# Stable $\infty$-Categories Week 8

George Cooper

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Throughout this document, HA refers to [Lur17] and HTT refers to [Lur06].

## 1 Prelude - Comparing Nerves with DG-Nerves

Recall from last week the construction of the dg-nerve of a dg-category. It will be useful in this talk to be able to compare dg-nerves with the usual notion of a nerve. More specifically, let $\mathcal{C}$ be a dg-category with underlying ordinary category $\mathcal{C}^\circ$.[1] Let $\sigma$ be the following $n$-simplex of the nerve $N(\mathcal{C}^\circ)$, where the $f_{ji} \in \mathrm{Mor}_{\mathcal{C}}(X_i, X_j)_0$ are 0-cycles:

$$X_0 \xrightarrow{f_{10}} X_1 \xrightarrow{f_{21}} X_2 \xrightarrow{f_{32}} \cdots \xrightarrow{f_{n,n-1}} X_n$$

We may associate to $\sigma$ an $n$-simplex $U(\sigma)$ of $N_{\mathrm{dg}}(\mathcal{C})$ given by $U(\sigma) = (\{X_i\}, \{f_I\})$, where

$$f_I = \begin{cases} f_{ji} & \text{if } I = \{j > i\}, \\ 0 & \text{otherwise.} \end{cases}$$

This gives a monomorphism of simplicial sets $U : N(\mathcal{C}^\circ) \to N_{\mathrm{dg}}(\mathcal{C})$ whose image is the simplicial subset of $N_{\mathrm{dg}}(\mathcal{C})$ spanned by those $n$-simplices $(\{X_i\}_{0 \le i \le n}, \{f_I\})$ such that $f_I = 0$ whenever $|I| > 2$. In particular, $U$ is bijective on 0 and 1-simplices.

## 2 Prelude - The Homotopy Category of a DG Category

**Definition 2.1.** *Let $\mathcal{C}$ be a dg category. We define the* homotopy category $\mathrm{Ho}(\mathcal{C})$ *of $\mathcal{C}$ as follows: the objects of* $\mathrm{Ho}(\mathcal{C})$ *are the same as those of $\mathcal{C}$, and for $X, Y \in \mathrm{Ob}(\mathcal{C})$, we set*

$$\mathrm{Mor}_{\mathrm{Ho}(\mathcal{C})}(X, Y) = \ker(d : \mathrm{Mor}_{\mathrm{C}}(X, Y)_0 \to \mathrm{Mor}_{\mathrm{C}}(X, Y)_{-1}) / \sim,$$

*where 0-cycles $f$ and $g$ are identified if there exists $h \in \mathrm{Mor}_{\mathcal{C}}(X, Y)_1$ such that $f - g = dh$. Composition of morphisms in* $\mathrm{Ho}(\mathcal{C})$ *is given by $[f] \circ [g] = [f \circ g]$.*

On the other hand, the dg-nerve $N_{\mathrm{dg}}(\mathcal{C})$ is an $\infty$-category, so we may form another homotopy category, namely $\mathrm{Ho}(N_{\mathrm{dg}}(\mathcal{C}))$. By the above construction, we have a map

$$\mathcal{C}^\circ \simeq \mathrm{Ho}(N(\mathcal{C}^\circ)) \to \mathrm{Ho}(N_{\mathrm{dg}}(\mathcal{C}))$$

which is bijective on objects and surjective on morphisms. Recall that a 2-simplex in $N_{\mathrm{dg}}(\mathcal{C})$ is given by the data of $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)_0$, $g \in \mathrm{Mor}_{\mathcal{C}}(Y, Z)_0$ and $j \in \mathrm{Mor}_{\mathcal{C}}(X, Z)_0$ such that $df = dg = dj = 0$, together with $h \in \mathrm{Mor}_{\mathcal{C}}(X, Z)_1$ such that $dh = (g \circ f) - j$. Combining this with some definition chasing, we have the following result.

