Functors of Artin Rings and Schlessinger's Criterion

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Conventions

- k is an algebraically closed field.
- $\mathcal{C}$ is the category of local Artin k-algebras with residue field k.
- $\widehat{\mathcal{C}}(\supset \mathcal{C})$ is the category of complete local k-algebras with residue field k.
- We will follow Hartshorne's scheme-theoretic treatment - for a stack-theoretic treatment, see Alper's notes.

Example: Deformations of the Node

Suppose we are looking at deforming the node $xy = 0$ in $\mathbb{A}^2 = \operatorname{Spec} k[x, y]$. Let $X = \{xy - t = 0\} \subset \mathbb{A}^3$ and $T = \operatorname{Spec} k[t] = \mathbb{A}^1$.
Let X'/S be a flat deformation of the node over $S = \operatorname{Spec} k[[s]]$. Again for simplicity assume X' is defined by a single equation $g(x, y, s) = 0$, with $g \in k[[s]][x, y]$ and $g(x, y, 0) = xy$.

"Claim"

There exists a morphism $S \to T$ (given by a homomorphism $\phi : k[t] \to k[[s]]$, with $\phi(t) = R(s)$ satisfying $R(0) = 0$) such that $X \times_T S \cong X'$.

Strategy

Find $P(x, y, s)$ and $Q(x, y, s)$ reducing to x and y when $s = 0$, and a unit $U(x, y, s)$ (all in $k[[s]][x, y]$) reducing to 1 when $s = 0$, such that $g = U(PQ - R)$.

Example: Deformations of the Node

Write $R = \sum_{i \geq 1} a_i s^i$, $P = x + \sum_{i \geq 1} b_i s^i$, $Q = y + \sum_{i \geq 1} c_i s^i$, $U = 1 + \sum_{i \geq 1} u_i s^i$, $g = xy + \sum_{i \geq 1} g_i s^i \in k[x, y][[s]]$ (where $a_i \in k$, $b_i, c_i, u_i, g_i \in k[x, y]$).
From the degree 1 part of $g = U(PQ - R)$:

$$xc_1 + yb_1 - a_1 + xyu_1 = g_1.$$

This uniquely determines a_1, but there are several possibilities for b_1, c_1 and u_1 (fix such choices).
Continuing inductively degree-by-degree, the a_i's are always uniquely determined, but there are several possibilities for the other polynomials. In any case, this gives $P, Q, R, U \in k[x, y][[s]]$ such that $g = U(PQ - R)$.

Example: Deformations of the Node

Unfortunately this isn't what we're after, as $k[[s]][x, y] \subsetneq k[x, y][[s]]$. That is, we haven't actually found an isomorphism (of S-schemes) between $X' = \{g(x, y, s) = 0\}$ and $X \times_T S$ - rather we have only found an isomorphism of their formal completions along the closed fibre over $s = 0$. In fact there is the following result:

Proposition

The deformation $X = \{xy - t = 0\}/T$ is miniversal in the following sense:

1. For any other deformation X'/S with S the spectrum of a complete local ring, there is a morphism $\phi : S \to T$ such that X' and $X \times_T S$ become isomorphic after completing along the closed fibre over zero.
2. ϕ is not uniquely determined, but the induced map on Zariski tangent spaces is.

Functors of Artin Rings

For the rest of today: we're interested in studying functors $F : \mathcal{C} \to \mathbf{Set}$.
Note that any functor $F : \mathcal{C} \to \mathbf{Set}$ admits a canonical extension $\hat{F}$ to $\widehat{\mathcal{C}}$ by setting $\hat{F}(R) = \varprojlim F(R/\mathfrak{m}^n)$.

Definition

$F : \mathcal{C} \to \mathbf{Set}$ is *pro-representable* if it is isomorphic to $h_R := \mathrm{Hom}_{\mathrm{loc}_k}(R, -)$ for some complete local k-algebra R.

