Backward Stochastic Difference Equations and nearly-time-consistent nonlinear expectations

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Abstract

We consider Backward Stochastic Difference Equations in discrete time with infinitely many states. This paper shows the existence and uniquenss of solutions to these equations in complete generality, and also derives a comparison theorem. Using these, time-consistent nonlinear evaluations and expectations are considered, and it is shown that every such evaluation or expectation corresponds to the solution of a BSDE, without any requirements for continuity or boundedness. The implications of these results in a continuous time context are then considered, and possible applications are discussed.

1 Introduction

The theory of Backward Stochastic Differential Equations (BSDEs) is an active area of research in both Mathematical Finance and Stochastic Control. In Mathematical Finance, BSDEs arise as the prices of assets under various assumptions, (see [9]), as the basis for dynamic risk measures, (see [17], [2]), and as a form of expected utility, (see [8], [3]). Through the work of [7] and [12], it has also become apparent that, under some technical assumptions, BSDEs provide the appropriate mathematical framework to describe all 'nonlinear expectations' (see [16]) in continuous time, where the filtration is generated by a finite-dimensional Wiener process. These nonlinear expectations encompass most of the above applications in a single mathematical construction. This paper shows that, in discrete time, the analogous 'Backward Stochastic Difference Equations' provide this framework, without the requirements needed in continuous time.

Beyond the world of finance, BSDEs, and the related Forward-Backward Stochastic Differential Equations, have been used extensively in control theory

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as a version of the HJB equation, (see [15], [18], [14] and [10]), and have also been used in the study of stochastic differential games (for example, [11]). The results of this paper are directly applicable to the discrete time versions of these problems.

Typically, one begins by defining processes (Y, Z) through an equation of the form

$$Y_t(\omega) - \int_{]t,T]} F(\omega, u, Y_{u-}(\omega), Z_u(\omega))du + \int_{]t,T]} Z_u(\omega)dM_u(\omega) = Q(\omega). \quad (1)$$

Here Q is a square-integrable terminal condition, F a progressively measurable 'driver' function, and M an N-dimensional Brownian Motion, all defined on an appropriate filtered probability space. The 'solutions' (Y, Z) are required to be adapted to the forward filtration, and Z is required to be predictable.

When the process M is not a Brownian motion generating the filtration, a BSDE based on M may not always have a solution. Essentially, this is because solutions to these BSDEs depend on the Martingale representation property of Brownian motion. In a previous paper, [5], we considered the analogous situation in discrete time, where the filtration was generated by a finite state process. In this context a martingale representation theorem does hold, and hence, in a broad degree of generality, one can obtain solutions to these equations.

In this paper, we shall consider the generic discrete-time BSDE (Backward Stochastic Difference Equation), without assuming the filtration is generated by a finite state process. We shall discuss under what conditions unique solutions to these equations exist, and the requirements for a comparison theorem to hold. Other authors have also considered BSDEs in discrete time, in particular as approximations of continuous time BSDEs, (see, for example, [13] and [4]).

Using these equations, we develop a theory of nonlinear expectations. This theory differs from previous developments in that it assumes time-consistency only on a discrete set of points. Applications and connections with optimal control with discrete observations are then discussed.

2 Existence and Uniqueness

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a discrete-time filtered probability space, with $\mathcal{F} = \mathcal{F}_T$ for some $T < \infty$. Let $L^1(\mathbb{R}^K; \mathcal{F}_t)$ denote the set of $\mathbb{R}^K$ valued, $\mathcal{F}_t$-measurable, integrable random variables.

In this paper, the fundamental equation considered is:

$$Y_t(\omega) - \sum_{t \leq u < T} F(\omega, u, Y_u(\omega), Z_{u+1}) + \sum_{t \leq u < T} Z_{u+1}(\omega) = Q(\omega). \quad (2)$$

Here $Q \in L^1(\mathbb{R}^K; \mathcal{F}_T)$ and for each $t \in \{0, 1, 2, ..., T-1\}$,

$$F(\cdot, t, \cdot, \cdot) : \Omega \times \mathbb{R}^K \times L^1(\mathbb{R}^K; \mathcal{F}_{t+1}) \to \overline{\mathbb{R}}^K$$

is an extended-real valued functional. Note particularly that F takes an argument $Z_{t+1} \in L^1(\mathbb{R}^K; \mathcal{F}_{t+1})$, that is, it considers the random variable Z_{t+1}, not its value $Z_{t+1}(\omega) \in \mathbb{R}^K$. A solution (Y, Z) of (2) is a pair of adapted $\mathbb{R}^K$-valued processes, where Y is integrable and Z is a martingale difference process. We shall assume that F is adapted, that is $F(\omega, t, Y_t(\omega), Z_{t+1})$ is an $\mathcal{F}_t$-measurable random variable taking values in $]-\infty, \infty]^K$, for all $Y_t \in L^1(\mathbb{R}^K; \mathcal{F}_t)$ and $Z_{t+1} \in L^1(\mathbb{R}^K; \mathcal{F}_{t+1})$. For simplicity, we shall henceforth omit the ω arguments of Y, Z and Q, and simply remember that F takes as arguments the *realised value* of Y_t, but the *random variable* Z_{t+1}.

Unlike in previous work, we here require Z to be a martingale difference process. When $\mathcal{F}_t$ is generated by a finite-state process, we obtain a representation

$$Z_{t+1} = z_t M_{t+1}$$

where z_t is an $\mathcal{F}_t$-measurable $\mathbb{R}^{K\times N}$ random matrix, and M is a $\mathbb{R}^N$-valued martingale difference process generated by the underlying finite-state process. In this case, we can express F, not as a functional of $Z_{t+1} \in L^1(\mathbb{R}^K, \mathcal{F}_{t+1})$, but rather as a function of $z_t(\omega) \in \mathbb{R}^{K\times N}$. In this case, the connection between (2) and (1) is clear. Not making this assumption, but allowing F to be a functional of Z_{t+1}, allows existence results to be obtained without reference to a martingale representation theorem, which is clearly not available in the discrete-time infinite-state context.

Also note that, unlike in previous work, we do not assume that F is finite. An assumption to this effect will be needed when we consider the existence and uniqueness of solutions to (2); however, the case when $F = \pm\infty$ for some values of Y will take on significance in reference to bounded expectations. This motivates the following definition.