**Proposition 2.2** (HA 1.3.11)**.** *The natural map $\mathcal{C}^\circ \to \mathrm{Ho}(N_{\mathrm{dg}}(\mathcal{C}))$ induces an isomorphism* $\mathrm{Ho}(\mathcal{C}) \simeq \mathrm{Ho}(N_{\mathrm{dg}}(\mathcal{C}))$.

[1] Morphisms are given by 0-cycles in $\mathcal{C}$.

## 3 Prelude - Localisations of $\infty$-Categories

**Definition 3.1.** *Let $\mathcal{C}$ be an $\infty$-category. A* localisation *of $\mathcal{C}$ is a full subcategory $\mathcal{C}_0$ of $\mathcal{C}$ such that the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ admits a left adjoint $L$.*

## 4 Recap - Ch($\mathcal{A}$) as a DG Category

Recall that for an additive category $\mathcal{A}$, the chain complex category $\mathbf{Ch}(\mathcal{A})$ has the structure of a dg-category, where

$$\mathrm{Mor}_{\mathbf{Ch}(\mathcal{A})}(M_\bullet, N_\bullet)_p = \prod_{n\in\mathbb{Z}} \mathrm{Mor}_{\mathcal{A}}(M_n, N_{n+p}),$$

and differential

$$(df)(x) = d_N(f(x)) + (-1)^{p+1} f(d_M x), \quad f \in \mathrm{Mor}_{\mathbf{Ch}(\mathcal{A})}(M_\bullet, N_\bullet)_p.$$

In particular, 0-cycles correspond to chain maps, and 0-boundaries correspond to null-homotopies; immediately from the definitions we see that

$$\mathrm{Ho}(\mathbf{Ch}(\mathcal{A})) = \mathbf{K}(\mathcal{A})$$

is the homotopy category of chain complexes in $\mathcal{A}$.

## 5 §1.3.4 - Inverting Quasi-Isomorphisms

Let $\mathcal{A}$ be an abelian category with enough projectives. The classical derived category $\mathbf{D}^-(\mathcal{A})$ can be described explicitly in terms of chain complexes of projectives or in terms of inverting $\mathbf{Ch}^-(\mathcal{A})$ with respect to the class of quasi-isomorphisms. We will now focus on the latter point of view.

**Definition 5.1.** *Let $\mathcal{C}$, $\mathcal{D}$ be $\infty$-categories and let $W$ be a collection of morphisms in $\mathcal{C}$. We say that the morphism $f : \mathcal{C} \to \mathcal{D}$* exhibits $\mathcal{D}$ as the $\infty$-category obtained by inverting $W$ *if for every $\infty$-category $\mathcal{E}$, the induced map*

$$\mathbf{Fun}(\mathcal{D}, \mathcal{E}) \to \mathbf{Fun}(\mathcal{C}, \mathcal{E})$$

*is a fully faithful embedding whose essential image is the collection of functors $F : \mathcal{C} \to \mathcal{E}$ sending each morphism in $W$ to an equivalence in $\mathcal{E}$.[2] $\mathcal{D}$ is determined uniquely up to equivalence by $\mathcal{C}$ and $W$; we denote this $\infty$-category as $\mathcal{C}[W^{-1}]$.*

*For $\mathcal{C}$ an ordinary category and $W$ a collection of morphisms in $\mathcal{C}$, we write $\mathcal{C}[W^{-1}] = N(\mathcal{C})[W^{-1}]$.*

**Proposition 5.2** (HTT §3.1)**.** *The $\infty$-category $\mathcal{C}[W^{-1}]$ is always defined.*

**Proposition 5.3** (HTT 5.2.7.12)**.** *Let $\mathcal{C}$ be an $\infty$-category, and let $\mathcal{C}_0$ be a localisation of $\mathcal{C}$. Let $W$ be the collection of all morphisms $\alpha$ in $\mathcal{C}$ such that $L(\alpha)$ is an equivalence; then the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C} \to \mathcal{C}[W^{-1}]$ is an equivalence of $\infty$-categories.*

Our main result is the following.

