Differentials in Scheme Theory

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Kähler Differentials

Let B be an A-module.

Definition

We set $\Omega_{B/A}$ to be the quotient of the free B-module on the symbols $\{db : b \in B\}$, quotiented out by the relations $d(b+b') = db + db'$, $d(bb') = bdb' + b'db$ and $da = 0$ for $a \in A$, and call $\Omega_{B/A}$ the *module of relative differential forms* of B over A.

Remark

$(\Omega_{B/A}, d)$ is universal amongst all A-derivations $d' : B \to M$, that is given such a pair (M, d'), there is a unique homomorphism $f : \Omega_{B/A} \to M$ such that $d' = f \circ d$.

Lots of Facts about Kähler Differentials

Proposition

Suppose B is generated by elements x_i subject to relations r_j. Then $\Omega_{B/A}$ is generated as a B-module by the elements dx_i, subject to the relations $dr_j = 0$.

Example

If $B = \mathbb{C}[x, y]/(y^2 - x^3)$ then

$$\Omega_{B/\mathbb{C}} = \frac{B \; dx \oplus B \; dy}{(2ydy - 3x^2dx)}.$$

Lots of Facts about Kähler Differentials

Proposition

Let $\mu : B \otimes_A B \to B$ be the multiplication map, and let $I = \ker \mu$. Then I/I^2 is naturally a B-module, and $\Omega_{B/A}$ can be described as I/I^2 together with the map

$$d : B \to I/I^2, \quad b \mapsto 1 \otimes b - b \otimes 1 \mod I^2.$$

Proposition

Suppose A' is another A-algebra and $B' = B \otimes_A A'$. Then $\Omega_{B'/A'} \cong \Omega_{B/A} \otimes_B B'$.

Proposition

Let $S \subset B$ be a multiplicative system. Then $\Omega_{S^{-1}B/A} \cong S^{-1}\Omega_{B/A}$.

Lots of Facts about Kähler Differentials

Proposition

Let $A \to B \to C$ be rings. Then there is a natural exact sequence of C-modules

$$\Omega_{B/A} \otimes C \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

Proposition

Suppose I is an ideal of B and $C = B/I$. Then there is a natural exact sequence of C-modules

$$I/I^2 \xrightarrow{\delta} \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0,$$

where $\delta : b \mapsto db \otimes 1$.

Lots of Facts about Kähler Differentials

Recall that a local ring $(B, \mathfrak{m})$ with residue field k is a *regular local ring* if B is Noetherian and $\dim B = \dim_k \mathfrak{m}/\mathfrak{m}^2$.

Theorem

Let B be a local ring containing a local ring k isomorphic to its residue field. Assume k is perfect and that B is the localisation of a finitely-generated k-algebra. Then the following are equivalent:

- *B is a regular local ring;*
- *$\Omega_{B/k}$ is a free B-module (of rank equal to $\dim B$).*

Sheaf of Differentials

Let $f : X = \mathrm{Spec}(B) \to Y = \mathrm{Spec}(A)$ be a morphism of affine schemes. Let $A \to B$ be the induced ring homomorphism; we set $\Omega_{X/Y} = (\Omega_{B/A})^{\sim}$.

Now let $f : X \to Y$ be a general morphism of schemes. Cover Y with open affines $U = \mathrm{Spec}(A)$ and X with open affines $V = \mathrm{Spec}(B)$, with the property that for any such V, $f(V)$ is contained in some U. Form the local sheaves $\Omega_{V/U}$. One can check using the compatibility of Ω with localisation that the $\Omega_{V/U}$ glue to give a sheaf of $\mathcal{O}_X$-modules $\Omega_{X/Y}$. Moreover the derivations $d : B \to \Omega_{B/A}$ glue to give a map $d : \mathcal{O}_X \to \Omega_{X/Y}$.

Remark

$(d, \Omega_{X/Y})$ satisfies the expected universal property. A global construction can be found in Hartshorne.

