb{P}$-a.s. equal to

$$Y^2_r - \int_{]r,t]} F^1(Y^2_{u-}, Z^2_u) du + \int_{]r,t]} Z^2_u dM_u + \int_{]r,t]} dL^2_u = Q^1.$$

Hence, in A, (Y^2, Z^2, L^2) is a solution at time r to the BSDE defining the triple (Y^1, Z^1, L^1).

By assumption, as $Q^1 \in \mathbf{Q}^{F^1}_{s,t}$, the solution to this BSDE is unique up to indistinguishability for $(Y, \int_{]s,\cdot]} Z_u dM_u, L)$ on $[s, t] \times \Omega$. It follows that, for each r, $(Y^1_r, \int_{]s,r]} Z^1_u dM_u, L^1_r)$ is unique up to equality $\mathbb{P}$-a.s., and therefore $\int_{]s,r]} Z^1_u dM_u = \int_{]s,r]} Z^2_u dM_u$ and $L^1_r = L^2_r$ $\mathbb{P}$-a.s. on A. As all of these processes are càdlàg, it follows from [11, Lemma 2.21] that they are indistinguishable on $[s, t] \times A$. □

Theorem 3 (Strict Comparison 1)**.** *Consider the scalar, ($K = 1$), case, where F^1 is such that Theorems 1 and 2 hold. Then the comparison is strict on* $[s, t]$

Proof. Again, as $K = 1$ we can omit e_i from all equations, and we omit the ω and t arguments of F^1 and F^2 for clarity. Let S_r be as defined in (6), and note that S is a nonnegative $\tilde{\mathbb{P}}$-supermartingale.

Taking a $\tilde{\mathbb{P}}|\mathcal{F}_s$ conditional expectation of (7) gives

$$\begin{aligned}
E_{\tilde{\mathbb{P}}}\left[Y_r^1 - Y_r^2 \middle| \mathcal{F}_s\right] &= E_{\tilde{\mathbb{P}}}\left[S_r + \int_{]s,t]} [F^1(Y_{u-}^1, Z_u^1) - F^1(Y_{u-}^2, Z_u^1)]du \middle| \mathcal{F}_s\right] \\
&\quad - E_{\tilde{\mathbb{P}}}\left[\int_{]s,r]} [F^1(Y_{u-}^1, Z_u^1) - F^1(Y_{u-}^2, Z_u^1)]du \middle| \mathcal{F}_s\right] \\
&\leq S_s + E_{\tilde{\mathbb{P}}}\left[\int_{]s,t]} [F^1(Y_{u-}^1, Z_u^1) - F^1(Y_{u-}^2, Z_u^1)]du \middle| \mathcal{F}_s\right] \\
&\quad + \int_{]s,r]} E_{\tilde{\mathbb{P}}}\left[|F^1(Y_{u-}^1, Z_u^1) - F^1(Y_{u-}^2, Z_u^1)| \middle| \mathcal{F}_s\right] du \\
&\leq S_s + E_{\tilde{\mathbb{P}}}\left[\int_{]s,t]} [F^1(Y_{u-}^1, Z_u^1) - F^1(Y_{u-}^2, Z_u^1)]du \middle| \mathcal{F}_s\right] \\
&\quad + c\int_{]s,r]} E_{\tilde{\mathbb{P}}}\left[|Y_{u-}^1 - Y_{u-}^2| \middle| \mathcal{F}_s\right] du
\end{aligned} \tag{8}$$

We know from (7) and the assumption $Y_s^1 - Y_s^2 = 0$ on A that

$$I_A S_s + I_A E_{\tilde{\mathbb{P}}}\left[\int_{]s,t]} [F^1(Y_{u-}^1, Z_u^1) - F^1(Y_{u-}^2, Z_u^1)]du \middle| \mathcal{F}_s\right] = I_A(Y_s^1 - Y_s^2) = 0,$$

and so, as $Y^1 - Y^2$ is nonnegative by Theorem 1, premultiplication of (8) by I_A and taking an expectation gives

$$E_{\tilde{\mathbb{P}}}[I_A(Y_r^1 - Y_r^2)] \leq c\int_{]s,r]} E_{\tilde{\mathbb{P}}}[I_A(Y_{u-}^1 - Y_{u-}^2)]du.$$

An application of (the forward version of) Grönwall's Lemma then yields

$$E_{\tilde{\mathbb{P}}}[I_A(Y_r^1 - Y_r^2)] \leq 0,$$

which, by nonnegativity, implies $Y_r^1 = Y_r^2$, $\tilde{\mathbb{P}}$-a.s. on A. Again, as $Y^1 - Y^2$ is càdlàg, this shows that $Y^1 = Y^2$ on $[s,t] \times A$, up to evanescence. □

Remark 4. Theorems 2, 4 and 3 can also be modified in the same way is done in Corollary 1; in this case the assumptions of the theorems will refer to $F^1(\omega, u, \cdot, Z_u^2)$.