**Theorem 5.4** (HA 1.3.4.4)**.** *Let $\mathcal{A}$ be an abelian category with enough projectives, let $\mathbb{A} = \mathbf{Ch}^-(\mathcal{A})$, regarded as an ordinary category, and let $W$ denote the class of quasi-isomorphisms in $\mathcal{A}$. Then there is a canonical equivalence of $\infty$-categories $\mathbb{A}[W^{-1}] \simeq \mathcal{D}^-(\mathcal{A})$.*

This result is proven using the follwing two technical results.

[2]An *equivalence* in an $\infty$-category $\mathcal{E}$ is a 1-morphism in $\mathcal{E}$ whose image in $\mathrm{Ho}(\mathcal{E})$ is an isomorphism.

**Proposition 5.5** (HA 1.3.4.5). *Let $\mathcal{A}$ be an additive category, let $\mathbf{Ch}'(\mathcal{A})$ be a full subcategory of $\mathbf{Ch}(\mathcal{A})$ and let $\mathbb{A}'$ be the underlying ordinary category of $\mathbf{Ch}'(\mathcal{A})$. Let $W$ be the collection of chain homotopy equivalences in $\mathbb{A}'$. Assume that $\mathbf{Ch}'(\mathcal{A})$ is closed under taking tensor products with $N_\bullet(\Delta^1)$, that is $\mathbf{Ch}'(\mathcal{A})$ is closed under taking mapping cylinders*[3] *of identity morphisms. Let*

$$\theta : N(\mathbb{A}') \to N_{\mathrm{dg}}(\mathbf{Ch}'(\mathcal{A}))$$

*denote the natural inclusion. Then $\theta$ induces an equivalence of $\infty$-categories $\mathbb{A}'[W^{-1}] \simeq N_{\mathrm{dg}}(\mathbf{Ch}'(\mathcal{A}))$.*

The proof of this proposition is quite involved, requiring several preliminary results; let me instead give some very brief intuition as to why such a result should hold. Recall from classical homological algebra, if $\mathrm{Cyl}(M)_\bullet$ denotes the mapping cylinder of the identity of $M$, then there is a one-to-one correspondence between morphisms $\mathrm{Cyl}(M)_\bullet \to N_\bullet$ and pairs of maps $f, g : M_\bullet \to N_\bullet$ together with a chain homotopy between $f$ and $g$; this teels us that we always need to be able to form mapping cylinders in $\mathbf{Ch}'(\mathcal{A})$ if we want to be able to localise with respect to all chain homotopy equivalences. Suppose we have chain maps $f : M_\bullet \to N_\bullet$ and $g : N_\bullet \to M_\bullet$ such that their composite is chain homotopic to the identity:

$$\begin{tikzcd} & N_\bullet \arrow[dr, "g"] \arrow[d, Rightarrow, "h"'] & \\ M_\bullet \arrow[ur, "f"] \arrow[rr, "\mathrm{id}_M"'] & {} & M_\bullet \end{tikzcd}$$

As $g \circ f$ need not equal $\mathrm{id}_M$, the 2-simplex in the ordinary nerve $N(\mathbf{Ch}(\mathcal{A})^\circ)$ corresponding to the above data is nothing more than

$$M_\bullet \xrightarrow{f} N_\bullet \xrightarrow{g} M_\bullet,$$

that is, we've forgotton about our choice of chain homotopy between $g \circ f$ and the identity, so $f$ and $g$ in general fail to be equivalences when viewed in $N(\mathbf{Ch}(\mathcal{A})^\circ)$.[4] However, in the dg-nerve $N(\mathbf{Ch}(\mathcal{A})^\circ)$, this data corresponds to a 2-simplex, with $h \in \mathrm{Mor}_{\mathbf{Ch}(\mathcal{A})}(M_\bullet, M_\bullet)_1$ satisfying $dh = (g \circ f) - \mathrm{id}_M$; passing to the homotopy category, the morphisms $g \circ f$ and $\mathrm{id}_M$ get identified. In this way, $f$ and $g$ are equivalences when viewed in the dg-nerve $N_{\mathrm{dg}}(\mathbf{Ch}(\mathcal{A}))$.