$\Omega_{X/Y}$ and Base Change

Proposition

Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms. Let $f' : X' = X \times_Y Y' \to Y'$ be the morphism obtained by base extension, and let $\pi : X' \to X$ be the projection. Then

$$\Omega_{X'/Y'} \cong \pi^* \Omega_{X/Y}.$$

Relative Cotangent and Conormal Sequences

Proposition (Relative Cotangent Sequence)

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms. Then there is an exact sequence of sheaves on X,

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

Proposition (Conormal Sequence)

Let $f : X \to Y$ be a morphism. Let Z be a closed subscheme of X with ideal sheaf $\mathcal{I}$. Then there is an exact sequence of sheaves on Z,

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \Omega_{X/Y} \otimes \mathcal{O}_Z \to \Omega_{Z/Y} \to 0.$$

Euler Sequence

Proposition

Let A be a ring, let $Y = \mathrm{Spec}(A)$ and let $X = \mathbb{P}^n_A$. Then there is an exact sequence of sheaves on X,

$$0 \to \Omega_{X/Y} \to \mathcal{O}_X(-1)^{\oplus(n+1)} \to \mathcal{O}_X \to 0.$$

Sketch Proof.

Set $S = A[x_0, \ldots, x_n]$, $E = S[-1]^{\oplus(n+1)}$, and give E the basis $e_0, \ldots, e_n$ in degree 1. Let $M = \ker(E \to S)$, where $e_i \mapsto x_i$. It is enough to show $\widetilde{M} \cong \Omega_{X/Y}$ by the functoriality of $\tilde{\cdot}$. But if $U_i = D_+(x_i)$ then $\widetilde{M}|_{U_i}$ is a free $\mathcal{O}_{U_i}$-module generated by the sections $(1/x_i^2)(x_i e_j - x_j e_i)$ for $j \neq i$. The isomorphism $\widetilde{M} \cong \Omega_{X/Y}$ is then given over U_i by associating to this section the element $d(x_j/x_i)$. □

Nonsingular Varieties

Definition

Let X be an abstract variety over an algebraically closed field k. Then X is *nonsingular* if all of the local rings of X are regular local rings.

Useful Technical Result

Any localisation of a regular local ring at a prime ideal is a regular local ring (hence it is enough to check the above condition at closed points).

Jacobian Criterion for Regularity

For finite type k-schemes, we have a linear algebraic characterisation of regularity:

Proposition

Suppose $X = \mathrm{Spec}(k[x_1, \ldots, x_n]/(f_1, \ldots, f_m))$ has pure dimension r. Then a closed point $p \in X$ is a regular point if and only if the Jacobian matrix

$$\mathrm{Jac}_X(p) = \left(\frac{\partial f_i}{\partial x_j}(p)\right)_{i,j}$$

has rank $n - r$.

Corollary

$\mathrm{Sing}(X)$ (set of non-regular points of X) is a closed subscheme of X, cut out by the vanishing of all $(n-r) \times (n-r)$ minors of Jac_X.

Nonsingular Varieties

Theorem

Let X be an irreducible separated scheme of finite type over an algebraically closed field k. The following are equivalent:

- *X/k is a nonsingular variety;*
- *$\Omega_{X/k}$ is a locally-free sheaf (of rank $n = \dim X$).*

Proof.

Suppose $x \in X$ is a closed point. Then $B = \mathcal{O}_{X,x}$ has dimension n, residue field k and is a localisation of a k-algebra of finite type. Moreover we have $(\Omega_{X/k})_x = \Omega_{B/k}$. Hence by a previous result, $(\Omega_{X/k})_x$ is free of rank n if and only if B is a regular local ring. The result now follows from the fact that the property of a coherent sheaf on X being locally-free is stalk-local. □

Subschemes of Nonsingular Varieties

Theorem

Let X/k be a nonsingular variety. Let $Y \subset X$ be an irreducible closed subscheme with ideal sheaf $\mathcal{I}$. Then Y is nonsingular if and only if:

- $\Omega_{Y/k}$ *is locally-free, and;*
- *the conormal sequence associated to* $Y \subset X \to \mathrm{Spec}(k)$ *is a short exact sequence.*

In this case, $\mathcal{I}$ *is locally generated by* $r = \mathrm{codim}_X(Y)$ *elements, and* $\mathcal{I}/\mathcal{I}^2$ *is a locally-free sheaf on* Y *of rank* r*.*

Bertini's Theorem

Theorem (Bertini)

Let X be a non-singular closed subvariety of $\mathbb{P}^n_k$, where k is an algebraically closed field. Then there is a dense open subset $U \subset (\mathbb{P}^n_k)^\vee$ of the dual projective space such that for any closed point $[H] \in U$, H doesn't contain any component of X, and the scheme $H \cap X$ is regular at every point.

