Embedding Curves in Projective Spaces

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Conventions

A *curve* means a complete, non-singular curve over an algebraically closed field k; fix once and for all such a field k. A *point* means a closed point, unless otherwise specified.

Aims

Theorem

Any curve can be embedded in $\mathbb{P}^3$.

Theorem

Any curve is birationally equivalent to a plane curve whose singularities are at worst nodes.

We will also introduce the canonical embedding (time permitting).

Projection from a Point

Fix a curve X of genus g and fix a non-degenerate embedding $X \hookrightarrow \mathbb{P}^n$; such an embedding exists since any divisor of degree $\geq 2g+1$ is very ample. Fix a hyperplane $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ and fix a point $O \in \mathbb{P}^n \setminus (X \cup H)$. Define a morphism $\phi : X \to H$ by

$$\phi : P \mapsto OP \cap H.$$

Observation

Suppose $\mathfrak{d}$ is the (basepoint-free) linear system on X cut out by the intersection of X with all hyperplanes H' containing O. Then one can check (e.g. by playing around with coordinates on $\mathbb{P}^n$) that ϕ is the morphism corresponding to the linear system $\mathfrak{d}$.

Question

When is ϕ a closed immersion?

Tangent and Secant Lines

Suppose $P, Q \in X$ are distinct points.

Definition

The *secant line* determined by P and Q is the line $PQ \subset \mathbb{P}^n$. The *tangent line* to X at P is the unique line $L \subset \mathbb{P}^n$ passing through P such that as subspaces of $T_P\mathbb{P}^n$, we have

$$T_P L = T_P X.$$

Definition

The *secant variety* $\operatorname{Sec} X$ of X is the union of all secant lines of X, and the *tangent variety* $\operatorname{Tan} X$ of X is the union of all tangent lines of X.

Projection from a Point

$$\phi : X \to H, \quad P \mapsto OP \cap H \quad \leftrightarrow \quad \mathfrak{d} = \{H' \cap X : O \in H'\}$$

Proposition

ϕ is a closed immersion if and only if O is not contained on any secant line or any tangent line of X.

Proof.

Need to show that $\mathfrak{d}$ separates points and separates tangent vectors. $\mathfrak{d}$ separates points if and only if for any $P \neq Q \in X$ there exists H' with $P \in H'$ and $Q \notin H'$; this is equivalent to O not being contained on any secant line of X. Similarly $\mathfrak{d}$ separates tangent vectors if and only if there exists H' containing the points O and P with $i(X, H; P) = 1$, if and only if O is not contained on any tangent line of X. □

Projection from a Point

Remark

Similarly one can show that the morphism ϕ is ramified at $P \in X$ if and only if OP is the tangent line to X at P (one way of seeing this is by playing around with local parameters after choosing suitable coordinates on $\mathbb{P}^n$).

Projection from a Point

Can we always find a point $O \in \mathbb{P}^n$ such that O is not contained on any tangent or secant line of X?

First Observation

$\operatorname{Sec} X$ is a locally closed subset of $\mathbb{P}^n$ with $\dim \operatorname{Sec} X \leq 3$ as locally it is the image of a morphism $(X \times X - \Delta) \times \mathbb{P}^1 \to \mathbb{P}^n$ sending the triple (P, Q, t) to the point on the line PQ at time $t \in \mathbb{P}^1$.

Second Observation

$\operatorname{Tan} X$ is a closed subset of $\mathbb{P}^n$ with $\dim \operatorname{Tan} X \leq 2$, as it is locally the image of a morphism $X \times \mathbb{P}^1 \to \mathbb{P}^n$.

Upshot

If $n \geq 4$ then $\operatorname{Sec} X \cup \operatorname{Tan} X \neq \mathbb{P}^n$, so such a point O always exists.

Any Curve Embeds in $\mathbb{P}^3$

Corollary

If X is a curve then there exists an embedding $X \hookrightarrow \mathbb{P}^3$. $\square$

Remark

Exercise III.5.6 shows that there are non-singular curves of any genus g. However, the genus of a degree d curve in $\mathbb{P}^2$ is $\frac{1}{2}(d-1)(d-2)$, and there are non-negative integers g not of the form $g = \frac{1}{2}(d-1)(d-2)$. Curves of such genera cannot be embedded in $\mathbb{P}^2$.

We now turn to showing that any curve is birational to a plane curve whose singularities are at worst nodes.

More Definitions

Definition

A *node* is a singular point of a plane curve of multiplicity 2 with distinct tangent directions.