The second result we need is given as follows.

**Proposition 5.6** (HA 1.3.4.6). *Let $\mathcal{A}$ be an abelian category with enough projectives.*

1. *The inclusion $\mathcal{D}^-(\mathcal{A}) \hookrightarrow N_{\mathrm{dg}}(\mathbf{Ch}^-(\mathcal{A}))$ admits a right adjoint $G$.*
2. *Let $\alpha$ be a morphism in $N_{\mathrm{dg}}(\mathbf{Ch}^-(\mathcal{A}))$. Then $G(\alpha)$ is an equivalence if and only if $\alpha$ is a quasi-isomorphism of chain complexes.*
3. *Let $W$ denote the collection of all morphisms in $N_{\mathrm{dg}}(\mathbf{Ch}^-(\mathcal{A}))$ which are quasi-isomorphisms. Then there is a canonical equivalence of $\infty$-categories*

$$N_{\mathrm{dg}}(\mathbf{Ch}^-(\mathcal{A}))[W^{-1}] \simeq \mathcal{D}^-(\mathcal{A}).$$

*Sketch proof.* The first assertion is a consequence of HTT 5.2.7.8 along with the fact that for any object $N_\bullet \in \mathbf{Ch}^-(\mathcal{A})$, there exists a quasi-isomorphism $M_\bullet \to N_\bullet$ with $M_\bullet \in \mathcal{D}^-(\mathcal{A})$. The "if" direction of the second assertion relies on the result that any quasi-isomorphism $M_\bullet \to N_\bullet$ of chain complexes induces, for any $P_\bullet \in \mathcal{D}^-(\mathcal{A})$, a quasi-isomorphism

[3] If $f : M_\bullet \to N_\bullet$ is a chain map then $\mathrm{Cyl}(f)_\bullet$ is the complex with degree $n$ term $M_n \oplus M_{n-1} \oplus N_n$ and differential $d_{\mathrm{Cyl}(f)} = \begin{pmatrix} d_M & 1_M & 0 \\ 0 & -d_M & 0 \\ 0 & -f & d_N \end{pmatrix}$.

[4] Whose homotopy category is canonically isomorphic to $\mathbf{Ch}(\mathcal{A})^\circ$.

$$\mathrm{Mor}_{\mathbf{Ch}(\mathcal{A})}(P_\bullet, M_\bullet) \to \mathrm{Mor}_{\mathbf{Ch}(\mathcal{A})}(P_\bullet, M_\bullet).$$

The "only if" direction follows by staring at the diagram

$$\begin{tikzcd} G(M_\bullet) \arrow[r, "G(\alpha)"] \arrow[d, "\text{q.is.}"'] & G(N_\bullet) \arrow[d, "\text{q.is.}"] \\ M_\bullet \arrow[r, "\alpha"'] & N_\bullet \end{tikzcd}$$

(the vertical arrows are quasi-isomrphisms by construction). The final assertion then follows by (a suitable adaptation of) Proposition 5.3. □