Definition 2.1. *Let $D_F(\omega, t, Z_{t+1})$ denote the set*

$$\{y \in \mathbb{R}^K : F(\omega, t, y, Z_{t+1}) \in \mathbb{R}^K\},$$

that is, for each ω, t, Z_{t+1}, the collection of values of y such that F is finite. The function F will be called $\mathbb{R}$-integrable *(in ω) if the product*

$$F(\cdot, t, y, Z_{t+1}) I_{D_F(\cdot,t,Z_{t+1})}(y)$$

*is integrable for all $t \in \{0, 1, ..., T-1\}$, $y \in \mathbb{R}^K$ and $Z_{t+1} \in L^1(\mathbb{R}^K; \mathcal{F}_{t+1})$. For simplicity, we shall call $D_F(\omega, t, Z_{t+1})$ the y-*domain *of F.*

This definition essentially implies that F is integrable, except when it takes the values $\pm\infty$, (in at least one component).

Theorem 2.1. *Let F be $\mathbb{R}$-integrable. Then (2) has a unique solution (Y, Z) for all $Q \in L^1(\mathbb{R}^K; \mathcal{F}_T)$ if the map*

$$\Phi(y) : D_F(\omega, t, Z_{t+1}) \to \mathbb{R}^K, y \mapsto y - F(\omega, t, y, Z_{t+1})$$

has a unique inverse, $\mathbb{P}$-a.s., for all martingale difference processes Z.

Proof. We prove the existence of a solution by induction. Clearly $Y_T = Q$ solves equation (2) at time T. For time t, suppose a solution Y_{t+1} exists for time $t+1$. Supposing that there exists a solution at time t, we can then rewrite (2) in the differenced, or one-step, form

$$Y_t - F(\omega, t, Y_t, Z_{t+1}) + Z_{t+1} = Y_{t+1} \tag{3}$$

and taking a conditional expectation gives

$$Y_t - F(\omega, t, Y_t, Z_{t+1}) = E[Y_{t+1}|\mathcal{F}_t]. \tag{4}$$

Taking the difference of these two equations yields

$$Z_{t+1} = Y_{t+1} - E[Y_{t+1}|\mathcal{F}_t] \tag{5}$$

which is the desired martingale difference process.

If there is to be a solution (Y, Z), which is required to be integrable, then a simple rearrangement of (4) shows that F is finite. Hence, the value of Y_t must lie in $D_F(\omega, t, Z_{t+1})$, where Z_{t+1} is as in (5). We know that, restricted to $D_F(\omega, t, Z_{t+1})$, the equation

$$\Phi(Y_t) = Y_t - F(\omega, t, Y_t, Z_{t+1}) = E[Y_{t+1}|\mathcal{F}_t]$$

has a unique solution, that is, there is a unique value of Y_t such that (4) is satisfied.

It is then straightforward to show that this pair (Y_t, Z_t) is a solution to (3), and hence, by backward induction, the processes (Y, Z) satisfy (2). □

Corollary 2.1.1. *Suppose F is finite-valued, that is, $D_F(\omega, t, Z_{t+1}) = \mathbb{R}^K$ for all ω, t and Z_{t+1}. Then the conditions of Theorem 2.1 are necessary for unique solutions to exist at all times.*

Proof. Suppose, for some Z_{t+1}, some $y \neq \bar{y} \in \mathbb{R}^K$, we had

$$y - F(\omega, t, y, Z_{t+1}) = \Phi(y) = \Phi(\bar{y}) = \bar{y} - F(\omega, t, \bar{y}, Z_{t+1})$$

with positive probability, that is, Φ is not injective. Then, let $Y_{t+1} = \Phi(y) + Z_{t+1}$. It is clear that the difference equation (4) will have two solutions $Y_t = y$ and $Y_t = \bar{y}$. Define the terminal condition $Q = Y_T$ through the equations

$$\begin{aligned} Y_{t+1} &= \Phi(y) + Z_{t+1}, \\ Y_s &= Y_{t+1} + \sum_{t \leq u < s} F(\omega, u, Y_u, 0). \end{aligned} \tag{6}$$

The equation with terminal condition $Q = Y_T$ will have solution Y_{t+1} at time $t + 1$, and hence multiple solutions at time t.

Now suppose, for some Z_{t+1}, some $k \in \mathbb{R}^K$, we had no solutions to $\Phi(y) = k$, that is Φ is not surjective. Let $Y_{t+1} = k + Z_{t+1}$, and again define the terminal condition $Q = Y_T$ through (6). Then the BSDE with terminal condition Q will either have multiple solutions at time $t+1$, (in which case we fail to have unique solutions), or will have the unique solution Y_{t+1} at time $t + 1$. In this case, the difference equation (4) will have no solutions, and hence no solution will exist at time t.

Therefore, it is clear that Φ must be both injective and surjective, and thus, has a unique inverse. □

Remark 2.1. The requirement, in Corollary 2.1.1, that F be finite-valued is because, if $Y_T = Q$ is a $\mathbb{R}^K$-valued random variable, then Y_{T-1} will be a $D(\omega, T-1, Z_T)$-valued random variable. Hence, it would be possible to define a set

$$\tilde{D}_F(\omega, T-2, Z_{T-1}) = \{y \in \mathbb{R}^K : y - F(\omega, T-2, y, Z_{T-1}) \in D(\omega, T-1, Z_T)\},$$

and it would then follow that $Y_{T-2} \in \tilde{D}_F(\omega, T-2, Z_{T-1})$.

For each (fixed) process Z, this would then lead, in a natural way, to a recursive stochastic sequence of sets

$$\tilde{D}_F(\omega, t, Z_{t+1}) = \{y \in \mathbb{R}^K : y - F(\omega, t, y, Z_{t+1}) \in \tilde{D}(\omega, t+1, Z_{t+2})\},$$

where, for any terminal condition, the solution (Y, Z) satisfies $Y_t \in \tilde{D}_F(\omega, t, Z_{t+1})$. For unique solutions to exist, it is then necessary and sufficient that the restriction of $y \mapsto y - F(\omega, t, y, Z_{t+1})$ to $\tilde{D}_F(\omega, t, Z_{t+1})$ has a unique inverse, where Z is recursively defined by (5).

The added complexity which such a condition generates is significant, and is of little benefit in practice. For this reason, we can view Theorem 2.1 as containing sufficient and *nearly* necessary conditions for the existence of a solution to (2).

3 Comparison results

A fundamental result for working with BSDEs is the comparison theorem, first obtained by [15]. We here present a version of the comparison theorem in this context.

Definition 3.1. *For any $\mathbb{R}$-valued random variable X, we define*

$$\operatorname{ess\,inf}_{\mathcal{F}_t}\{X\} = \inf\{x : x \text{ is } \mathcal{F}_t\text{-measurable}, \mathbb{P}(X \leq x) > 0\},$$

the $\mathcal{F}_t$-conditional essential infimum. Note that $\operatorname{ess\,inf}_{\mathcal{F}_t} X$ can take the value $-\infty$, even for X real-valued (if X is conditionally unbounded from below). We can also define $\operatorname{ess\,sup}_{\mathcal{F}_t}$ in a similar way.