Remark 5. In a vector setting, it is easy to see that, if the ith component of F^1 depends only on the ith component of Y, that is, we can write

$$e_i^* F^1(\omega, t, Y_{t-}, Z_t) = F_i(\omega, t, e_i^* Y_{t-}, Z_t)$$

for some F_i, and if F_i is uniformly Lipschitz in $e_i^* Y_{t-}$, then the proofs of Theorems 2 and 3 can be extended to cover these cases, simply by considering each component separately.

In this case, the strict comparison will apply *componentwise*, that is, if for some $A \in \mathcal{F}_s$ we have $e_i^* Y_s^1 = e_i^* Y_s^2$, then $e_i^* Y_u^1 = e_i^* Y_u^2$ on $[s,t] \times A$, up to evanescence.

If F^1 does not depend on Y_{t-}, then this is clearly the case, (as F^1 is uniformly Lipschitz with constant $c = 0$).

In some situations, particularly in the vector context, this result may be insufficient. The following theorem addresses some such cases.

Theorem 4 (Strict Comparison 2)**.** *Suppose we have two BSDEs satisfying the conditions of Theorem 1. Suppose furthermore that Assumption (iv) is satisfied in such a way that, for all $r \in [s,t]$, $A \in \mathcal{F}_s$*

$$\begin{aligned} e_i^* Y_r^1 - E_{\tilde{\mathbb{P}}_i}\left[\int_{]r,t]} e_i^* F^1(\omega, u, Y_{u-}^1, Z_u^1)du \middle| \mathcal{F}_r\right] \\ = e_i^* Y_r^2 - E_{\tilde{\mathbb{P}}_i}\left[\int_{]r,t]} e_i^* F^1(\omega, u, Y_{u-}^2, Z_u^1)du \middle| \mathcal{F}_r\right] \end{aligned}$$

$\mathbb{P}$-a.s. on A for all i, if and only if $Y_r^1 = Y_r^2$ $\mathbb{P}$-a.s. on A. Then the comparison is strict on $[s,t]$.

Proof. As in Theorem 1, the weak comparison holds, that is, $Y_r^1 \geq Y_r^2$ $\mathbb{P}$-a.s. on $[s,t]$. Recall that the measures $\mathbb{P}$ and $\tilde{\mathbb{P}}_i$ were assumed to be equivalent, and hence any statement up to equality $\tilde{\mathbb{P}}_i$-a.s. could equivalently be made $\mathbb{P}$-a.s.

As before, $e_i^*\tilde{X}$, defined in (5), is a $\tilde{\mathbb{P}}_i$-supermartingale with $e_i^*\tilde{X}_t = 0$. We know that $Y_s^1 = Y_s^2$ on A, and hence, by the stronger version of Assumption (iv),

$$Y_s^1 - Y_s^2 - E_{\tilde{\mathbb{P}}}\left[\int_{]s,t]} [F^1(Y_{u-}^1, Z_u^1) - F^1(Y_{u-}^2, Z_u^1) \middle| \mathcal{F}_s\right] du = 0.$$

As in (4), this implies

$$\begin{aligned} 0 \geq E_{\tilde{\mathbb{P}}_i}&\left[\int_{]s,t]} [F^1(Y_{u-}^2, Z_u^2) - F^2(Y_{u-}^2, Z_u^2)]du \middle| \mathcal{F}_s\right] \\ &+ E_{\tilde{\mathbb{P}}_i}\left[\int_{]s,t]} [F^1(Y_{u-}^2, Z_u^1) - F^1(Y_{u-}^2, Z_u^2)]du \right. \\ &\qquad \left. - \int_{]s,t]} [Z_u^1 - Z_u^2]^* dM_u - L_s^1 + L_s^2 \middle| \mathcal{F}_s\right]. \end{aligned}$$

Premultiplying by e_i^*, we have

$$0 \geq E_{\tilde{\mathbb{P}}_i}\left[\int_{]s,t]} e_i^*[F^1(Y_{u-}^2, Z_u^2) - F^2(Y_{u-}^2, Z_u^2)]du \middle| \mathcal{F}_s\right] + e_i^*\tilde{X}_s.$$

Both of the terms on the right are nonnegative, and hence both must be zero on A.