Let $X \subset \mathbb{P}^3$ be a curve. If $P \in X$, let L_P be the tangent line to X at P.

Definition

A *multisecant* of X is a line in $\mathbb{P}^3$ meeting X in at least 3 distinct points.

Definition

A *secant with coplanar tangent lines* is a secant of X joining distinct points P, Q with L_P and L_Q coplanar; equivalently L_P and L_Q intersect.

Projection from a Point Revisited

Suppose $X \subset \mathbb{P}^3$ is a curve, $O \in \mathbb{P}^3 \setminus X$ is a point and $\phi : X \to \mathbb{P}^2$ denotes projection from O.

Proposition

ϕ is birational onto its image and $\phi(X)$ has at worst nodes as singularities, if and only if the following conditions are satisfied:

1. O lies on finitely many secants of X;
2. O does not lie on any tangent line of X;
3. O does not lie on any multisecant of X; and
4. O is not on any secant with coplanar tangent lines.

Projection from a Point Revisited

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4. O is not on any secant with coplanar tangent lines.

Proof.

(1) $\iff$ ϕ is 1-1 a.e. $\iff$ ϕ is birational. If O lies on a secant line L, then $(2), (3), (4)$ for L is equivalent to requiring that L meets X at distinct points P and Q, $L_P \neq L \neq L_Q$ and that L_P and L_Q are mapped to distinct lines in $\mathbb{P}^2$. In turn this is equivalent to $\phi(P) = \phi(Q)$ being a node of $\phi(X)$. $\square$

Projection from a Point Revisited

Proposition

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1. O lies on finitely many secants of X;
2. O does not lie on any tangent line of X;
3. O does not lie on any multisecant of X; and
4. O is not on any secant with coplanar tangent lines.

We will now show that we can always find a point $O \in \mathbb{P}^3 \setminus X$ satisfying $(1) - (4)$.

A Technical Result

Lemma

Let $X \subset \mathbb{P}^3$ be a non-denegerate curve. Suppose

1. every secant of X is a multisecant; or
2. for any two points $P, Q \in X$ the tangent lines L_P and L_Q are coplanar.

Then there exists a point $A \in \mathbb{P}^3$ which lies on every tangent line of X.

We break this proof into several stages.
Step 1: We show (1) $\Rightarrow$ (2). Fix a hyperplane $H \subset \mathbb{P}^3$. For each $R \in X \setminus (X \cap H)$ let $\psi_R : X \to H$ denote projection from R. (1) implies each ψ_R is many-to-one.

To continue, we state without proof the following result from §IV.4.2.

Lemma

Suppose $f : X \to Y$ is a finite inseparable morphism of curves (meaning the field extension $K(Y) \hookrightarrow K(X)$ is inseparable). Then every point of X is ramified.

We now split into cases.

Case 1: If ψ_R is inseparable for some R, then every $P \in X$ is a ramification point of ψ_R, so $R \in L_P$ for all P.

Case 2: Suppose ψ_R is separable for all R. Fixing R, there exists a non-singular point $T \in \psi_R(X)$ which is not a branch point. Then for any two $P, Q \in \psi_R^{-1}(T)$, ψ_R takes the lines L_P and L_Q to the tangent line L_T to $\psi(X)$ at T. Then L_P and L_Q lie in the plane spanned by RT and L_T.

A Technical Result

Case 2: (ctd.) Hence for any $R \in X$, for almost all $P, Q \in X$ with P, Q, R collinear, the lines L_P and L_Q are coplanar, so there is an open set of pairs $(P, Q) \in X \times X$ for which the lines L_P and L_Q are coplanar. But coplanarity is a closed condition. Hence for all $P, Q \in X$, the lines L_P and L_Q are coplanar.
Step 2: We have reduced the problem to establising the following lemma.

Lemma

Let $X \subset \mathbb{P}^3$ be a non-denegerate curve. Suppose for any two points $P, Q \in X$ the tangent lines L_P and L_Q are coplanar. Then there exists a point $A \in \mathbb{P}^3$ which lies on every tangent line of X.

A Technical Result

Lemma

Let $X \subset \mathbb{P}^3$ be a non-denegerate curve. Suppose for any two points $P, Q \in X$ the tangent lines L_P and L_Q are coplanar. Then there exists a point $A \in \mathbb{P}^3$ which lies on every tangent line of X.

Proof.