*Proof of Theorem 5.4.* Let $\mathbb{A}$ denote the ordinary category underlying $\mathbf{Ch}^-(\mathcal{A})$, let $W$ denote the set of quasi-isomorphisms in $\mathbb{A}$ and let $W_0 \subset W$ denote the set of chain-homotopy equivalences. Then we have equivalences of $\infty$-categories

$$N_{\mathrm{dg}}(\mathbf{Ch}^-(\mathcal{A})) \overset{1.3.4.5}{\simeq} \mathbb{A}[W_0^{-1}] \quad \text{and} \quad \mathcal{D}^-(\mathcal{A}) \overset{1.3.4.6}{\simeq} \mathbb{A}[W^{-1}],$$

from which we obtain an equivalence $\mathcal{D}^-(\mathcal{A}) \simeq \mathbb{A}[W^{-1}]$. □

# 6 §1.3.5 - Grothendieck Abelian Categories

We begin by introducing a special class of abelian categories.

**Definition 6.1.** *Let $\mathcal{A}$ be an abelian category. We say $\mathcal{A}$ is* Grothendieck *if the following hold:*

- *$\mathcal{A}$ is presentable; in particular there is a small collection of objects $X_i \in \mathcal{A}$ which generates $\mathcal{A}$ under small colimits.*
- *the collection of monomorphisms in $\mathcal{A}$ is closed under small filtered colimits.*

**Example 6.2.** The following are all examples of GAC's:

- $\mathbf{Mod}(R)$ for any ring $R$;
- $\mathbf{QCoh}(X)$ for any scheme $X$;
- $\mathbf{Ab}(\mathcal{C}, \mathcal{S})$ for any small site $(\mathcal{C}, \mathcal{S})$.

We are interested in studying the category of chain complexes $\mathbf{Ch}(\mathcal{A})$, where $\mathcal{A}$ is a GAC.

**Proposition 6.3** (HA 1.3.5.3)**.** *Let $\mathcal{A}$ be a GAC. Then $\mathbf{Ch}(\mathcal{A})$ admits a (left, proper, combinatorial) model structure, given as follows:*

- *Cofibrations: (degree-wise) monomorphisms in $\mathbf{Ch}(\mathcal{A})$.*
- *Weak equivalences: quasi-isomorphisms.*
- *Fibrations: maps satisfying the right lifting property with respect to acyclic cofibrations:*

$$\begin{tikzcd} A_\bullet \arrow[r] \arrow[d, hook, "\text{q.is.}"'] & M_\bullet \arrow[d, "f"] \\ B_\bullet \arrow[r] \arrow[ur, dashed, "\exists"] & N_\bullet \end{tikzcd}$$

In particular, every object of $\mathbf{Ch}(\mathcal{A})$ is cofibrant. We can also say something about the fibrant objects:

**Proposition 6.4.** *If $M_\bullet$ is a fibrant object of $\mathbf{Ch}(\mathcal{A})$ then each $M_n$ is injective. Conversely, if each $M_n$ is injective and if $M_n \simeq 0$ for $n \gg 0$ then $M_\bullet$ is a fibrant chain complex.*

*Proof.* For the first statement, for any $X \in \mathcal{A}$ form the chain complex $E(X,n)_\bullet$ consisting of $\mathrm{id}_X : X \to X$ supported in degrees $n$ and $n-1$. For any monomorphism $X \to Y$ in $\mathcal{A}$, the induced map $E(X,n)_\bullet \to E(Y,n)_\bullet$ is an acyclic cofibration; the right lifting property applied to the square

$$\begin{tikzcd} E(X,n)_\bullet \arrow[r] \arrow[d, hook] & M_\bullet \arrow[d] \\ E(Y,n)_\bullet \arrow[r] \arrow[ur, dashed] & 0 \end{tikzcd}$$

shows that $M_n$ is injective.

For the second statement, suppose we are given an acyclic cofibration $u : A_\bullet \to A'_\bullet$ and a chain map $f : A_\bullet \to M_\bullet$; we need to construct a lift of this map to $A'_\bullet$. We introduce the following notation:

- $Z_n(A) = \ker(A_n \to A_{n-1})$;
- $B_n(A) = \mathrm{im}(A_{n+1} \to A_n)$;
- $A(n)_\bullet = (\cdots \to A_{n+2} \to A_{n+1} \to B_n(A) \to 0 \to 0 \to \cdots)$;
- $f_n = f|A(n)_\bullet$.