If $e_i^*\tilde{X}_s = 0$ on A, then $e_i^*\tilde{X}_r = 0$ on A for all $r \geq s$, as $e_i^*\tilde{X}$ is a nonnegative $\tilde{\mathbb{P}}_i$ supermartingale. Similarly,

$$E_{\tilde{\mathbb{P}}_i}\left[\int_{]s,t]} e_i^*[F^1(Y_{u-}^2, Z_u^2) - F^2(Y_{u-}^2, Z_u^2)]du \middle| \mathcal{F}_s\right] = 0$$

and, therefore, as $e_i^*[F^1(Y_{u-}^2, Z_u^2) - F^2(Y_{u-}^2, Z_u^2)]$ is nonnegative it must be zero $du \times \tilde{\mathbb{P}}_i$-a.s. on $[s,t] \times A$. Hence

$$\begin{aligned}
0 &= e_i^*[Y_s^1 - Y_s^2] - E_{\tilde{\mathbb{P}}_i}\left[\int_{]s,t]} e_i^*[F^1(Y_{u-}^1, Z_u^1) - F^1(Y_{u-}^2, Z_u^1)]du \middle| \mathcal{F}_s\right] \\
&\quad + E_{\tilde{\mathbb{P}}_i}\left[\int_{]s,t]} e_i^*[F^1(Y_{u-}^2, Z_u^2) - F^2(Y_{u-}^2, Z_u^2)]du \middle| \mathcal{F}_s\right] + e_i^*\tilde{X}_s \\
&= E_{\tilde{\mathbb{P}}_i}[e_i^*(Q^1 - Q^2)|\mathcal{F}_s].
\end{aligned}$$

We know $Q^1 - Q^2$ is nonnegative $\mathbb{P}$-a.s. and, therefore, combining these results for all i,

$$Q^1 - Q^2 = 0 \ \mathbb{P}\text{-a.s. on } A$$

Finally, we see that, for all i, all $r \in [s,t]$,

$$\begin{aligned}
0 &= E_{\tilde{\mathbb{P}}_i}[e_i^*(Q^1 - Q^2)|\mathcal{F}_r] \\
&= e_i^*[Y_r^1 - Y_r^2] - E_{\tilde{\mathbb{P}}_i}\left[\int_{]r,t]} e_i^*[F^1(Y_{u-}^1, Z_u^1) - F^1(Y_{u-}^2, Z_u^1)]du \middle| \mathcal{F}_r\right] \\
&\quad + E_{\tilde{\mathbb{P}}_i}\left[\int_{]r,t]} e_i^*[F^1(Y_{u-}^2, Z_u^2) - F^2(Y_{u-}^2, Z_u^2)]du \middle| \mathcal{F}_r\right] + e_i^*\tilde{X}_r
\end{aligned}$$

and hence,

$$e_i^*[Y_r^1 - Y_r^2] - E_{\tilde{\mathbb{P}}_i}\left[\int_{]r,t]} e_i^*[F^1(Y_{u-}^1, Z_u^1) - F^1(Y_{u-}^2, Z_u^1)]du \middle| \mathcal{F}_r\right] = 0,$$

$\mathbb{P}$-a.s. By the stronger version of Assumption (iv) assumed in the theorem, this proves that, for all $r \in [s,t]$, $Y_r^1 - Y_r^2 = 0$ $\mathbb{P}$-a.s. on A

As $Y^1 - Y^2$ is càdlàg, this shows that $Y^1 = Y^2$ on $[s,t] \times A$, up to evanescence. □

Remark 6. Theorem 1 helps distinguish between the understanding of dominance in the classical case and in the nonlinear case generated by BSDEs. In the classical case, no-dominance (or, in a financial setting, 'no-arbitrage') is loosely equivalent to the existence of an equivalent martingale measure for the processes Y (see [8] for more details). Here, we have assumed the existence of an equivalent supermartingale measure for the processes $e_i^* X$. One key difference is that, in the classical linear case, the equivalent martingale measure $\tilde{\mathbb{P}}$ is the same for all terminal values Q. In this nonlinear context, Assumption (iii) of the theorem states, in some sense, that there exists an equivalent (super)martingale measure corresponding to each differenced pair of terminal conditions $Q^1 - Q^2$.