Take any two points $P, Q \in X$ with distinct tangents and set $A = L_P \cap L_Q$. If Π is the plane spanned by L_P and L_Q then by non-degeneracy $X \cap \Pi$ is a finite set of points. For any $R \in X \setminus (X \cap \Pi)$, L_R meets L_P and L_Q but is not contained in Π, so $A \in L_R$. Hence there is an open set of points $R \in X$ with $A \in L_R$. But this is a closed condition on X, so $A \in L_R$ for all $R \in X$. $\square$

Strange Curves

Definition

A curve $X \subset \mathbb{P}^n$ is *strange* if there is a point $A \in \mathbb{P}^n$ such that A lies on all of the tangent lines of X.

Example

1. $\mathbb{P}^1$ is strange, as for any $P \in \mathbb{P}^1$ we have $L_P = \mathbb{P}^1$.
2. A plane conic in $\mathbb{P}^2$ over a field of characteristic 2 is strange. For instance, all tangent lines to the curve $C = \mathbb{V}(xy - z^2)$ pass through the point $P = [0 : 0 : 1]$.

Theorem (Samuel)

These are the only examples of strange curves.

Proof of Samuel's Theorem

Suppose A lies on all tangent lines of $X \subset \mathbb{P}^n$. Without loss of generality assume $n = 3$. Choose an $\mathbb{A}^3_{x,y,z} \subset \mathbb{P}^3_{x,y,z,w}$ such that:

- $A = [1 : 0 : 0 : 0]$ is the point at infinity on the x-axis;
- if $A \in X$ then L_A is not contained in the xz-plane;
- the z-axis does not meet X; and
- if X meets the line at infinity of the xz-plane, namely $\mathbb{V}(y, w)$, then X must meet this line at the point A.

Proof of Samuel's Theorem

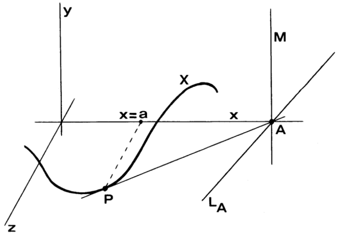

Figure 14. Proof of (3.9).

Let $\psi : X \to \mathbb{P}^2$ be the morphism given by projecting from A to the yz-plane. This is ramified everywhere, so it's image is either a point (in which case X is a line) or is inseparable (as separable morphisms have finitely many ramification points). Thus the restrictions of the functions y and z to X lie in $K(X)^p$, where $p = \text{char}\, k > 0$.

Proof of Samuel's Theorem

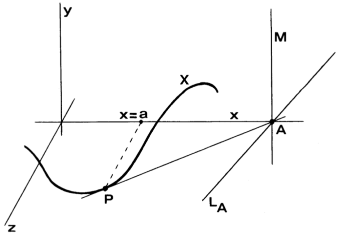

Figure 14. Proof of (3.9).

Let M be the line at infinity in the xy-plane and define $\phi : X \to M$ by setting $\phi(P)$ to be the intersection of the plane spanned by OP and the z-axis with M. ϕ is a morphism of degree $d = \deg X$, ramified exactly at the points of $X \setminus \{A\}$ lying in $(xz - \text{plane}) \cap \mathbb{A}^3$, since this is when the line $AP = L_P$ lies in the plane spanned by the z-axis and the line OP.

Proof of Samuel's Theorem

Idea: Apply Riemann-Hurwitz to the morphism ϕ.

Theorem (Riemann-Hurwitz)

Let $f : X \to Y$ be a finite separable morphism of curves of degree $n = \deg f$. Then

$$2g(X) - 2 = n(2g(Y) - 2) + \sum_{P \in X} \nu_P(dt/du),$$

where for each P, u is a local parameter at P and t is a local parameter at $f(P)$.

Proof of Samuel's Theorem

Suppose P is a ramification point with x-coordinate a. Take $u = x - a$ (where $a \in k^\times$) as a local coordinate at P on X and $t = y/x$ a local coordinate at A on M. We have $t = y(u+a)^{-1}$. As $y \in K(X)^p$ and char $k = p$, we know $dy/du = 0$, so

$$\frac{dt}{du} = -y(u+a)^{-2}.$$

But $u + a = x \in \mathcal{O}_{X,P}^\times$, hence $\nu_P(dt/du) = \nu_P(y)$.
If $P_1, \ldots, P_r$ are the ramification points of ϕ, then by Riemann-Hurwitz

$$2g(X) - 2 = -2d + \sum_{i=1}^{r} \nu_{P_i}(y).$$

Proof of Samuel's Theorem

Key Formula

$2g - 2 = -2d + \sum_{i=1}^{r} \nu_{P_i}(y)$, $d = \deg X$, $g = \text{genus}(X)$.