Formal Families

Suppose we have a morphism $\phi : h_R \to F$ with R a complete local k-algebra. For each n, we have a map $\mathrm{Hom}(R, R/\mathfrak{m}^n) \to F(R/\mathfrak{m}^n)$. Let $\xi_n \in F(R/\mathfrak{m}^n)$ be the image of the quotient map $R \to R/\mathfrak{m}^n$. The ξ_n's are compatible with respect to the natural maps $R/\mathfrak{m}^{n+1} \to R/\mathfrak{m}^n$, so $\xi = \{\xi_n\} \in \varprojlim F(R/\mathfrak{m}^n) = \hat{F}(R)$.

Definition

ξ is known as a *formal family* of F over the ring R.

Formal Families

Suppose instead we have $\xi = \{\xi_n\} \in \hat{F}(R)$. For any $A \in \mathcal{C}$ and $f \in h_R(A)$, the morphism f factors through some $R/\mathfrak{m}^n$ (as A is Artinian). Letting $\phi(f)$ be the image of ξ_n under $F(R/\mathfrak{m}^n \to A)$, we obtain a homomorphism $\phi : h_R \to F$. This sets up a well-defined inverse construction:

Proposition

There is a natural 1-1 correspondence between formal families of F over R and morphisms of functors $h_R \to F$.

From now on: we use the notation (R, ξ) (where $R \in \widehat{\mathcal{C}}$, $\xi \in \hat{F}(R)$) to denote a formal family of F.

Smooth Morphisms

Definition

A morphism of functors $G \to F$ is *smooth* if:

1. for every $A \in \mathcal{C}$, $G(A) \to F(A)$ is surjective.
2. For every surjection $B \to A$ in $\mathcal{C}$, $G(B) \to G(A) \times_{F(A)} F(B)$ is surjective:

$$\begin{tikzcd}
G(B) \arrow[dr, two heads, red] \arrow[drr, bend left] \arrow[ddr, bend right, two heads] & & \\
& G(A) \times_{F(A)} F(B) \arrow[r] \arrow[d] & G(A) \arrow[d, two heads] \\
& F(B) \arrow[r] & F(A)
\end{tikzcd}$$

Smooth Morphisms

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In the case where $G = h_R$, this says the following:

- Any $\eta \in F(A)$ is induced by some map $R \to A$.
- Given any map $R \to A$ inducing $\eta \in F(A)$, any surjection $B \to A$ and any $\theta \in F(B)$ mapping to η, there exists a lift $R \to B$ of $R \to A$ which induces θ.

Various Kinds of Formal Families

Fix $F : \mathcal{C} \to \mathbf{Set}$. Let $D = k[\varepsilon]/\varepsilon^2$.

Definition

(R, ξ) is a:

- *versal family* if $h_R \to F$ is smooth;
- *miniversal family* (a.k.a. *prorepresentable hull*) if in addition $h_R(D) \to F(D)$ is a bijection;
- *universal family* if $h_R \to F$ is an isomorphism.

Various Kinds of Formal Families

Proposition

If (R, ξ) is a versal family (corr. to $\phi : h_R \to F$), then for any other formal family (S, η) there is a ring homomorphism $f : R \to S$ such that $\hat{F}(R) \to \hat{F}(S)$ sends ξ to η.

Sketch proof.

Use the surjectivity of ϕ to lift each $\eta_n \in F(S/\mathfrak{m}_S^n)$ to $f_n \in h_R(S/\mathfrak{m}_S^n)$ with the property that the induced map $F(R/\mathfrak{m}_R^n) \to F(S/\mathfrak{m}_S^n)$ sends ξ_n to η_n. Smoothness allows the choices of f_n's to be made compatibly (arguing by induction on n), so they determine a homomorphism $f : R \to \varprojlim S/\mathfrak{m}_S^n \overset{S \in \widehat{\mathcal{C}}}{=} S$, which by construction sends ξ to η. $\square$

Various Kinds of Formal Families

Proposition

If (R, ξ) is a versal family (corr. to $\phi : h_R \to F$), then for any other formal family (S, η) there is a ring homomorphism $f : R \to S$ such that $\hat{F}(R) \to \hat{F}(S)$ sends ξ to η.