4 Applications to Nonlinear Expectations

A useful consequence of this result is that it allows us to develop a theory of nonlinear expectations, in the same way as [17]. These are closely related to the theory of dynamic risk measures, as in [3], [18], [1] and others, as each concave nonlinear expectation $\mathcal{E}(\cdot|\mathcal{F}_t)$ corresponds to a dynamic convex risk measure through the relationship

$$\rho_t(Q) = -\mathcal{E}(Q|\mathcal{F}_t).$$

A further discussion of this relationship can be found in [18].

For simplicity, we shall, for the remainder of this paper, assume that the F considered are such that

$$\mathbf{Q}^F_{s,t} = \{Q \in L^0(\mathcal{F}_t) : E[\|Q\|^2|\mathcal{F}_s] < +\infty \ \mathbb{P}\text{-a.s.}\}.$$

That is, for any $\mathcal{F}_t$ measurable random variable with $\mathbb{P}$-a.s. finite $\mathcal{F}_s$ conditional variance, there exists a unique solution to the BSDE (2) on $[s,t]$, (satisfying certain conditions, such as square integrability, which we shall leave as implicit). For simplicity, this set shall be denoted $L^2_s(\mathcal{F}_t)$. For consistency, this requires that, for all $r \in [s,t]$, the solution Y_r to (2) satisfies $Y_r \in \mathbf{Q}^F_{s,r} = L^2_s(\mathcal{F}_r)$. Note that $L^0(\mathcal{F}_r) \subseteq L^2_r(\mathcal{F}_t) \subseteq L^2_s(\mathcal{F}_t)$ for all $r \leq s \leq t$.

As in [6], make the following generalisation of a definition of [17].

Definition 2. *For $s \leq t \leq T$, fix the sets $\mathcal{Q}_{s,t} \subseteq L^2_s(\mathcal{F}_t)$. A system of operators*

$$\mathcal{E}_{s,t} : L^2_s(\mathcal{F}_t) \to L^0(\mathcal{F}_s), 0 \leq s \leq t \leq T$$

*is called an $\mathcal{F}$-consistent **nonlinear evaluation** for $\{\mathcal{Q}_{s,t}\}$ defined on $[0,T]$ if $\mathcal{E}_{s,t}$ satisfies the following properties.*

1. *For $Q, Q' \in \mathcal{Q}_{s,t}$, if $Q \geq Q'$ $\mathbb{P}$-a.s. componentwise then*

   $$\mathcal{E}_{s,t}(Q) \geq \mathcal{E}_{s,t}(Q') \ \mathbb{P}\text{-a.s.}$$

   componentwise, with equality iff $Q = Q'$ $\mathbb{P}$-a.s.

2. *For $Q \in L^0(\mathcal{F}_t)$, $\mathcal{E}_{t,t}(Q) = Q$ $\mathbb{P}$-a.s.*

3. *For any $r \leq s \leq t$, any $Q \in L^2_r(\mathcal{F}_t)$,*

   $$\mathcal{E}_{s,t}(Q) \in L^2_r(\mathcal{F}_s)$$

   and

   $$\mathcal{E}_{r,s}(\mathcal{E}_{s,t}(Q)) = \mathcal{E}_{r,s}(Q) \ \mathbb{P}\text{-a.s.}$$

4. *For any $s \leq t$, $A \in \mathcal{F}_s$, $Q \in L^2_s(\mathcal{F}_t)$,*

   $$I_A\mathcal{E}_{s,t}(Q) = I_A\mathcal{E}_{s,t}(I_A Q) \ \mathbb{P}\text{-a.s.}$$

Definition 3. *Fix a driver F and time points $s \leq t$. For any*

$$Q^1, Q^2 \in \mathbf{Q}^F_{s,t} = L^2_s(\mathcal{F}_t)$$

let (Y^1, Z^1) and (Y^2, Z^2) be the associated solutions of the BSDE (2).