Case 1: Suppose $A \notin X$. Then we can compute d as the number of intersection points of the xz-plane (defined by $y = 0$) with X (with appropriate multiplicites), that is

$$d = \sum_{i=1}^{r} \nu_{P_i}(y).$$

Then $2g - 2 = -d$, which implies $g = 0$ and $d = 2$. Consequently $X \cong \mathbb{P}^1$ as abstract curves, and is embedded by a divisor D of degree 2. By Riemann-Roch $\dim |D| = 2$, so X is embedded as a conic in some $\mathbb{P}^2$. As X is strange then necessarily $\operatorname{char} k = 2$.

Proof of Samuel's Theorem

Key Formula

$2g - 2 = -2d + \sum_{i=1}^{r} \nu_{P_i}(y)$, $d = \deg X$, $g = \text{genus}(X)$.

Case 2: Suppose $A \in X$. As L_A is not in the xz-plane, the xz-plane meets X transversally at A, so computing d with the xz-plane gives

$$d = \sum_{i=1}^{r} \nu_{P_i}(y) + 1.$$

Hence $2g - 2 = -d - 1$, so $g = 0$ and $d = 1$, which implies X is a line. This completes the proof of Samuel's theorem.

Any Curve is Birational to a Plane Nodal Curve

Suppose $X \subset \mathbb{P}^3$ is a curve, $O \in \mathbb{P}^3 \setminus X$ is a point and $\phi : X \to \mathbb{P}^2$ denotes projection from O.

Proposition

ϕ is birational onto its image and $\phi(X)$ has at worst nodes as singularities, if and only if the following conditions are satisfied:

1. O lies on finitely many secants of X;
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Proposition

Such a point O always exists.

Any Curve is Birational to a Plane Nodal Curve

Proposition

Such a point O always exists.

Proof.

We may assume $X \subset \mathbb{P}^3$ is non-denegerate. Then X is not a conic or a line, so by Samuel's theorem cannot be strange, and so X has a secant which is not a multisecant and has a secant without coplanar tangent lines. Both of these are open conditions, hence there is a non-empty open subset of $X \times X$ consisting of pairs (P, Q) such that the secant line PQ is not a multisecant and doesn't have coplanar tangent lines.
The complement of this set is proper of dimension ≤ 1, so the union of the corresponding secant lines in $\mathbb{P}^3$ has dimension ≤ 2. We also have $\dim \operatorname{Tan} X \leq 2$, so there is an open subset of $\mathbb{P}^3$ consisting of points O satisfying conditions $(2), (3)$ and (4).

Any Curve is Birational to a Plane Nodal Curve

Proof (ctd).

We still need to show that O can be chosen to lie on finitely many secants of X. Recall that $\operatorname{Sec} X$ is locally the image of a morphism $(X \times X - \Delta) \times \mathbb{P}^1 \to \mathbb{P}^3$ sending the triple (P, Q, t) to the point on the line PQ at time $t \in \mathbb{P}^1$. If the dimension of $\operatorname{Sec} X$ is < 3 then we can choose O to lie on no secant. If $\dim \operatorname{Sec} X = 3$ then we apply the following result to see that there is an open subset of $\mathbb{P}^3$ consisting of points lying on finitely many secants of X. $\square$

Hartshorne Exercise II.3.7.

Let $f : X \to Y$ be a morphism of integral schemes which is dominant, of finite type and generically finite. Then there is an open dense subset $U \subset Y$ with $f^{-1}(U) \to U$ finite.

Any Curve is Birational to a Plane Nodal Curve

It seems natural to ask whether every plane nodal curve arises from projecting a non-singular curve in $\mathbb{P}^3$.

Hartshorne Exercise IV.3.7.

Assume char $k \neq 2$. Then the nodal curve $C = \mathbb{V}(xyz^2 + x^4 + y^4) \subset \mathbb{P}^2$ does not arise in this way.

Reason: Any (non-degenerate, non-singular) curve $X \subset \mathbb{P}^3$ projecting to C would be of degree 4 and genus 2. Suppose D is a hyperplane divisor on X, so $\deg D = 4$ and thus Riemann-Roch gives $h^0(X, \mathcal{O}_X(D)) = 3$. But the non-degeneracy of $X \subset \mathbb{P}^3$ gives $h^0(X, \mathcal{O}_X(D)) = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) = 4$, a contradiction.

The Canonical Embedding

Let X be a non-singular curve of genus g defined over an algebraically closed field. Here we always assume $g \geq 2$. We study in more detail the canonical linear system $|K|$.

Lemma

If $g \geq 2$ then $|K|$ has no base points.

Proof.