For $n \gg 0$ we have $f_n = 0$, so there is an extension $f'_n : A'(n)_\bullet \to M_\bullet$. Fix this $n$. It suffices (by taking limits) to show that given $i \leq n$ and an extension $f'_i : A'(i)_\bullet \to M_\bullet$ of $f_n$, there is a chain map $f'_{i-1} : A'(i-1)_\bullet \to M_\bullet$ extending both $f'_i$ and $f_{i-1}$. As $u$ is a quasi-isomorphism then

$$\begin{tikzcd} B_i(A) \arrow[r] \arrow[d] & Z_i(A) \arrow[d] \\ B_i(A') \arrow[r] & Z_i(A') \end{tikzcd}$$

is a pushout square, so $f|Z_i(A)$ and $f'_i|B_n(A')$ determine a unique map $g : Z_i(A') \to M_i$. We have $g(Z_i(A)) \subset \ker(d : M_i \to M_{i-1})$ and $g(B_i(A')) \subset \mathrm{im}(d : M_{i+1} \to M_i)$. However $B_{i-1}(A') \cong A'_n/Z_n(A')$, so we conclude that we are done if we can find a map $\bar{g} : A'_i \to M_i$ extending $g$, such that the composite of this map with $u : A_i \to A'_i$ is equal to $f$. But $M_i$ is injective, so such a lift exists if we can show that the map

$$\theta : A_i \sqcup_{Z_i(A)} Z_i(A') \to A'_i$$

is a monomorphism. To do this, we apply the snake lemma to the diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & Z_i(A') \arrow[r] \arrow[d, "="'] & A_i \sqcup_{Z_i(A)} Z_i(A') \arrow[r] \arrow[d] & B_{i-1}(A) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Z_i(A') \arrow[r] & A'_i \arrow[r] & B_{i-1}(A') \arrow[r] & 0 \end{tikzcd}$$

and note that the right-most vertical arrow is a monomorphism since $u$ is a cofibration. $\square$

**Corollary 6.5.** *If $\mathcal{A}$ is a GAC then $\mathcal{A}$ has enough injectives.*

*Proof.* If $X \in \mathcal{A}$, pick an acyclic cofibration $X[0] \to Q_\bullet$ with $Q_\bullet$ fibrant. The induced map $X \to Q_0$ is then a monomorphism into an injective object. □

**Definition 6.6.** *Let $\mathcal{A}$ be a GAC. We let $\mathbf{Ch}(\mathcal{A})^f$ denote the full subcategory of $\mathbf{Ch}(\mathcal{A})$ spanned by the fibrant objects. Then the* derived $\infty$-category *of $\mathcal{A}$ is the dg-nerve $\mathcal{D}(\mathcal{A}) = N_{\mathrm{dg}}(\mathbf{Ch}(\mathcal{A})^f)$.*

**Proposition 6.7.** *$\mathcal{D}(\mathcal{A})$ is a stable $\infty$-category.*

*Proof.* We know from last week that $N_{\mathrm{dg}}(\mathbf{Ch}(\mathcal{A}))$ is stable, so it is enough to show that $\mathcal{D}(\mathcal{A})$ is a stable subcategory of $N_{\mathrm{dg}}(\mathbf{Ch}(\mathcal{A}))$. By HA 1.1.3.3 and the observation that $\mathcal{D}(\mathcal{A})$ is evidently invariant under translation, it suffices to show that $\mathcal{D}(\mathcal{A})$ is closed under taking cofibres. This will follow if we can show that for any map $f : M_\bullet \to N_\bullet$ between fibrant complexes, the mapping cone $C_\bullet(f)$ is also fibrant; as $M_\bullet[1]$ is fibrant then it is enough to show that the map $C_\bullet(f) \to M_\bullet[1]$ is a fibration. Suppose we have a diagram