Suppose that, for some set $\mathcal{Q}_{s,t} \subseteq L^2_s(\mathcal{F}_t)$, for all $Q^1, Q^2 \in \mathcal{Q}_{s,t}$, *(1) For each i, there exists a measure $\tilde{\mathbb{P}}_i$ equivalent to $\mathbb{P}$ such that the ith component of X, as defined for $r \in [s,t]$ by*

$$\begin{aligned} e_i^* X_r := &- \int_{]s,r]} e_i^*[F(\omega, u, Y_{u-}^2, Z_u^1) - F(\omega, u, Y_{u-}^2, Z_u^2)]du \\ &+ \int_{]s,r]} e_i^*[Z_u^1 - Z_u^2]dM_u + e_i^*[L_r^1 - L_r^2] \end{aligned}$$

is a $\tilde{\mathbb{P}}_i$ supermartingale on $[s,t]$,

(2) F is componentwise Lipschitz continuous in Y, in the sense of Remark 5. (This includes the case when F does not depend on Y.)

*Then F will be called **balanced** on $\mathcal{Q}_{s,t}$. Additionally, F will be called balanced on a family $\{\mathcal{Q}_{s,t}\}$ if it is balanced on each member of the family.*

Remark 7. The distinction between the sets $\mathcal{Q}_{s,t}$ and $L_s^2(\mathcal{F}_t)$ is that $L_s^2(\mathcal{F}_t)$ is the set on which solutions to the BSDE exist, whereas $\mathcal{Q}_{s,t}$ is the set on which a (strict) comparison theorem holds. In some cases, these sets may be identical. However, it is useful in general to distinguish between them.

Remark 8. Again, it is possible to modify Definition 3 in the same way as done in Corollary 1.

Theorem 5. *Fix a driver F balanced on some family $\{\mathcal{Q}_{s,t}\}$. Define the 'F-evaluation', a system of operators on $L_s^2(\mathcal{F}_t)$, for all $s < t$, by*

$$\mathcal{E}_{s,t}(Q) = Y_s \tag{9}$$

where Y_s is the solution to the BSDE (2). Then $\mathcal{E}_{s,t}$ is an $\mathcal{F}_t$-consistent nonlinear evaluation for $\{\mathcal{Q}_{s,t}\}$.

Proof. We verify conditions 1-4 of Definition 2 are satisfied.

1. The statement $\mathcal{E}_{s,t}(Q_1) \geq \mathcal{E}_{s,t}(Q_2)$ $\mathbb{P}$-a.s. whenever $Q_1 \geq Q_2$ $\mathbb{P}$-a.s. is simply the result of the Strict Comparison theorem, (Theorem 1 with Theorems 2 and 3 and Remark 5), which holds as F is balanced on $\mathcal{Q}_{s,t}$.

2. The fact $\mathcal{E}_{t,t}(Q) = Q$, $\mathbb{P}$-a.s. for any $Q \in L^0(\mathcal{F}_t)$ is trivial, as we have defined $\mathcal{E}_{t,t}(Q)$ by the solution to a BSDE, which reaches its terminal value Q at time t by construction.

3. To show $\mathcal{E}_{r,s}(\mathcal{E}_{s,t}(Q)) = \mathcal{E}_{r,t}(Q)$ $\mathbb{P}$-a.s. for any $r \leq s \leq t$, let Y denote the solution to the relevant BSDE. Then a simple rearrangement of (2) gives

   $$Y_s = Y_r - \int_{]r,s]} F(\omega, u, Y_{u-}, Z_u)du + \int_{]r,s]} Z_u^* dM_u + \int_{]r,s]} dL_u.$$

   Hence Y_r is also the time r value of a solution to the BSDE with terminal time s and value Y_s. Therefore, it is clear that $Y_s \in L_r^2(\mathcal{F}_s)$, and by the uniqueness of BSDE solutions,

   $$\mathcal{E}_{r,s}(\mathcal{E}_{s,t}(Q)) = \mathcal{E}_{r,t}(Q)$$

   $\mathbb{P}$-a.s. as desired.