Theorem 3.1 (Comparison Theorem)**.** *Consider two discrete time BSDEs as in (2), corresponding to coefficients F^1, F^2 and terminal values Q^1, Q^2. Suppose the conditions of Theorem 2.1 are satisfied for both equations, let (Y^1, Z^1) and (Y^2, Z^2) be the associated solutions. Suppose the following conditions hold, all vector inequalities being taken componentwise.*

(i) $Q^1 \geq Q^2$ *$\mathbb{P}$-a.s.*

(ii) *$\mathbb{P}$-a.s., for all times t,*

$$F^1(\omega, t, Y_t^2, Z_{t+1}^2) \geq F^2(\omega, t, Y_t^2, Z_{t+1}^2).$$

(iii) *for all t, for all i, the ith component of F^1, given by $e_i^* F^1$, satisfies*

$$e_i^* F^1(\omega, t, Y_t^2, Z_{t+1}^1) - e_i^* F^1(\omega, t, Y_t^2, Z_{t+1}^2) \geq \operatorname{ess\,inf}_{\mathcal{F}_t}\{e_i^*(Z_{t+1}^1 - Z_{t+1}^2)\}.$$

(iv) *$\mathbb{P}$-a.s., for all t, if*

$$Y_t^1 - F^1(\omega, t, Y_t^1, Z_{t+1}^1) \geq Y_t^2 - F^1(\omega, t, Y_t^2, Z_{t+1}^1)$$

then $Y_t^1 \geq Y_t^2$.

(v) *$F^1(\omega, t, Y_t^2, Z_{t+1}^2)$ is finite.*

It is then true that $Y^1 \geq Y^2$ $\mathbb{P}$*-a.s.*

Proof. We shall establish this theorem using backward induction. For $t = T$ it is clear that $Y^1_t - Y^2_t = Q^1 - Q^2 \geq 0$ $\mathbb{P}$-a.s. as desired.

Taking the one step equation, as in (3) we have

$$Y^i_t - F^i(\omega, t, Y^i_t, Z^i_{t+1}) + Z^i_{t+1} = Y^i_{t+1}$$

for all $0 \leq t < T$. Note that we know $F^1(\omega, t, Y^2_t, Z^2_{t+1})$, $F^1(\omega, t, Y^1_t, Z^1_{t+1})$ and $F^2(\omega, t, Y^2_t, Z^2_{t+1})$ are all finite, the latter two as they are part of the (finite) solutions of the original BSDEs.

For a given t, suppose we know $Y^1_{t+1} - Y^2_{t+1} \geq 0$ $\mathbb{P}$-a.s. Then, omitting the ω and t arguments of F^1 and F^2,

$$Y^1_t - Y^2_t - F^1(Y^1_t, Z^1_{t+1}) + F^2(Y^2_t, Z^2_{t+1}) + (Z^1_{t+1} - Z^2_{t+1}) = Y^1_{t+1} - Y^2_{t+1} \geq 0. \quad (7)$$

It is then clear that

$$e^*_i(Y^1_t - Y^2_t) \geq e^*_i(F^1(Y^1_t, Z^1_{t+1}) - F^2(Y^2_t, Z^2_{t+1})) - \operatorname{ess\,inf}_{\mathcal{F}_t}\{e^*_i(Z^1_{t+1} - Z^2_{t+1}))\}.$$

Hence, again $\mathbb{P}$-a.s., assumptions (ii) and (iii) imply

$$\begin{aligned}
&e^*_i(Y^1_t - Y^2_t - F^1(Y^1_t, Z^1_{t+1}) + F^1(Y^2_t, Z^1_{t+1})) \\
&\geq e^*_i(F^1(Y^2_t, Z^2_{t+1}) - F^2(Y^2_t, Z^2_{t+1})) \\
&\qquad + e^*_i F^1(Y^2_{t+1}, Z^1_{t+1}) - e^*_i F^1(Y^2_t, Z^2_{t+1}) - \operatorname{ess\,inf}_{\mathcal{F}_t}\{e^*_i(Z^1_{t+1} - Z^2_{t+1}))\} \\
&\geq 0.
\end{aligned} \quad (8)$$

That is, the inequality being taken componentwise,

$$Y^1_t - Y^2_t - F^1(Y^1_t, Z^1_{t+1}) + F^1(Y^2_t, Z^1_{t+1}) \geq 0,$$

and hence, by Assumption (iv),

$$Y^1_t \geq Y^2_t$$

$\mathbb{P}$-a.s. as desired. The general statement follows by backward induction. □

Corollary 3.1.1. *Suppose Theorem 3.1 holds with the following, stronger assumptions*

(iii') The inequality in assumption (iii) is strict componentwise, that is, for all i*,*

$$e^*_i F^1(Y^2_t, Z^1_{t+1}) - e^*_i F^1(Y^2_t, Z^2_{t+1}) \geq \operatorname{ess\,inf}_{\mathcal{F}_t}\{e^*_i(Z^1_{t+1} - Z^2_{t+1})\},$$

with equality only if $e^*_i Z^1_{t+1} = e^*_i Z^2_{t+1}$ *almost surely.*

(iv') For almost all (ω, t, Z_{t+1})*, the* i*th component of* $F^1(\omega, t, y, Z_{t+1})$ *depends only on the* i*th component of* y*, and, for* $y \in D(\omega, t, Z_{t+1})$*, the map*

$$e^*_i y_i \mapsto e^*_i \Phi(y) = e^*_i y - e^*_i F(\omega, t, y, Z_{t+1})$$

is strictly increasing. (This implies assumption (iv) of Theorem 3.1 and the invertibility required for the existence result of Theorem 2.1.)

Then this comparison is strict componentwise, that is, if on some $A \in \mathcal{F}_t$, for some i, we have $e_i^* Y_t^1 = e_i^* Y_t^2$ $\mathbb{P}$-a.s. on A, then $e_i^* Q^1 = e_i^* Q^2$ $\mathbb{P}$-a.s. on A, and for all $s \in \{t, ..., T-1\}$, $\mathbb{P}$-a.s. on A, $e_i^* F^1(\omega, s, Y_s^2, Z_{s+1}^2) = e_i^* F^2(\omega, s, Y_s^2, Z_{s+1}^2)$, $e_i^* Z_{s+1}^1 = e_i^* Z_{s+1}^2$ and $e_i^* Y_s^1 = e_i^* Y_s^2$.

Proof. Throughout this proof we shall omit the ω and t arguments of F^1 and F^2, and all (in-)equalities are assumed to hold $\mathbb{P}$-a.s. on A.