Remark

The closed points of $(\mathbb{P}^n_k)^\vee$ correspond to hyperplanes in $\mathbb{P}^n_k$ via the usual projective duality.

Proof of Bertini's Theorem

For simplicity assume X is irreducible. Let Z be the "bad" locus

$$Z = \{(p \in X, [H] \in (\mathbb{P}^n_k)^\vee) : (p \in H) \text{ and } (p \in \mathrm{Sing}(H \cap X) \text{ or } X \subset H)\}.$$

Using the Jacobi criterion, Z can be described in terms of polynomials on $\mathbb{P}^n_k \times_k (\mathbb{P}^n_k)^\vee$, so is a closed subscheme.

Claim

$\dim Z \leq n - 1$

For each closed point p, let

$$Z_p = \{[H] : (X \subset H) \text{ or } (p \in X \cap H \text{ and } p \in \mathrm{Sing}(X \cap H))\} \subset (\mathbb{P}^n_k)^\vee.$$

Suppose $\dim X = r$. The restrictions on the hyperplanes in Z_p correspond to $r + 1$ linear conditions, so Z_p is a projective space of codimension $r + 1$, i.e. $\dim Z_p = n - r - 1$.

Proof of Bertini's Theorem

Next, we apply the following result to the projection morphism $Z \to X$:

Proposition (Vakil 11.4.A)

Let $f : X \to Y$ be a morphism of locally Noetherian schemes and suppose $y = f(x)$. Then

$$\operatorname{codim}_X(x) \leq \operatorname{codim}_Y(y) + \operatorname{codim}_{X_y}(x).$$

This implies $\dim Z \leq n - 1$ as claimed. To end the proof, use the fact that the structure morphism $\mathbb{P}^n_k \to \operatorname{Spec}(k)$ is universally closed, so the image of Z in $(\mathbb{P}^n_k)^\vee$ is a closed subscheme of dimension at most $n - 1$, and so its complement is a dense subset of $(\mathbb{P}^n_k)^\vee$.

Hypersurfaces in $\mathbb{P}^n_k$ (Example of Bertini)

Let $X \subset \mathbb{P}^n_k$ be a non-singular closed subvariety as before. Then a generic degree $d > 0$ hypersurface intersects X in a regular subvariety of codimension 1; to see this, replace $X \subset \mathbb{P}^n_k$ with the embedding $X \to \mathbb{P}^n_k \xrightarrow{\nu_d} \mathbb{P}^N_k$, where ν_d is the dth Veronese embedding and apply Bertini's theorem in $\mathbb{P}^N_k$.

Taking $X = \mathbb{P}^n_k$, we get for free that there are nonsingular hypersurfaces of degree d in $\mathbb{P}^n_k$, and the locus of nonsingular hypersurfaces forms a dense open subset of the complete linear system $|\mathcal{O}(d)|$.

Invariants from Differentials

Let X/k be a nonsingular variety. We can define the following sheaves:

- The *tangent sheaf* $\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$;
- The *canonical sheaf* $\omega_X = \det(\Omega_{X/k})$.

Suppose in addition X is projective. Then we can define the following invariants of X:

- The *plurigenera* $P_n = h^0(X, \omega_X^{\otimes n})$ for $n \geq 1$ (note $P_1 = p_g$ is the geometric genus);
- The *Hodge numbers* $h^{p,q} = h^q(X, \Lambda^p \Omega_{X/k})$.

Invariants from Differentials

Theorem

The plurigenera P_n and the Hodge numbers $h^{p,0}$ are birational invariants of X.

Remark

The Hodge numbers $h^{p,q}$ for $q \neq 0$ need not be birational invariants. For example, take $k = \mathbb{C}$ and consider blowing up a smooth projective surface X at a point to obtain Y. Then X and Y are birational, so have equal $h^{1,0}$ (and hence equal $h^{0,1}$, as for Kähler manifolds $h^{p,q} = h^{q,p}$), but $h^{1,1}(Y) = h^{1,1}(X) + 1$, since Y is homeomorphic to $X \# \overline{\mathbb{P}^2_{\mathbb{C}}}$ and hence $b_2(Y) = b_2(X) + 1$ (by Hodge theory we have $b_2 = h^{1,0} + h^{1,1} + h^{0,1}$).