4. We wish to show that for $A \in \mathcal{F}_s$, $I_A \mathcal{E}_{s,t}(Q) = I_A \mathcal{E}_{s,t}(I_A Q)$ $\mathbb{P}$-a.s. For $Q \in L^2_s(\mathcal{F}_t)$, let Y^1 be the solution to the BSDE (2) with terminal condition Q. Premultiplying by I_A, we have a BSDE with terminal condition $I_A Q$, driver $I_A F$ and solution $I_A Y^1_s = I_A \mathcal{E}_{s,t}(Q)$. If Y^2 is the solution to the BSDE with terminal condition $I_A Q$, then we similarly obtain $I_A Y^2_s = I_A \mathcal{E}_{s,t}(I_A Q)$. We can now write

$$Y^3_u = I_A Y^2_u + I_{A^c} Y^1_u$$

for $u \in [s,t]$. It follows that

$$\begin{aligned} Y^3_s - \int_{]s,t]} [I_A F(\omega, t, Y^3_u, Z^3_u) + I_{A^c} F(\omega, t, Y^3_u, Z^3_u)] du + \int_{]s,t]} Z^3_u dM_u \\ = I_A Q + I_{A^c} Q, \end{aligned}$$

which clearly implies Y^1 and Y^3 are both solutions at s to the BSDE with driver F, terminal condition $Q \in L^2_s(\mathcal{F}_s)$, and hence $Y^1 = Y^3$ up to indistinguishability. Therefore $I_A Y^2_s = I_A Y^1_s$ $\mathbb{P}$-a.s. as desired.

□

With a slight modification, we also define nonlinear expectations.

Definition 4. *For $t \leq T$, fix the sets $\mathcal{Q}_t \subseteq L^2_t(\mathcal{F}_T)$. A system of operators*

$$\mathcal{E}(\cdot|\mathcal{F}_t) : L^2_t(\mathcal{F}_T) \to L^0(\mathcal{F}_t), 0 \leq t \leq T$$

*is called an $\mathcal{F}_t$-consistent **nonlinear expectation** for $\{\mathcal{Q}_t\}$ defined on $[0,T]$ if $\mathcal{E}(\cdot|\mathcal{F}_t)$ satisfies the following properties.*

1. *For $Q, Q' \in \mathcal{Q}_t$, if $Q \geq Q'$ $\mathbb{P}$-a.s. componentwise,*

$$\mathcal{E}(Q|\mathcal{F}_t) \geq \mathcal{E}(Q'|\mathcal{F}_t) \ \mathbb{P}\text{-}a.s.$$

componentwise, with equality iff $Q = Q'$ $\mathbb{P}$-a.s.

2. *For $Q \in L^0(\mathcal{F}_t)$, $\mathcal{E}(Q|\mathcal{F}_t) = Q$ $\mathbb{P}$-a.s.*

3. *For any $s \leq t$, any $Q \in L^2_s(\mathcal{F}_T)$,*

$$\mathcal{E}(Q|\mathcal{F}_t) \in L^2_s(\mathcal{F}_t) \subseteq L^2_s(\mathcal{F}_T)$$

and

$$\mathcal{E}(\mathcal{E}(Q|\mathcal{F}_t)|\mathcal{F}_s) = \mathcal{E}(Q|\mathcal{F}_s) \ \mathbb{P}\text{-}a.s.$$

4. *For any $A \in \mathcal{F}_t$, $Q \in L^2_t(\mathcal{F}_T)$,*

$$I_A \mathcal{E}(Q|\mathcal{F}_t) = \mathcal{E}(I_A Q|\mathcal{F}_t) \ \mathbb{P}\text{-}a.s.$$

Theorem 6. *Consider a driver F balanced on a family $\{\mathcal{Q}_{s,t}\}$ with*

$$F(\omega, u, Y_{u-}, 0) = 0 \ du \times \mathbb{P}\text{-}a.s.$$

on $[0,T] \times \Omega$.

For each $Q \in L^2_s(\mathcal{F}_T)$, define

$$\mathcal{E}(Q|\mathcal{F}_s) := Y_s,$$

where Y_s is the solution to (2) with $t = T$.

Then $\mathcal{E}(\cdot|\mathcal{F}_s)$ is a nonlinear expectation on $\{\mathcal{Q}_s := \mathcal{Q}_{s,T}\}$. In this case it will also be called an F-expectation.

Proof. Properties 1 and 3 follow exactly as for nonlinear evaluations.

2. Consider the BSDE (2) on $[t, T]$,

$$Y_t - \int_{]t,T]} F(\omega, u, Y_{u-}, Z_u)du + \int_{]t,T]} Z_u dM_u + \int_{]s,t]} dL_u = Q.$$

This has a solution $(Y, 0, 0)$ with $Y = Y_t = Q$. As $Q \in L^0(\mathcal{F}_t) \subset L^2_t(\mathcal{F}_T)$, this solution is adapted and unique. Therefore $\mathcal{E}(Q|\mathcal{F}_t) = Y_t = Q$ as desired.