In this case, for a given t, by the same argument as used to show (8), we can establish the strict inequality, for each i,

$$\begin{aligned}
&e_i^*(Y_t^1 - Y_t^2 - F^1(Y_t^1, Z_{t+1}^1) + F^1(Y_t^2, Z_{t+1}^1)) \\
&= e_i^*(Y_{t+1}^1 - Y_{t+1}^2) + e_i^*(F^1(Y_t^2, Z_{t+1}^2) - F^2(Y_t^2, Z_{t+1}^2)) \\
&\quad + e_i^* F^1(Y_t^2, Z_{t+1}^1) - e_i^* F^1(Y_t^2, Z_{t+1}^2) - e_i^*(Z_{t+1}^1 - Z_{t+1}^2) \\
&\geq e_i^*(F^1(Y_t^2, Z_{t+1}^2) - F^2(Y_t^2, Z_{t+1}^2)) \\
&\quad + e_i^* F^1(Y_t^2, Z_{t+1}^1) - e_i^* F^1(Y_t^2, Z_{t+1}^2) - \operatorname{ess\,inf}_{\mathcal{F}_t}\{e_i^*(Z_{t+1}^1 - Z_{t+1}^2))\} \\
&> 0,
\end{aligned} \tag{9}$$

unless $e_i^* Z_{t+1}^1 = e_i^* Z_{t+1}^2$. Hence, if $e_i^* Y_t^1 = e_i^* Y_t^2$, the first term of this inequality is zero, which is only the case if $e_i^* Z_{t+1}^1 = e_i^* Z_{t+1}^2$.

If $e_i^* Z_{t+1}^1 = e_i^* Z_{t+1}^2$, we know that

$$e_i^*(F^1(Y_t^2, Z_t^1) - F^1(Y_t^2, Z_t^2)) - e_i^*(Z_{t+1}^1 - Z_{t+1}^2) = 0,$$

and so, from (7), and assumption (i) of Theorem 3.1,

$$\begin{aligned}
0 &\leq e_i^*(Y_{t+1}^1 - Y_{t+1}^2) \\
&= e_i^*(Y_t^1 - Y_t^2 - F^1(Y_t^1, Z_{t+1}^1) + F^2(Y_t^2, Z_{t+1}^2) + (Z_{t+1}^1 - Z_{t+1}^2)) \\
&= -e_i^*(F^1(Y_t^2, Z_{t+1}^2) - F^2(Y_t^2, Z_{t+1}^2)) \\
&\leq 0
\end{aligned}$$

and hence $e_i^* F^1(Y_t^2, Z_{t+1}^2) = e_i^* F^2(Y_t^2, Z_{t+1}^2)$. Substituting this into (7), it follows that $e_i^* Y_{t+1}^1 = e_i^* Y_{t+1}^2$. The result follows by forward induction. □

Remark 3.1. It is easy to show that each of conditions (i)-(iv) of Theorem 3.1 is necessary when the other conditions hold with equality. However, in general, these conditions are sufficient but not necessary. This is simply because each condition, if it holds with a strict inequality, is capable of 'compensating' for another condition which does not hold.

Remark 3.2. In the scalar case, the final assumption is simply that the map $\Phi(y) = y - F(\omega, t, y, Z_{t+1})$ is strictly increasing on its y-domain. This is clearly true if F is small relative to y, as will occur when approximating a continuous time BSDE on a fine mesh in the time dimension. Furthermore, this assumption is often satisfied in applications, as we have already assumed in Theorem 2.1 that Φ is invertible. For example, in Mathematical Finance, for classical pricing theory one obtains a driver of the form

$$F(\omega, t, y, Z_{t+1}) = -r_t Y_t + E_\pi[Z_{t+1}|\mathcal{F}_t],$$

where r_t is the interest rate at time t, and π is the risk-neutral measure for the market. In this case, our assumption is simply that the interest rate r_t is higher than -100%.

Definition 3.2. *A function F^1 which satisfies the assumptions of Corollary 3.1.1, for all pairs (Y, Z) with $Y_t \in D(\omega, t, Z_{t+1})$ for all t, will be called* balanced.

Balanced drivers are a useful concept in the analysis of solutions of BSDEs, as they are the drivers for which a (componentwise) comparison theorem holds. As we shall see in Theorem 4.3, the requirement that a driver is balanced arises naturally from the assumption of monotonicity of the solutions of a BSDE.

These conditions are essentially the same as those used in [5], for the discrete-time finite-state case, and [6], for the general continuous time case. In particular, we have the following result, which directly links the conditions of Corollary 3.1.1 with those used in [6] in continuous time.

Theorem 3.2. *The following statements are equivalent (omitting ω and t for simplicity):*

(a) For all t, for all i, the ith component of F^1, given by $e_i^* F^1$, satisfies

$$e_i^*(F^1(Y_t^2, Z_{t+1}^1) - F^1(Y_t^2, Z_{t+1}^2)) > \operatorname{ess\,inf}_{\mathcal{F}_t}\{e_i^*(Z_{t+1}^1 - Z_{t+1}^2)\},$$

unless $e_i^* Z_{t+1}^1 = e_i^* Z_{t+1}^2$ almost surely.

(b) For each i, there exists a measure $\tilde{\mathbb{P}}_i$, equivalent to $\mathbb{P}$, such that the process defined by

$$\begin{aligned} e_i^* X_t := &- \sum_{0 \le u < t} e_i^*(F^1(Y_t^2, Z_t^1) - F^1(Y_t^2, Z_t^2)) \\ &+ \sum_{0 \le u < t} e_i^*(Z_{t+1}^1 - Z_{t+1}^2) \end{aligned}$$

is a $\tilde{\mathbb{P}}_i$ supermartingale.

Proof. To show (b) implies (a), we note that if (a) is false, then

$$0 > -e_i^*(F^1(Y_t^2, Z_t^1) - F^1(Y_t^2, Z_t^2)) + e_i^*(Z_{t+1}^1 - Z_{t+1}^2)$$

with probability one. Hence the process $e_i^* X_t$ described in (b) is almost surely nondecreasing, and hence cannot be a supermartingale under any equivalent measure. Hence, (b) is false, and the implication follows by contradiction.

To show (a) implies (b), we note that either the process $e_i^* X$ is non-increasing with probability one, or there is a positive probability of it increasing. For each t, let A_t be set on which $e_i^* X_{t+1} \le e_i^* X_t$ almost surely given $\mathcal{F}_t$. Define the random variable $\lambda_{t+1} = 1$ on A_t.

