GIT Constructions of Compactified Universal Jacobians over $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}(X,\beta)$

George Cooper (cooperg@maths.ox.ac.uk)

UNIVERSITY OF OXFORD
Mathematical Institute

Throughout we work with separated finite-type schemes over $\mathbb{C}$. **Let $\mathcal{M}$ be one of the stacks $\mathcal{M}_{g,n}$ or $\mathcal{M}_{g,n}(X,\beta)$**, parametrising smooth n-marked genus g stable curves/maps. Sitting over $\mathcal{M}$ is the stack $\mathcal{J}^{d,r}_{\mathcal{M}}$, parametrising **families of objects of $\mathcal{M}$** together with degree d, rank r **slope-semistable vector bundles**.

Aim. Find ways of completing the diagram

$$\begin{tikzcd} \mathcal{J}^{d,r}_{\mathcal{M}} \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, dashed] \\ \mathcal{M} \arrow[r, hook] & \overline{\mathcal{M}} \end{tikzcd}$$

to obtain a proper algebraic stack $\overline{\mathcal{J}}^{d,r}_{\overline{\mathcal{M}}}$ over $\overline{\mathcal{M}}$ **admitting a modular description** and **containing the stack of slope-semistable vector bundles over objects of $\mathcal{M}$ as an open substack.**

For our stacks to be universally closed, we ask that $\overline{\mathcal{J}}^{d,r}_{\overline{\mathcal{M}}}$ **parametrises flat families of rank r torsion-free sheaves** over families of objects of $\overline{\mathcal{M}}$. We extend the slope-stability condition by using a relatively ample line bundle on the universal family $\pi_{\overline{\mathcal{M}}} : \overline{\mathcal{C}} \to \overline{\mathcal{M}}$ to **define a Gieseker-stability condition.**

Definition 1. *Let $\mathcal{L}$ be a $\pi_{\overline{\mathcal{M}}}$-ample $\mathbb{Q}$-line bundle on $\overline{\mathcal{C}}$. The collection of objects of the stack $\overline{\mathcal{J}ac}^{d,r,(s)s}_{\overline{\mathcal{M}}}(\mathcal{L})$ over a scheme S is given by all flat families of degree d rank r torsion-free sheaves $\mathcal{F}$ over objects of $\overline{\mathcal{M}}(S)$, such that for each $s \in S$, the sheaf $\mathcal{F}_s$ is Gieseker-(semi)stable with respect to the polarisation $\mathcal{L}_s$. Morphisms in $\overline{\mathcal{J}ac}^{d,r,(s)s}_{\overline{\mathcal{M}}}(\mathcal{L})$ are given by morphisms in $\overline{\mathcal{M}}$ together with isomorphisms of sheaves.*

Let $\overline{\mathcal{J}}^{d,r,(s)s}_{\overline{\mathcal{M}}}(\mathcal{L})$ denote the $\mathbb{G}_m$-rigidification of $\overline{\mathcal{J}ac}^{d,r,(s)s}_{\overline{\mathcal{M}}}(\mathcal{L})$.

Remark. If E is a vector bundle on a smooth curve C and if L is an ample line bundle on C, then E is Gieseker-(semi)stable with respect to L if and only if E is slope-(semi)stable.

Assuming for simplicity that $r = 1$ and all degree d semistable sheaves are stable, **the following properties are well-known** (see for instance [6] and [7]):

- The stack $\overline{\mathcal{J}}^{d,1,ss}_{\overline{\mathcal{M}}}(\mathcal{L})$ is proper and Deligne-Mumford, and admits a projective coarse moduli space $\overline{J}^{d,1,ss}_{\overline{M}}(\mathcal{L})$.
- The forgetful morphism to $\overline{\mathcal{M}}$ is representable.
- If $n \geq 1$ then $\overline{\mathcal{J}}^{d,1,ss}_{\overline{\mathcal{M}}}(\mathcal{L})$ admits a universal family.
- If $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{g,n}$ then $\overline{\mathcal{J}}^{d,1,ss}_{\overline{\mathcal{M}}}(\mathcal{L})$ is smooth, irreducible and flat over $\overline{\mathcal{M}}_{g,n}$ and has dimension $4g-3+n$.

It is also known that $\overline{\mathcal{J}}^{d,r,ss}_{\overline{\mathcal{M}}_g}(\omega)$ admits a projective good moduli space, which **is a GIT quotient** - this is the main result of Pandharipande [8]. In the $r = 1$ case a separate construction exists due to Caporaso [2]. Both of these constructions rely on Gieseker's construction of $\overline{M}_g$ [4].