4. We know $I_A F(\omega, t, Y_{t-}, Z_t) = F(\omega, t, I_A Y_{t-}, I_A Z_t)$ $dt \times \mathbb{P}$-a.s. and $I_A Q \in L^2_t(\mathcal{F}_T)$. Therefore, if (Y, Z, L) is the unique solution to the BSDE with driver F and terminal value Q, then we can premultiply the BSDE (2) by I_A to see that $(I_A Y, I_A Z, I_A L)$ is the unique solution to the BSDE with driver F and terminal value $I_A Q$. That is, $I_A \mathcal{E}(Q|\mathcal{F}_t) = I_A Y_t = \mathcal{E}(I_A Q|\mathcal{F}_t)$.

□

4.1 Geometry of F-Evaluations

The comparison theorem establishes various geometric properties of the BSDE solutions, or equivalently of the F-evaluations. Some of these properties are explored in this section.

Theorem 7. *Suppose $\mathcal{Q}_{s,t}$ is a convex set, with F a balanced driver on $\mathcal{Q}_{s,t}$. Suppose F is concave on $\mathcal{Q}_{s,t}$, that is, for any $\lambda \in [0, 1]$, any $(Y^1, Z^1), (Y^2, Z^2)$ corresponding to $Q^1, Q^2 \in \mathcal{Q}_{s,t}$, $dt \times \mathbb{P}$-a.s. on $[s, t]$,*

$$\begin{aligned} &F(\omega, t, \lambda Z^1_{t-} + (1-\lambda)Z^2_{t-}, \lambda Y^1_t + (1-\lambda)Y^2_t) \\ &\qquad \geq \lambda F(\omega, t, Z^1_{t-}, Y_t) + (1-\lambda)F(\omega, t, Z^2_{t-}, Y^2_t). \end{aligned}$$

the inequality being taken componentwise.

Then for any $\lambda \in [0, 1]$ and any $Q^1, Q^2 \in \mathcal{Q}_{s,t}$, the F-evaluation is strictly concave, that is, it satisfies:

$$\mathcal{E}_{s,t}(\lambda Q^1 + (1-\lambda)Q^2) \geq \lambda \mathcal{E}_{t,T}(Q^1) + (1-\lambda)\mathcal{E}_{s,t}(Q^2),$$

with equality if and only if $Q^1 = Q^2$ $\mathbb{P}$-a.s.

Proof. Taking a convex combination of the BSDEs with terminal conditions Q^1 and Q^2 gives the equation

$$\begin{aligned} \lambda Z^1_s + (1-\lambda)Z^2_s - &\int_{]s,t]} [\lambda F(\omega, u, Z^1_{u-}, Y_u) + (1-\lambda)F(\omega, u, Z^2_{u-}, Y^2_u)]du \\ + \int_{]s,t]} &[\lambda Y^1_u + (1-\lambda)Y^2_u]dM_u = \lambda Q^1 + (1-\lambda)Q^2, \end{aligned}$$

which is a BSDE with terminal condition $\lambda Q^1 + (1-\lambda)Q^2$ and driver

$$\tilde{F} = \lambda F(\omega, u, Z^1_{u-}, Y_u) + (1-\lambda)F(\omega, u, Z^2_{u-}, Y^2_u).$$

Consider the BSDE with terminal condition $\lambda Q^1 + (1-\lambda)Q^2$ and driver F. Denote the solution to this by Z^λ. We can compare these BSDEs using Theorem 1. The assumptions are all satisfied as F is balanced on $\mathcal{Q}_{s,t}$. Hence, the solutions satisfy

$$Y^\lambda \geq \lambda Y^1 + (1-\lambda)Y^2.$$

By the strict comparison, which holds as F is balanced, we have equality if and only if the terminal conditions are equal with conditional probability one. The result follows. □

Theorem 8. *Consider a driver F balanced on a family $\{\mathcal{Q}_{s,t}\}$ (which may be empty). Suppose, for some $s < t$, for all deterministic λ in some set $\mathcal{C}$ where all products are assumed to be well defined,*

$$F(\omega, u, \lambda Y_{u-}, \lambda Z_u) = \lambda F(\omega, u, Y_{u-}, Z_u)\ du \times \mathbb{P}\text{-a.s.}$$

on $[s,t] \times \Omega$. Then the nonlinear evaluation generated by F satisfies

$$\mathcal{E}_{s,t}(\lambda Q) = \lambda \mathcal{E}_{s,t}(Q)$$

for all $Q \in L^2_s(\mathcal{F}_t)$ and all $\lambda \in \mathcal{C}$.

