SIMPLICIAL HOMOLOGY AND THE EULER CHARACTERISTIC OF A SURFACE

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Recall the following theorem from lectures:

Theorem 1. *Let X be a compact connected topological surface with a given cellular decomposition. Then the Euler characteristic $\chi(X)$ is independent of the choice of decomposition.*

This note aims to explain very roughly where this result comes from (for more details, see C3.1 Algebraic Topology).

Let X be any topological space. You may have already come across the group $\pi_1(X, x_0)$, the *fundamental group* of (X, x_0). There are other groups that can be attached to X, the *(singular) homology groups* $H_n(X)$, which have the property that if X and Y are homeomorphic (or even homotopy equivalent) spaces then they have the same homology groups.

The definition of the groups $H_n(X)$ seems very unwieldy at first, but they are defined in such a way that it's clear they depend only on the homeomorphism class of X.[1] For each integer $n \geq 0$, let Δ^n be the *standard n-simplex*:

$$\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} : \text{each } t_i \geq 0, \sum_i t_i = 1 \right\}.$$

For each $i = 0, \ldots, n$ let Δ^n_i be the *ith facet* of Δ^n:

$$\Delta^n_i = \{(t_0, \ldots, t_n) \in \Delta^n : t_i = 0\}.$$

We identify Δ^n_i with Δ^{n-1} via

$$(t_0, \ldots, t_{n-1}) \in \Delta^{n-1} \leftrightarrow (t_0, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_{n-1}) \in \Delta^n_i \subset \Delta^n.$$

Definition 2. *We let $C_n(X)$ be the free $\mathbb{Z}$-module with basis given by the set of all continuous maps $\sigma : \Delta^n \to X$. Elements of $C_n(X)$ are known as* singular n-chains.

For all but the very simplest of spaces, $C_n(X)$ is a group that's far too big to work with in practice. However, by construction $C_n(X)$ depends only on the space X itself and not on any additional choices, such as the choice of a cellular decomposition.

Next, we define the homomorphism $\partial_n = \partial^X_n : C_n(X) \to C_{n-1}(X)$ by setting

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma|_{\Delta^n_i}$$

and extending $\mathbb{Z}$-linearly. Here we use our identification $\Delta^n_i \equiv \Delta^{n-1}$ to view $\sigma|_{\Delta^n_i}$ as a map $\Delta^{n-1} \to X$.

We now define two important subgroups of $C_n(X)$.

Definition 3. *The group of* n-cycles *is $Z_n(X) = \ker(\partial_n : C_n(X) \to C_{n-1}(X))$, and the group of* n-boundaries *is $B_n(X) = \mathrm{im}(\partial_{n+1} : C_{n+1}(X) \to C_n(X))$.*

A very crucial property of the maps ∂_n is that for all n, $\partial_n \circ \partial_{n+1} = 0$, so $B_n(X) \subset Z_n(X)$.

Definition 4. *The* nth (singular) homology group of X *is the group $H_n(X) = Z_n(X)/B_n(X)$.*

It turns out that H_0 and H_1 admit nice descriptions:

- $H_0(X)$ is isomorphic to $\mathbb{Z}^N$, where N is the cardinality of the set of path-components of X.

[1] That they depend only on the *homotopy class* of X requires much more work.

- If X is path-connected then $H_1(X)$ is isomorphic to the abelianisation[2] of $\pi_1(X)$; this is known as *Hurewicz's theorem*. More generally, for each $n \geq 1$ there is a natural homomorphism $\pi_n(X) \to H_n(X)$, and Hurewicz's theorem gives sufficient conditions for this map to be an isomorphism.

Let's ignore the question of how to actually compute homology groups in general, and instead explain why $H_n(X)$ is a homeomorphism invariant of X. Suppose $f : X \to Y$ is a continuous map between topological spaces. Then we have induced homomorphisms $f_*^n : C_n(X) \to C_n(Y)$ given by setting $f_*^n(\sigma) = f \circ \sigma$ and extending $\mathbb{Z}$-linearly. As an exercise, try to prove the following result yourself:

Lemma 5. *For each n, we have $f_*^n \circ \partial_{n+1}^X = \partial_{n+1}^Y \circ f_*^{n+1}$ as maps $C_{n+1}(X) \to C_n(Y)$.*