$$\begin{tikzcd} A_\bullet \arrow[r] \arrow[d, "i"'] & C_\bullet(f) \arrow[d] \\ B_\bullet \arrow[r] & M_\bullet[1] \end{tikzcd}$$

where $i$ is an acyclic cofibration. The induced map $j : C_\bullet(i) \to C_\bullet(\mathrm{id}_B)$ is an acyclic cofibration. As $N_\bullet$ is fibrant, the induced map $C_\bullet(i) \to N_\bullet[1]$ admits a lift to $C_\bullet(\mathrm{id}_B)$; this in turn gives a lift $B_\bullet \to C_\bullet(f)$. □

*Remark.* As $\mathcal{A}$ has enough injectives, we can form the $\infty$-category $\mathcal{D}^+(\mathcal{A}) = N_{\mathrm{dg}}(\mathbf{Ch}^+(\mathcal{A}_{\mathrm{inj}}))$. Our characterisation of the fibrant objects of $\mathbf{Ch}(\mathcal{A})$ gives that $\mathcal{D}^+(\mathcal{A})$ is a full subcategory of $\mathcal{D}(\mathcal{A})$.

**Proposition 6.8** (HA 1.3.5.13)**.** *$\mathcal{D}(\mathcal{A})$ is a localisation of the $\infty$-category $N_{\mathrm{dg}}(\mathbf{Ch}(\mathcal{A}))$.*

*Sketch proof.* We need to show that the inclusion $\mathcal{D}(\mathcal{A}) \hookrightarrow N_{\mathrm{dg}}(\mathbf{Ch}(\mathcal{A}))$ admits a left adjoint. However, given $M_\bullet \in \mathbf{Ch}(\mathcal{A})$, we may pick an acyclic cofibration $f : M_\bullet \to Q_\bullet$. HA 1.3.5.12 then tells us that for any fibrant chain complex $Q'_\bullet$, the image of $f$ under the functor $\mathrm{Mor}_{\mathbf{Ch}(\mathcal{A})}(\cdot, Q'_\bullet)$ is a quasi-isomorphism of chain complexes, so $f$ exhibits $Q_\bullet$ as a $\mathcal{D}(\mathcal{A})$-localisation of the complex $M_\bullet$. □

As one would hope for, $\mathcal{D}(\mathcal{A})$ can be regarded as the $\infty$-category obtained from the ordinary category of chain complexes over $\mathcal{A}$ by inverting all quasi-isomorphisms. To fix notation, let $\mathcal{A}$ be a GAC and let $\mathbb{A}$ denote $\mathbf{Ch}(\mathcal{A})$ regarded as an ordinary category. Let $L$ denote a left adjoint to the inclusion $\mathcal{D}(\mathcal{A}) \hookrightarrow N_{\mathrm{dg}}(\mathbf{Ch}(\mathcal{A}))$. We then have the following result.

**Proposition 6.9** (HA 1.3.5.15)**.** *The composite map $N(\mathbb{A}) \to N_{\mathrm{dg}}(\mathbf{Ch}(\mathcal{A})) \xrightarrow{L} \mathcal{D}(\mathcal{A})$ induces an equivalence of $\infty$-categories $N(\mathbb{A})[W^{-1}] \simeq \mathcal{D}(\mathcal{A})$, where $W$ is the collection of all quasi-isomorphisms.*

This result is related to (but doesn't quite follow from)[5] a result known as the *Dwyer-Kan theorem* (see HA 1.3.4.20). This result needs some preliminaries before it can be fully stated, but in summary the theorem states that if $\mathbb{A}$ is a simplicial model category and if $W$ denotes the collection of weak equivalences in $\mathbb{A}^c$, then there is a natural equivalence of $\infty$-categories $\mathbb{A}^c[W^{-1}] \simeq N(\mathbb{A}^{fc})$. Again, the proof of this result is fairly technical, but the following intermediary proposition should help clarify that we are in fact looking at the right thing with $\mathcal{D}(\mathcal{A})$:

**Proposition 6.10** (HA 1.2.5.14)**.** *Let $\mathcal{A}$ be a GAC and let $f : M_\bullet \to M'_\bullet$ be a map of chain complexes. If $f$ is a quasi-isomorphism and $Q_\bullet$ is a fibrant complex, then the image of $f$ under $\mathrm{Mor}_{\mathbf{Ch}(\mathcal{A})}(\cdot, Q_\bullet)$ is a quasi-isomorphism. In particular,[6] if $f$ is a quasi-isomorphism between fibrant complexes, then $f$ is a chain homotopy equivalence, so induces an equivalence in $\mathcal{D}(\mathcal{A})$.*

[5] As $\mathbf{Ch}(\mathcal{A})$ is not a simplicial model category.

[6] Read: by Yoneda, after first passing to the homotopy category of $\mathbf{Ch}(\mathcal{A})^f$.

For each integer $n$, we can consider the full subcategory $N_{\mathrm{dg}}(\mathbf{Ch}(\mathcal{A}))_{\geq n}$ (resp. $N_{\mathrm{dg}}(\mathbf{Ch}(\mathcal{A}))_{\geq n}$) spanned by complexes $M_\bullet$ whose homology is concentrated in degrees $\geq n$ (resp. $N_{\mathrm{dg}}(\mathbf{Ch}(\mathcal{A}))_{\leq n}$). Let $\mathcal{D}(\mathcal{A})_{\geq n} = \mathcal{D}(\mathcal{A}) \cap N_{\mathrm{dg}}(\mathbf{Ch}(\mathcal{A}))_{\geq n}$ and similarly define $\mathcal{D}(\mathcal{A})_{\leq n} = \mathcal{D}(\mathcal{A}) \cap N_{\mathrm{dg}}(\mathbf{Ch}(\mathcal{A}))_{\leq n}$.

**Proposition 6.11** (HA 1.3.5.21)**.** *$(\mathcal{D}(\mathcal{A})_{\geq 0}, \mathcal{D}(\mathcal{A})_{\leq 0})$ determines a (right-complete, accessible) t-structure on $\mathcal{D}(\mathcal{A})$, which is compatible with filtered colimits, in the sense that $\mathcal{D}(\mathcal{A})_{\leq 0}$ is closed under small filtered colimits in $\mathcal{D}(\mathcal{A})$.*

To end the talk, suppose $\mathcal{A}$ is a GAC with enough projectives (eg. $\mathcal{A} = \mathbf{Mod}(R)$ for some ring $R$). Then we can form $\mathcal{D}^-(\mathcal{A})$ by considering complexes of projectives, or we can form $\mathcal{D}(\mathcal{A})$ by considering fibrant complexes (ie. complexes of injectives). Ideally we'd like to be able to somehow relate these two constructions. Fortunately we are in luck.

**Proposition 6.12** (HA 1.3.5.24)**.** *Let $\mathcal{A}$ be a GAC with enough projectives, and let $L : N_{\mathrm{dg}}(\mathbf{Ch}(\mathcal{A})) \to \mathcal{D}(\mathcal{A})$ be a left adjoint to the inclusion. Then the composite*

$$F : \mathcal{D}^-(\mathcal{A}) \hookrightarrow N_{\mathrm{dg}}(\mathbf{Ch}(\mathcal{A})) \xrightarrow{L} \mathcal{D}(\mathcal{A})$$

*is a fully faithful embedding, whose essential image is the subcategory $\bigcup_{n\geq 0} \mathcal{D}(\mathcal{A})_{\geq -n} \subset \mathcal{D}(\mathcal{A})$.*

# References


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