On A_t^c, there is a positive conditional probability of $e_i^* X$ increasing from t to $t+1$, but from (a), we have the inequality

$$\begin{aligned} 0 &> -e_i^*(F^1(Y_t^2, Z_t^1) - F^1(Y_t^2, Z_t^2)) + \operatorname{ess\,inf}_{\mathcal{F}_t}\{e_i^*(Z_{t+1}^1 - Z_{t+1}^2)\} \\ &= \operatorname{ess\,inf}_{\mathcal{F}_t}\{e_i^*(X_{t+1} - X_t)\}, \end{aligned}$$

as if $e_i^* Z_{t+1}^1 = e_i^* Z_{t+1}^2$ a.s. then $e_i^*(X_{t+1} - X_t) = 0$ almost surely. Let B_{t+1} be the event $e_i^*(X_{t+1} - X_t) < 0$. It follows that, on A_t^c, $E[-I_{B_{t+1}} e_i^*(X_{t+1} - X_t)|\mathcal{F}_t] > 0$.

Now, on A_t^c, define

$$\lambda_{t+1} = k^{-1} E[-I_{B_{t+1}} e_i^*(X_{t+1} - X_t)|\mathcal{F}_t] I_{B_{t+1}^c} + k^{-1} E[I_{B_{t+1}^c} e_i^*(X_{t+1} - X_t)|\mathcal{F}_t] I_{B_{t+1}}$$

where all expectations are taken under $\mathbb{P}$ and

$$k = E[-I_{B_{t+1}} e_i^*(X_{t+1}-X_t)|\mathcal{F}_t]E[I_{B_{t+1}^c}|\mathcal{F}_t]+E[I_{B_{t+1}^c} e_i^*(X_{t+1}-X_t)|\mathcal{F}_t]E[I_{B_{t+1}}|\mathcal{F}_t]$$

is a normalising constant. Note that, by construction, each of these terms is nonnegative, and λ_{t+1} is strictly positive and integrable.

Combining these definitions of λ_{t+1}, we have $E[\lambda_{t+1}|\mathcal{F}_t] = 1$ and $E[\lambda_{t+1} e_i^*(X_{t+1} - X_t)|\mathcal{F}_t] \leq 0$. Hence if we define

$$\frac{d\tilde{\mathbb{P}}_i}{d\mathbb{P}} = \prod_{0\leq t<T} \lambda_{t+1},$$

we have constructed an equivalent measure under which $e_i^* X$ is a supermartingale. □

4 Nonlinear evaluations and expectations

A significant application of BSDEs is in the development of value functions for stochastic optimal control problems. A useful way of expressing these functions has been developed by [16], [7], [12] and others, in the language of *nonlinear evaluations* and *nonlinear expectations*.

Definition 4.1. *A system of operators*

$$\mathcal{E}_{s,t} : L^1(\mathbb{R}^K; \mathcal{F}_t) \to L^1(\mathbb{R}^K; \mathcal{F}_s), 0 \leq s \leq t \leq T$$

is called an $\mathcal{F}_t$-consistent nonlinear evaluation *defined on $[0,T]$ if it satisfies the following properties.*

1. *For $Q, Q' \in L^1(\mathbb{R}^K; \mathcal{F}_t)$, if $Q \geq Q'$ $\mathbb{P}$-a.s. componentwise, then*

   $$\mathcal{E}_{s,t}(Q) \geq \mathcal{E}_{s,t}(Q')$$

   $\mathbb{P}$-a.s. componentwise, with, for each i,

   $$e_i^* \mathcal{E}_{s,t}(Q) = e_i^* \mathcal{E}_{s,t}(Q')$$

   only if $e_i^* Q = e_i^* Q'$ $\mathbb{P}$-a.s.

2. *$\mathcal{E}_{t,t}(Q) = Q$ $\mathbb{P}$-a.s.*

3. *$\mathcal{E}_{r,s}(\mathcal{E}_{s,t}(Q)) = \mathcal{E}_{r,t}(Q)$ $\mathbb{P}$-a.s. for any $0 \leq r \leq s \leq t \leq T$.*

4. *For any $A \in \mathcal{F}_s$, $I_A \mathcal{E}_{s,t}(Q) = I_A \mathcal{E}_{s,t}(I_A Q)$ $\mathbb{P}$-a.s.*

Theorem 4.1. *In the scalar, $K = 1$, case, if $\mathcal{E}_{s,t}(\cdot)$ is a $\mathcal{F}_t$-consistent nonlinear evaluation for all $0 \leq s \leq t \leq T$, then there exists a balanced driver F such that $Y_s^t = \mathcal{E}_{s,t}(Q)$ satisfies a BSDE*

$$Y_s^t - \sum_{s\leq u<t} F(\omega, u, Y_u^t, Z_{u+1}^t) + \sum_{s\leq u<t} Z_{u+1}^t = Q.$$

Furthermore, F is unique.

Proof. Leaving t fixed, and omitting it for clarity, it is sufficient to construct F such that the one-step equation

$$Y_s - F(\omega, s, Y_s, Z_{s+1}) + Z_{s+1} = Y_{s+1}.$$

is satisfied. Recursivity then allows this to be extended to the full BSDE.

For each $Y_s \in L^1(\mathcal{F}_s)$, $Z_{s+1} \in L^1(\mathcal{F}_{s+1})$ we consider the equation

$$Y_s = \mathcal{E}_{s,s+1}(Y_s + Z_{s+1} + c_s)$$

for $c_s \in L^1(\mathcal{F}_s)$. By the strict monotonicity Property 1 of $\mathcal{E}$, there is at most one solution c_s to this equation. Now let

$$F(\omega, s, Y_s, Z_{s+1}) = \begin{cases} -c_s & \text{if } c_s \text{ exists} \\ \infty & \text{otherwise} \end{cases}.$$

By the monotonicity property 1 of $\mathcal{E}$, it follows that F is monotone in Y_s over its y-domain, which corresponds to those values of Y_s where c_s exists.

For any Y_{s+1}, we can decompose Y_{s+1} as

$$Y_{s+1} = E[Y_{s+1}|\mathcal{F}_s] + Z_{s+1} = Y_s + \bar{c}_s + Z_{s+1}$$

for some martingale difference process Z and some adapted process $\bar{c}$. Hence we know $Y_s = \mathcal{E}_{s,s+1}(Y_{s+1}) = \mathcal{E}_{s,s+1}(Y_s + \bar{c}_s + Z_{s+1})$ which implies the above constructed F satisfies $-\bar{c}_s = F(\omega, s, Y_s, Z_{s+1})$. Finally, it follows that

$$Y_{s+1} = Y_s - F(\omega, s, Y_s, Z_{s+1}) + Z_{s+1}$$

is satisfied. It is clear that F must satisfy these equations, if it exists, and the strict monotonicity of $\mathcal{E}_{s,s+1}$ then guarantees uniqueness of F.