Using the work of Greb, Ross and Toma [5] **another GIT quotient construction exists**; this has the advantage that moduli spaces arising from different stability conditions are GIT quotients of the **same variety**. This construction **also extends naturally to stable maps**, using the GIT construction of $\overline{M}_{g,n}(X,\beta)$ by Baldwin and Swinarski [1].

Notation. Let $\mathcal{L}_1, \ldots, \mathcal{L}_k$ be $\pi_{\overline{\mathcal{M}}}$-ample line bundles on the universal curve $\overline{\mathcal{C}}$. For $\sigma \in \mathfrak{S} := (\mathbb{Q}^{\geq 0})^k \setminus \{0\}$, let $\mathcal{L}_\sigma = \bigotimes_i \mathcal{L}_i^{\sigma_i}$.

Theorem 2. *Given a finite subset $\Sigma \subset \mathfrak{S}$, there exists a quasi-projective scheme R (depending on the $\mathcal{L}_i$ and Σ), a reductive group G acting on R and linearisations $(N_\sigma)_{\sigma \in \Sigma}$ for this action such that the following holds for each $\sigma \in \Sigma$:*

- *The stack $\overline{\mathcal{J}}^{d,r,(s)s}_{\overline{\mathcal{M}}}(\mathcal{L}_\sigma)$ is universally closed and isomorphic to the quotient stack $[R^{(s)s}(N_\sigma)/G]$.*
- *Each $\overline{\mathcal{J}}^{d,r,ss}_{\overline{\mathcal{M}}}(\mathcal{L}_\sigma)$ admits a projective good moduli space $\overline{J}^{d,r,ss}_{\overline{M}}(\mathcal{L}_\sigma)$, which is isomorphic to the GIT quotient $R /\!\!/_{N_\sigma} G$.*
- *The fibre of $\overline{J}^{d,r,ss}_{\overline{M}}(\mathcal{L}_\sigma)$ over a point $[\zeta] \in \overline{M}$ is isomorphic to $\overline{J}^{d,r,ss}_{\zeta}((\mathcal{L}_\sigma)_\zeta)/\mathrm{Aut}(\zeta)$.*

Proof Idea. Use Baldwin and Swinarski [1] to write $\overline{M}_{g,n}(X,\beta) = \overline{J} /\!\!/ SL(W)$ for an appropriate subscheme J of a product of a Hilbert scheme of curves inside $\mathbb{P}(W) \times X$ (for some vector space W) with $(\mathbb{P}(W) \times X)^{\times n}$. Let $\psi : U\overline{J} \to \overline{J}$ denote the universal family. Apply a relative version of the construction of Greb, Ross and Toma [5] to ψ, along with the pullback of the $\mathcal{L}_i$'s to $U\overline{J}$, to express the moduli space of $\overline{J}$-flat σ-semistable sheaves over $U\overline{J}$ as $R /\!\!/_{N'_\sigma} H$. The desired good moduli spaces are GIT quotients of this R by an induced action of $G = H \times SL(W)$, and R can be chosen to be independent of $\sigma \in \Sigma$.

After fixing $\mathcal{L}_1, \ldots, \mathcal{L}_k$, the space of stability conditions $\mathfrak{S}$ **admits a finite wall-chamber decomposition**. In the case where $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{g,n}$ and where $r = 1$, the following proposition is essentially a special case of the more general main result of [6].

Definition 3. *A stability parameter σ is **non-degenerate** if all $\mathcal{L}_\sigma$-semistable sheaves are stable.*

Proposition 4. *The set $\mathfrak{S}$ is cut into chambers by a finite number of rational linear walls, such that the moduli stack $\overline{\mathcal{J}}^{d,r,ss}_{\overline{\mathcal{M}}}(\mathcal{L}_\sigma)$ is unchanged as σ varies in the interior of a single chamber. σ is non-degenerate if and only if σ is not contained in any wall. If σ_1 and σ_2 are non-degenerate and lie either side of a wall, the corresponding coarse moduli spaces are related by a Thaddeus flip through some moduli space $\overline{J}^{d,r,ss}_{\overline{M}}(\mathcal{L}_{\sigma'})$.*

In the case when $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{g,n}(X,\beta)$, the stacks $\overline{\mathcal{J}}^{d,1,(s)s}_{\overline{\mathcal{M}}}(\mathcal{L})$ will in general not be smooth, however they often still **possess natural virtual fundamental classes.**

Proposition 5. *Assume all semistable sheaves are stable and that the variety X is smooth. Then the stack $\overline{\mathcal{J}}^{d,r,ss}_{\overline{\mathcal{M}}}(\mathcal{L})$ is Deligne-Mumford, and when $r = 1$ admits a natural perfect obstruction theory and a virtual fundamental class. The forgetful morphism to $\overline{\mathcal{M}}$ is virtually smooth.*

References

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