Proof. Simply take the BSDE with terminal condition Q, premultiply by λ and factor the driver F term. It is clear that this is then the BSDE with terminal condition λQ, and that the solution is $(\lambda Y, \lambda Z, \lambda L)$. □

Remark 9. Note that this theorem applies for both scalar and square matrix valued λ, and it is clear that, with appropriate modifications to allow for dimensionality of F, would apply for vector valued λ as well.

Definition 5. *For any time t, we define $H_t(Q)$, the* ***essential convex hull*** *of Q at time t, to be the smallest, $\mathcal{F}_t$ measurable, convex set such that $P(Q \in H_t(Q)|\mathcal{F}_t) = 1$.*

Definition 6. *We denote by $r.i.H_t(Q)$ the* ***relative interior*** *of $H_t(Q)$, that is, the interior of $H_t(Q)$ viewed as a subset of the affine hull it generates.*

Remark 10. The interested reader is referred to any good book on elementary stochastic finance for a more detailed definition, for example, [13, p.27] or [12, p.65].

Theorem 9. *Consider a nonlinear F-expectation $\mathcal{E}(\cdot|\mathcal{F}_t)$ for $\{\mathcal{Q}_t\}$, with*

$$L^0(\mathcal{F}_t) \subseteq \mathcal{Q}_t$$

for all t. Then for all $Q \in \mathcal{Q}_t$, each component $e_i^*\mathcal{E}(Q|\mathcal{F}_t)$ satisfies

$$e_i^*\mathcal{E}(Q|\mathcal{F}_t) \in r.i.H_t(e_i^*Q).$$

Proof. For a fixed t, define a random variable Q^{min} by

$$e_i^* Q^{min} = \inf H_t(e_i^* Q).$$

Note that $Q > Q^{min}$ componentwise. As Q^{min} is $\mathcal{F}_t$ measurable, the solution to the BSDE with driver F and terminal condition Q^{min} is simply $Y_s = Q^{min}$ for $s \geq t$. As $Q^{min} \in L^0(\mathcal{F}_t) \subseteq \mathcal{Q}_t$, Property 1 of Definition 2 implies that either $H_t(Q)$ contains only a single point, in which case

$$Q = Q^{min} \in H_t(Q) = \text{r.i.} H_t(Q),$$

or

$$\mathcal{E}(Q|\mathcal{F}_t) > Q^{min}$$

componentwise. We can then repeat this argument with Q^{max} defined by $e_i^* Q^{max} = \sup H_t(e_i^* Q)$, which shows that either $H_t(Q)$ contains a single point, or $\mathcal{E}(Q|\mathcal{F}_t) < Q^{max}$ componentwise. Hence, in the latter case, $e_i^* \mathcal{E}(Q|\mathcal{F}_t)$ lies strictly within the interior of $H_t(e_i^* Q)$, which is the same as r.i.$H_t(e_i^* Q)$. In either case this shows

$$e_i^* \mathcal{E}(Q|\mathcal{F}_t) \in \text{r.i.} H_t(e_i^* Q).$$

□

5 Conclusion

In this paper we have presented a comparison theorem for Backward Stochastic Differential Equations in which the stochastic term is given by an arbitrary martingale. This result is a generalisation of the result of [16], as it allows for martingales other than Brownian motion, and also applies to the case of vector-valued equations. We have shown how, under some conditions, for example Lipschitz continuity, the conditions of this theorem can be simplified.

We have defined the concept of a balanced driver for a BSDE, which is essentially a condition on the driver such that a comparison theorem holds. By expressing this condition in terms of equivalent (super-)martingale measures, the links with previous work on arbitrage theory are more apparent.

Using these results, we have developed a theory of nonlinear expectations, which can now lie in a general probability space. These are closely related to dynamic risk measures, as emphasised in [18]. Various applications of this theory are possible, as we have not assumed that the martingale M used to define the BSDEs will generate the filtration of the probability space. We have also outlined some general geometric properties of these nonlinear expectations.

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