We now need to show that F is balanced. Suppose Assumption (iii') of Corollary 3.1.1 does not hold, that is, for some Z^1_{t+1}, Z^2_{t+1}, some $Y_t \in D(\omega, t, Z^2_{t+1})$, we have

$$F(Y_t, Z^1_{t+1}) - F(Y_t, Z^2_{t+1}) < \operatorname{ess\,inf}_{\mathcal{F}_t}\{Z^1_{t+1} - Z^2_{t+1}\}$$

or

$$F(Y_t, Z^1_{t+1}) - F(Y_t, Z^2_{t+1}) = \operatorname{ess\,inf}_{\mathcal{F}_t}\{Z^1_{t+1} - Z^2_{t+1}\}$$

with $Z^1_{t+1} \neq Z^2_{t+1}$ with positive probability. (Note this implies that $F(Y_t, Z^1_{t+1})$ is finite.) Then it is clear that, defining

$$Y^i_{t+1} = Y_t - F(Y_t, Z^i_{t+1}) + Z^i_{t+1}, \qquad i = 1, 2,$$

we have

$$\operatorname{ess\,inf}_{\mathcal{F}_t}\{Y^1_{t+1} - Y^2_{t+1}\} = -F(Y_t, Z^1_{t+1}) + F(Y_t, Z^2_{t+1}) + \operatorname{ess\,inf}_{\mathcal{F}_t}\{Z^1_{t+1} - Z^2_{t+1}\}$$

and so $Y^1_{t+1} \geq Y^2_{t+1}$ $\mathbb{P}$-a.s., and $Y^1_{t+1} > Y^2_{t+1}$ with positive probability, yet

$$Y_t = \mathcal{E}_{t,t+1}(Y^1_{t+1}) = \mathcal{E}_{t,t+1}(Y^2_{t+1})$$

contradicting the strict monotonicity of $\mathcal{E}$. By contradiction, F must satisfy Assumption (iii') of Corollary 3.1.1.

Now suppose Assumption (iv') of Corollary 3.1.1 is not satisfied. Then, for some Z_{t+1}, the map

$$y \mapsto y - F(y, Z_{t+1})$$

is not strictly increasing on $D(\omega, t, Z_{t+1})$. Let $y^1, y^2 \in D(\omega, t, Z_{t+1})$ be two real values such that $y^1 > y^2$, but $y - F(y, Z_{t+1}) \leq y' - F(y', Z_{t+1})$. Then defining the terminal conditions $Y^i_{t+1} = y^i - F(Y^i, Z_{t+1}) + Z_{t+1}$ for $i = 1, 2$, we see that $Y^1_{t+1} \leq Y^2_{t+1}$, yet

$$\mathcal{E}_{t,t+1}(Y^1_{t+1}) = y^1 > y^2 = \mathcal{E}_{t,t+1}(Y^2_{t+1}),$$

again contradicting the monotonicity of $\mathcal{E}$.

Hence, it is clear that the monotonicity of $\mathcal{E}$ ensures that the driver it generates is balanced. □

Remark 4.1. This theorem is significantly stronger than that given in [5], an analogue of which is given as Theorem 4.3 below, as it refers to *nonlinear evaluations*, not nonlinear expectations (Definition 4.2), and requires no assumptions of translation invariance. In continuous time, under the assumption of translation invariance, analogous theorems have been shown in [7] and [12] for nonlinear expectations.

Remark 4.2. Clearly, if $\mathcal{E}_{s,t}(\cdot)$ is bounded, then there will exist values of Y_t such that there is no Y_{t+1} satisfying the equation $Y_t = \mathcal{E}_{t,t+1}(Y_{t+1})$. In these cases, we see that the equation $Y_t = \mathcal{E}_{t,t=1}(Y_t + Z_{t+1} + c_t)$ has no solutions, and hence the driver F will be infinite for these values of Y_t. This makes explicit the motivation for allowing the driver to take non-real values, as without this generalisation, the above representation would not be possible.

We also have the following converse result, which applies in both the scalar and vector cases.

Theorem 4.2. *Suppose F is a balanced driver, then the functional defined by $\mathcal{E}_{s,t}(Q) = Y^t_s$, for $Q \in L^1(\mathbb{R}^K; \mathcal{F}_t)$, where*

$$Y^t_s - \sum_{s \leq u < t} F(\omega, u, Y^t_u, Z^t_{u+1}) + \sum_{s \leq u < t} Z^t_{u+1} = Q$$

is a nonlinear evaluation.

Proof. We shall show that each of the properties of a nonlinear expectation is satisfied.

1. The statement $\mathcal{E}_{s,t}(Q^1) \geq \mathcal{E}_{s,t}(Q^2)$ $\mathbb{P}$-a.s. whenever $Q^1 \geq Q^2$ $\mathbb{P}$-a.s. is the main result of Theorem 3.1, which holds as F is balanced. The strict comparison of Corollary 3.1.1 then establishes the second statement.

2. For $s = t$, it is clear that the BSDE degenerates to the equation $Y_t = Q$, and so $Y_t = \mathcal{E}_{t,t}(Q) = Q$.

3. For $r \leq s \leq t$, as Y is the solution to the relevant BSDE (with terminal time t and terminal value Q), we can deduce

$$Y_s = Y_r - \sum_{r \leq u < s} F(\omega, u, Y_u, Z_{u+1}) + \sum_{r \leq u < s} Z_{u+1}.$$

   Hence Y_r is also the time r value of a solution to the BSDE with terminal time s and value Y_s. Hence $\mathcal{E}_{r,s}(\mathcal{E}_{s,t}(Q)) = \mathcal{E}_{r,t}(Q)$ $\mathbb{P}$-a.s. as desired.

4. We know that

$$I_A Q = I_A Y_t - \sum_{t \le u < T} I_A F(\omega, u, Y_u, Z_u) + \sum_{t \le u < T} I_A Z_u M_{u+1}$$

and clearly

$$I_A F(\omega, u, Y_u, Z_u) = I_A F(\omega, u, I_A Y_u, I_A Z_u).$$

Hence $(I_A Y, I_A Z)$ is the solution to a BSDE with driver $I_A F$ and terminal condition $I_A Q$, and hence, again premultiplying by I_A,

$$I_A \mathcal{E}(Q|\mathcal{F}_t) = I_A Y_t = I_A \mathcal{E}(I_A Q|\mathcal{F}_t)$$

as desired.

□

In many applications, we require these functionals to obey further rules, in particular, we wish to have $\mathcal{E}_{s,t}$ not depend on the terminal time t, and we wish to have the translation invariance property $\mathcal{E}_{s,t}(Q+q) = \mathcal{E}_{s,t}(Q) + q$ for all $q \in L^1(\mathbb{R}^K; \mathcal{F}_s)$. This motivates the following definition.