Proof that p_g is a Birational Invariant

Suppose $f : X \overset{\simeq}{\dashrightarrow} X'$ is a birational equivalence. Let $V \subset X$ be the largest open subset where f is defined. Then we have an induced morphism of sheaves $f^*\omega_{X'/k} \to \omega_{V/k}$ given by pulling back forms, and thus a map $F : H^0(X', \omega_{X'/k}) \to H^0(V, \omega_{V/k})$ on global sections. The map F is injective (as f gives an isomorphism between some open subset of V and some open subset of X', and a non-zero global section cannot vanish on a dense open subset). If we can show $H^0(X, \omega_{X/k}) \cong H^0(V, \omega_{V/k})$ then by symmetry we are done.

Proof that p_g is a Birational Invariant

Claim

$\mathrm{codim}_X(X \setminus V) \geq 2$

Suppose $x \in X$ is a point of codimension 1. Then $\mathcal{O}_{X,x}$ is regular and of dimension 1, so is a discrete valuation ring (as X is nonsingular) with field of fractions $K(X)$. If $\xi = \mathrm{Spec}(K(X))$ is the generic point of X then we have a morphism $\mathrm{Spec}(K(X)) \to X'$, which by the valuative criterion for properness extends to give a morphism of k-schemes $\mathrm{Spec}(\mathcal{O}_{X,x}) \to X'$:

Proof that p_g is a Birational Invariant

Claim

$\operatorname{codim}_X(X \setminus V) \geq 2$

$$\begin{tikzcd} \operatorname{Spec}(K(X)) \arrow[r, "f"] \arrow[d] & X' \arrow[d] \\ \operatorname{Spec}(\mathcal{O}_{X,x}) \arrow[r] \arrow[ur, dashed] & \operatorname{Spec}(k) \end{tikzcd}$$

This gives a map of k-algebras $A' \to \mathcal{O}_{X,x}$ where $\operatorname{Spec}(A')$ is an open affine containing $f(x)$. As $\mathcal{O}_{X,x}$ and A' are finitely-generated, one can find an affine neighbourhood $\operatorname{Spec}(A)$ containing x and a lift $A' \to A$ of $A' \to \mathcal{O}_{X,x}$, giving a morphism $\operatorname{Spec}(A) \to X'$ which agrees with f. Hence $x \in V$.

Proof that p_g is a Birational Invariant

Claim

The restriction map $H^0(X, \omega_{X/k}) \to H^0(V, \omega_{V/k})$ is an isomorphism.

It is enough to assume $\omega_{X/k} = \mathcal{O}_X$ is trivial (as this sheaf is locally free). But this is then an immediate consequence of the following result from commutative algebra:

Proposition

If R is an integrally closed Noetherian domain then

$$R = \bigcap_{\mathrm{height}(\wp)=1} R_{\wp}.$$

Sheaves Associated to Closed Subvarieties

Suppose Y is a nonsingular subvariety of a nonsingular variety X, defined over k. Then we can define the following locally-free sheaves on Y:

- The *conormal sheaf* $\mathcal{I}/\mathcal{I}^2$;
- The *normal sheaf* $\mathcal{N}_{Y/X} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$.

Proposition (Adjunction Formula)

Suppose $r = \mathrm{codim}_X(Y)$. Then

$$\omega_Y \cong \omega_X \otimes \Lambda^r \mathcal{N}_{Y/X}.$$

Proof: Take determinants in the conormal sequence.

Canonical Sheaves of $X = \mathbb{P}^n_k$ and Projective Hypersurfaces (Example)

Recall we have the Euler sequence

$$0 \to \Omega_{X/k} \to \mathcal{O}_X(-1)^{\oplus(n+1)} \to \mathcal{O}_X \to 0.$$

Taking determinants gives $\omega_X \cong \mathcal{O}_X(-n-1)$. In particular $p_g(X) = 0$, and so any projective nonsingular rational variety must have geometric genus 0.

Now suppose $Y \subset X$ is a projective hypersurface of degree d. Then by the adjunction formula

$$\omega_Y \cong \mathcal{O}_Y(d-n-1).$$

In particular if $n = 3$ and $d = 4$ then ω_Y is trivial; that is quartic surfaces in $\mathbb{P}^3_k$ are K3 surfaces.