We need to show that for every $P \in X$, $\dim |K - P| = \dim |K| - 1$. We know $\dim |K| = h^0(X, \omega_X) - 1 = g - 1$. On the other hand, as X is not rational then $\ell(P) = 1$, so Riemann-Roch gives $|K - P| = g - 2$. □

More on Linear Systems

Definition

A g^r_d on X is a linear system of dimension r and degree d.

Example

Recall that X is said to be *hyperelliptic* if it admits a degree 2 morphism $X \to \mathbb{P}^1$. In the language of linear systems, X is hyperelliptic if and only if X has a g^1_2. With a bit more work, one can show that the g^1_2 on a hyperelliptic curve is unique.

Example

Suppose X is a curve of genus 2. Then $|K|$ is a g^1_2, so X is hyperelliptic (this follows easily from Riemann-Roch).

The Canonical Embedding

Proposition

Suppose X is a curve of genus $g \geq 2$. Then $|K|$ is very ample if and only if X is not hyperelliptic.

Proof.

Recall that $|K|$ is very ample if and only if for any two points $P, Q \in X$ we have $\dim |K - P - Q| = \dim |K| - 2 = g - 3$. By Riemann-Roch,

$$\dim |P + Q| - \dim |K - P - Q| = 3 - g,$$

so the question becomes determining when $\dim |P + Q| = 0$. If X is hyperelliptic then for any $P + Q \in g_2^1$ we have $\dim |P + Q| = 1 \neq 0$. Conversely, if $\dim |P + Q| > 0$ then there exists a non-constant section $f \in H^0(X, \mathcal{O}_X(P + Q))$; as X is not rational then f must have poles at P and Q, so f gives a degree 2 map $X \to \mathbb{P}^1$, and hence X is hyperelliptic. $\square$

The Canonical Embedding

Definition

Let X be a non-hyperelliptic curve of genus $g \geq 3$. The embedding $X \to \mathbb{P}^{g-1}$ (defined up to the action of $PGL(g, k)$) corresponding to $|K|$ is called the *canonical embedding*, and its image, a curve of degree $2g - 2$, is called a *canonical curve*.

Example

Let X be a non-hyperelliptic curve of genus 3. The canonical map embeds X as a quartic in $\mathbb{P}^2$.
Conversely, if $X \subset \mathbb{P}^2$ is a non-singular plane quartic then by the adjunction formula, $\omega_X \cong \mathcal{O}_X(1)$, so X is a canonical curve.

Genus 4 Canonical Curves

Example

Let X be a non-hyperelliptic curve of genus 4. The canonical map embeds X as a degree 6 curve in $\mathbb{P}^3$.

Let us explore this example in more detail. Suppose $X \subset \mathbb{P}^3$ is a canonical curve (so $\mathcal{O}_X(1)$ corresponds to the divisor K) with ideal sheaf $\mathcal{I}$. By twisting the ideal sheaf sequence by $\mathcal{O}_X(2)$ and taking cohomology, one sees that $h^0(\mathbb{P}^3, \mathcal{I}(2)) \geq 1$, so there is a degree 2 surface $Q \subset \mathbb{P}^3$ containing X. As X does not lie in any $\mathbb{P}^2$ then Q must be irreducible and reduced. Moreover Q is the only such surface containing X; if $X \subset Q'$ then $Q \cap Q'$ is a curve of degree 4 containing the degree 6 curve X, a contradiction.

Genus 4 Canonical Curves

Example

Let X be a non-hyperelliptic curve of genus 4. The canonical map embeds X as a degree 6 curve in $\mathbb{P}^3$.

We also have $h^0(\mathbb{P}^3, \mathcal{I}(3)) \geq 5$; the subspace consisting of the preceding quadratic form times a linear form is of dimension 4, so X is also contained in an irreducible cubic surface F. It follows that $X = Q \cap F$ is a complete intersection.

Conversely, if X is the complete intersection of a quadric and a cubic in $\mathbb{P}^3$ then X is a curve of genus 4 with $\omega_X \cong \mathcal{O}_X(1)$, by Exercise II.8.4.

The Canonical Map for Hyperelliptic Curves

Proposition

Let X be a hyperelliptic curve of genus $g \geq 2$.

1. X has a unique g_2^1 (let $f_0 : X \to \mathbb{P}^1$ be the corresponding morphism).
2. The canonical map f consists of f_0 followed by the Veronese embedding $\nu_{g-1} : \mathbb{P}^1 \to \mathbb{P}^{g-1}$.
3. Every effective canonical divisor on X is a sum of $g-1$ divisors in the unique g_2^1.