Definition 4.2. *A system of operators*

$$\mathcal{E}(\cdot|\mathcal{F}_t) : L^1(\mathbb{R}^K; \mathcal{F}_T) \to L^1(\mathbb{R}^K; \mathcal{F}_t),\ 0 \le t \le T$$

is called an $\mathcal{F}_t$-consistent, translation-invariant, nonlinear expectation *if it satisfies the following properties:*

1. *For $Q, Q' \in L^1(\mathbb{R}^K; \mathcal{F}_T)$, if $Q \ge Q'$ $\mathbb{P}$-a.s. componentwise, then*

$$\mathcal{E}(Q|\mathcal{F}_t) \ge \mathcal{E}(Q'|\mathcal{F}_t)$$

   $\mathbb{P}$-a.s. componentwise, with, for each i,

$$e_i^* \mathcal{E}(Q|\mathcal{F}_t) = e_i^* \mathcal{E}(Q'|\mathcal{F}_t)$$

   only if $e_i^* Q = e_i^* Q'$ $\mathbb{P}$-a.s.

2. *$\mathcal{E}(Q|\mathcal{F}_t) = Q$ $\mathbb{P}$-a.s. for any $\mathcal{F}_t$-measurable Q.*

3. *$\mathcal{E}(\mathcal{E}(Q|\mathcal{F}_t)|\mathcal{F}_s) = \mathcal{E}(Q|\mathcal{F}_s)$ $\mathbb{P}$-a.s. for any $s \le t$*

4. *For any $A \in \mathcal{F}_t$, $I_A \mathcal{E}(Q|\mathcal{F}_t) = \mathcal{E}(I_A Q|\mathcal{F}_t)$ $\mathbb{P}$-a.s.*

5. *For any $\mathcal{F}_t$-measurable q with values in $\mathbb{R}^K$, any $Q \in L^1(\mathbb{R}^K; \mathcal{F}_T)$, then*

$$\mathcal{E}(Q + q|\mathcal{F}_t) = \mathcal{E}(Q|\mathcal{F}_t) + q.$$

Remark 4.3. Property 5 is not usually included in the requirements for a nonlinear expectation. However, it will prove very useful here. There is a slight abuse of notation as we have assumed that Property 5 holds without assuming q is integrable. This can be formally dealt with by defining $\mathcal{E}(\cdot|\mathcal{F}_t)$ on the set $\{Q : E[\|Q\|\,|\mathcal{F}_t] < +\infty\ \mathbb{P}\text{-a.s.}\}$, which is slightly larger than the set $L^1(\mathbb{R}^K; \mathcal{F}_t)$. This involves very slight modifications throughout the existence and uniqueness results for BSDEs, however is on benefit when proving Theorem 4.3. We will take such modifications as implicit.

Remark 4.4. In [5] we have given results, in the finite state case, for a variant of these operators, where monotonicity is only assumed to hold in some subset of $L^1(\mathbb{R}^K;\mathcal{F}_t)$. This discussion carries over directly to the infinite state case, and will not be repeated here.

In this situation, we can give the following version of a theorem, proven in the finite-state case in [5].

Theorem 4.3. *The following statements are equivalent.*

1. $\mathcal{E}(\cdot|\mathcal{F}_t)$ *is an* $\mathcal{F}_t$*-consistent, translation invariant, nonlinear expectation.*
2. *There exists a balanced driver* F*, which is independent of* Y*, and satisfies the normalisation condition* $F(\omega,t,Y_t,0)=0$*, such that, for all* Q*,* $Y_t = \mathcal{E}(Q|\mathcal{F}_t)$ *is the solution to a BSDE with terminal condition* Q *and driver* F*.*

Furthermore, these two statements are related by the equation

$$F(\omega,t,Y_t,Z_{t+1}) = \mathcal{E}(Z_{t+1}|\mathcal{F}_t).$$

Proof. (2. implies 1.) As every nonlinear expectation is also a nonlinear evaluation, with $\mathcal{E}_{s,t}(\cdot) := \mathcal{E}(\cdot|\mathcal{F}_s)$, we need only show properties 2, 4 and 5. For property 2, by the normalisation of F, the solution to the BSDE with $\mathcal{F}_t$-measurable terminal condition Q will be $(Y_u,Z_u)=(Q,0)$ for $u\geq t$. By the uniqueness result of Theorem 2.1 this is then the value of $\mathcal{E}(Q|\mathcal{F}_t)$. For property 4, we note that by normalisation, $I_AF(\omega,u,Y_u,Z_u) = F(\omega,u,I_AY_u,I_AZ_u)$, and the result follows as for nonlinear evaluations. Property 5 follows as $F(\omega,t,Y_t+q,Z_{t+1}) = F(\omega,t,Y_t,Z_{t+1})$, and so, if Y_t solves the BSDE with terminal condition Q, Y_t+q solves the BSDE with terminal condition $Q+q$.

(1. implies 2.) We define

$$F(\omega,t,Y_t,Z_{t+1}) := \mathcal{E}(Z_{t+1}|\mathcal{F}_t),$$

and note that this does not depend on Y_t, satisfies $F(\omega,t,Y_t,0)=0$ and, in the scalar case, is equal (by translation invariance) to the value of F defined for nonlinear evaluations in Theorem 4.1. It follows that, for any Y_{t+1}, we can write

$$Y_t = \mathcal{E}(Y_{t+1}|\mathcal{F}_t) = \mathcal{E}(Y_{t+1} - E[Y_{t+1}|\mathcal{F}_t]|\mathcal{F}_t) + E[Y_{t+1}|\mathcal{F}_t].$$

and hence, for $Z_{t+1} = Y_{t+1} - E[Y_{t+1}|\mathcal{F}_t]$,

$$E[Y_{t+1}|\mathcal{F}_t] = Y_t - \mathcal{E}(Z_{t+1}|\mathcal{F}_t)$$

from which the one-step equation

$$Y_{t+1} = Y_t - F(\omega,t,Y_t,Z_{t+1}) + Z_{t+1}$$

follows directly. Recursivity then extends this directly to the full BSDE.

We need only to show that the driver F is balanced. As F is independent of Y and $\mathcal{E}(\cdot|\mathcal{F}_t)$ takes only finite values, the only relevant requirement for F to be balanced is that, for each component i, for any Z^1_{t+1}, Z^2_{t+1},

$$e_i^*F(\omega,t,Y_t^2,Z^1_{t+1}) - e_i^*F(\omega,t,Y_t^2,Z^2_{t+1}) \geq \operatorname{ess\,inf}_{\mathcal{F}_t}\{e_i^*(Z^1_{t+1}-Z^2_{t+1})\},$$

with equality only if $e_i^* Z_{t+1}^1 = e_i^* Z_{t+1}^2$ $\mathbb{P}$-a.s.