As a consequence of this lemma, we have $f_*^n(Z_n(X)) \subset Z_n(Y)$ and $f_*^n(B_n(X)) \subset B_n(Y)$, so f_*^n descends to the quotient H_n:

$$f_*^n : H_n(X) \to H_n(Y), \quad [\sigma] \mapsto [f \circ \sigma].$$

Now, if f is a homeomorphism with inverse $g : Y \to X$, then $g_*^n = (f_*^n)^{-1}$, both as maps $C_n(Y) \to C_n(X)$ and $H_n(Y) \to H_n(X)$. Therefore $H_n(X)$ and $H_n(Y)$ are isomorphic abelian groups.

From this point on, **assume X is a finite CW-complex of dimension k.**[3] Then:

(1) $H_n(X) = 0$ if $n < 0$ or if $n > k$.
(2) Each $H_n(X)$ is a finitely-generated abelian group, and in particular has a finite rank $b_n(X) = \operatorname{rank}(H_n(X)) = \dim_{\mathbb{Q}} H_n(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. $b_n(X)$ is known as the *nth Betti number* of X.

This allows us to define the *Euler characteristic* of X:

$$\chi(X) := \sum_i (-1)^i b_i(X) = b_0(X) - b_1(X) + \cdots + (\pm 1)^k b_k(X).$$

As $\chi(X)$ depends only on the homology groups of X, it is a homotopy/homeomorphism invariant of X.

Now suppose we are given a cellular decomposition of X, with m_n-many n-cells for $0 \leq n \leq k$.[4] Then the groups $H_n(X)$ can be computed using *cellular homology*.

Theorem 6. *There exists a chain complex*[5]

$$0 \longrightarrow C_k^{\mathrm{CW}}(X) \xrightarrow{\partial_k^{\mathrm{CW}}} C_{k-1}^{\mathrm{CW}}(X) \xrightarrow{\partial_{k-1}^{\mathrm{CW}}} \cdots \xrightarrow{\partial_2^{\mathrm{CW}}} C_1^{\mathrm{CW}}(X) \xrightarrow{\partial_1^{\mathrm{CW}}} C_0^{\mathrm{CW}}(X) \longrightarrow 0$$

whose homology is isomorphic to the singular homology $H_\bullet(X)$ of X. The group $C_n^{\mathrm{CW}}(X)$ is isomorphic to $\mathbb{Z}^{m_n}$ and is freely generated by the n-cells in X. Moreover

$$\chi(X) = \sum_i b_i(X) = \sum_{n=0}^{k} (-1)^n m_n.$$

Unlike the group of singular n-chains, the groups $C_n^{\mathrm{CW}}(X)$ and the morphisms ∂_n^{CW} can easily be computed from a given cellular decomposition, giving a much simpler way to compute the homology $H_\bullet(X)$ of X. In the case where X is a compact connected surface with a cellular decomposition with V vertices, E edges and F faces, we recover the well-known formula

$$\chi(X) = V - E + F.$$

We end with listing the homology groups of compact connected surfaces.

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[2] $G_{\mathrm{ab}} := G/[G, G]$.

[3] Don't worry about what this means - any "reasonable" compact space is homeomorphic to a finite CW-complex, including all compact connected smooth manifolds as well as (the geometric realisation of) any finite simplicial complex.

[4] The definition given in lectures for when $k = 2$ generalises in the way you'd expect it to.

[5] Meaning: each $C_n^{\mathrm{CW}}(X)$ is a $\mathbb{Z}$-module, each ∂_n^{CW} is a homomorphism and $\partial_n^{\mathrm{CW}} \circ \partial_{n+1}^{\mathrm{CW}} = 0$.

Proposition 7. *Let* $X = \#_g T^2$ *be the compact orientable surface of genus* $g \geq 0$ *and let* $Y = \#_h \mathbb{RP}^2$ *be the compact non-orientable surface of genus* $h \geq 1$. *Then*

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0, 2, \\ \mathbb{Z}^{2g} & n = 1, \\ 0 & \textit{otherwise}, \end{cases} \qquad H_n(Y) = \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z}^{h-1} \oplus \mathbb{Z}/2 & n = 1, \\ 0 & \textit{otherwise}. \end{cases}$$

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