Define an $\mathcal{F}_t$-measurable random variable q by

$$e_i^* q = \operatorname{ess\,inf}_{\mathcal{F}_t}\{e_i^*(Z_{t+1}^1 - Z_{t+1}^2)\}$$

Then $Z_{t+1}^1 - q \geq Z_{t+1}^2$ componentwise, and hence we know

$$\mathcal{E}(Z_{t+1}^1 - q|\mathcal{F}_t) \geq \mathcal{E}(Z_{t+1}^2|\mathcal{F}_t)$$

By translation invariance of $\mathcal{E}$, we then have,

$$e_i^*\mathcal{E}(Z_{t+1}^1|\mathcal{F}_t) - e_i^*\mathcal{E}(Z_{t+1}^2|\mathcal{F}_t) \geq e_i^* q$$

and hence,

$$e_i^* F(\omega, t, Y_t^2, Z_{t+1}^1) - e_i^* F(\omega, t, Y_t^2, Z_{t+1}^2) \geq \operatorname{ess\,inf}_{\mathcal{F}_t}\{e_i^*(Z_{t+1}^1 - Z_{t+1}^2)\}.$$

To show the strict inequality, note that, for each i, if

$$e_i^*\mathcal{E}(Z_{t+1}^1 - q|\mathcal{F}_t) = e_i^*\mathcal{E}(Z_{t+1}^2|\mathcal{F}_t)$$

then $e_i^* Z_{t+1}^1 - e_i^* q = e_i^* Z_{t+1}^2$ $\mathbb{P}$-a.s. by the strict monotonicity of $\mathcal{E}$. It follows that

$$e_i^*(Z_{t+1}^1 - Z_{t+1}^2) = e_i^* q = \operatorname{ess\,inf}_{\mathcal{F}_t}\{e_i^*(Z_{t+1}^1 - Z_{t+1}^2)\}$$

and so, as $Z_{t+1}^1 - Z_{t+1}^2$ is a martingale difference process, $e_i^*(Z_{t+1}^1 - Z_{t+1}^2) = 0$ $\mathbb{P}$-a.s. □

Corollary 4.3.1. *For $\mathcal{E}$ an $\mathcal{F}_t$ consistent nonlinear expectation and F the corresponding BSDE driver, the following statements hold.*

- *F is independent of Y*
- *F is concave with respect to Z if and only if $\mathcal{E}$ is concave.*
- *F is (Lipschitz) continuous in L^1 norm with respect to Z if and only if $\mathcal{E}$ is (Lipschitz) continuous in L^1 norm.*
- *F is linear if and only if $\mathcal{E}$ is linear.*
- *F is positively homogenous (that is $F(\omega, t, \lambda Y_t, \lambda Z_{t+1}) = \lambda F(\omega, t, Y_t, Z_t)$ for all t, all $\lambda \geq 0$) if and only if $\mathcal{E}$ is positively homogenous.*

Proof. Each of these statements, and many others, is trivial once one realises that $F(\omega, t, Y_t, Z_t) = \mathcal{E}(Z_{t+1}|\mathcal{F}_t)$. □

5 Nearly-time-consistent nonlinear expectations

We shall now consider the BSDE (2) as being a discrete equation in a continuous time-setting. By this we mean that there exists a continuous time filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, where T is finite. We wish to consider the solutions, for $t \in \{0 = t_0, t_1, t_2, ..., t_N = T\} = \mathcal{T}$, of the equation

$$Y_t - \sum_{u\in\mathcal{T}\cap\{t\leq u<T\}} F(\omega, u, Y_u, Z_{u+}) + \sum_{u\in\mathcal{T}\cap\{t\leq u<T\}} Z_{u+} = Q$$

where $u+$ denotes the element of $\mathcal{T}$ following u. Note that we could equivalently take the sum over i from 0 to N, where the sums are taken of the values $F(\omega, t_i, Y_{t_i}, Z_{t_{i+1}})$ and $Z_{t_{i+1}}$ respectively.

For a fixed mesh $\mathcal{T}$, for $t \in [t_i, t_{i+1}[$, we can define $\bar{Y}_t = Y_{t_i}$. Under this definition, $\bar{Y}$ is a right-continuous step function, which changes only at times in $\mathcal{T}$. It follows easily that, if F is balanced, $\bar{Y}_t = \mathcal{E}(\cdot|\mathcal{F}_t)$ defines a *nearly-time-consistent nonlinear expectation* in the following sense.

Definition 5.1. *For a fixed set of dates $\mathcal{T} \subset [0, t]$, a family of operators*

$$\mathcal{E}(\cdot|\mathcal{F}_t) : L^1(\mathbb{R}^K; \mathcal{F}_T) \to L^1(\mathbb{R}^K; \mathcal{F}_t); t \in [0, T]$$

is called a nearly-time-consistent nonlinear expectation *if for $t \in \mathcal{T}$, $\mathcal{E}$ is a nonlinear expectation.*

Clearly every $\mathcal{F}_t$-consistent nonlinear expectation in continuous time is a nearly-time-consistent nonlinear expectation. However, the converse is not true.

A nearly-time-consistent nonlinear expectation is then recursive, but only at the points in $\mathcal{T}$. It satisfies the Zero-One law, ($\mathcal{E}(I_A Q|\mathcal{F}_t) = I_A \mathcal{E}(Q|\mathcal{F}_t)$ for all $A \in \mathcal{F}_t$), only for $t \in \mathcal{T}$. Overall, each of the properties of nonlinear expectations holds, but is only guaranteed to do so on the mesh $\mathcal{T}$. These properties, as discussed in [1], form an appropriate description of time-consistency, and can be shown to be equivalent to Bellman's principle, (under some conditions).

This type of consistency arises naturally in many situations where, even though time may be continuous, decisions can only be made at a discrete set of points. In these circumstances, there is no reason to require the optimal value function to be consistent between decision times. On the other hand, we do require consistency in decision making, and hence, we require time-consistency on those dates where decisions can be made.

Another example where a discrete-time-consistency requirement is natural is when decisions are being made on the basis of time-series data. Typically, new data is only available at discrete time-points and, therefore, it is only at these time-points that new decisions need to be made. Therefore, the natural type of time-consistency that is needed is only to be considered at a discrete set of points, and hence a nearly-time-consistent nonlinear evaluation arises.

Overall, these nearly-time-consistent nonlinear expectations and evaluations have various natural applications. We have shown that the theory of BSDEs is an appropriate tool to use to study these functionals, as we have given representation theorems in this context. These results are very suggestive of the existence of analogous results in continuous time, for very general probability spaces. However, this remains an open problem.

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