Exercise

Show that if (R, ξ) is a miniversal family, then the induced homomorphism $\bar{f} : R/\mathfrak{m}_R^2 \to S/\mathfrak{m}_S^2$ is independent of f. Show also that any miniversal family is unique up to (not necessarily unique) isomorphism.

Some More Terminology

We write $t_F = F(D)$ and $t_R = h_R(D) = (\mathfrak{m}/\mathfrak{m}^2)^\vee$. A *small extension* in $\mathcal{C}$ is a surjection whose kernel is a one-dimensional k-vector space.
If A' and A'' are local Artin k-algebras with morphisms to A, we set

$$A' \times_A A'' = \{(a', a'') \in A' \times A'' : f'(a) = f''(a'')\} \in \mathcal{C}.$$

The Cartesian product $M' \times_M M''$ of modules is defined analogously.

Existence of Versal Families

Proposition

If F has a versal family, then $F(k)$ consists of a single element.

Proof.

$\mathrm{Hom}(R, k) \to F(k)$ is surjective, and the domain has a single element. $\square$

Existence of Versal Families

Proposition

If F has a versal family, then for any morphisms $A' \to A$ and $A'' \to A$ in $\mathcal{C}$, the natural map

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$$

is surjective.

Proof.

Suppose $\eta' \in F(A')$, $\eta'' \in F(A'')$ map to $\eta \in F(A)$. These arise from homomorphisms $R \to A$, $R \to A'$ and $R \to A''$, and by smoothness these can be chosen to all be compatible, so that there is an induced homomorphism $R \to A' \times_A A''$. This defines an element $\xi \in F(A' \times_A A'')$ which restricts to η' and η''. $\square$

Existence of Miniversal Families

Proposition

If F has a miniversal family, then for any $A \in \mathcal{C}$, the natural map

$$F(A \times_k D) \to F(A) \times_{F(k)} F(D)$$

is bijective.

Proof.

Suppose θ_1, $\theta_2 \in F(A \times_k D)$ lie over $(\eta, \xi) \in F(A) \times_{F(k)} F(D)$. Pick $u : R \to A$ inducing η; as $A \times_k D = A[\varepsilon]/\varepsilon^2$ surjects onto A, we can lift u to v_1, $v_2 : R \to A \times_k D$ inducing θ_1 and θ_2 respectively. The projections of the v_i to D induce ξ so must coincide (since by hypothesis $t_R \xrightarrow{\sim} t_F$), and the projections to A are both u, so $v_1 = v_2$. Hence $\theta_1 = \theta_2$. □

Existence of Miniversal Families

Proposition

If F has a miniversal family, then t_F has the structure of a finite-dimensional k-vector space.

Proof.

Use the isomorphism $t_F \cong t_R = (\mathfrak{m}/\mathfrak{m}^2)^\vee$. $\square$

Exercise

Show that it is possible to define the vector space structure on t_F intrinsically.

Existence of Miniversal Families

Proposition

If F has a miniversal family, then for any small extension $p : A' \to A$ (with kernel I) and any $\eta \in F(A)$, the set $p^{-1}(\eta) \subset F(A')$ is acted on transitively by t_F.

Proof.

We have an isomorphism $A' \times_A A' \xrightarrow{\sim} A' \times_k k[I]$ given by $(x, y) \mapsto (x, x_0 + y - x)$, where $x_0 = x \bmod \mathfrak{m}$. Hence $F(A' \times_A A') = F(A' \times_k k[I]) \cong F(A') \times t_F$, so we have a surjective map

$$F(A') \times t_F \to F(A') \times_{F(A)} F(A')$$

which is an isomorphism on the first factor. If $\eta' \in p^{-1}(\eta)$, this gives a surjective map $\{\eta'\} \times t_F \to \{\eta'\} \times p^{-1}(\eta)$. $\square$

Existence of Miniversal Families

Proposition

If F has a miniversal family, then for any small extension $p : A' \to A$ (with kernel I) and any $\eta \in F(A)$, the set $p^{-1}(\eta) \subset F(A')$ is acted on transitively by t_F. If F is pro-representable then this action is bijective and $p^{-1}(\eta) \neq \emptyset$.

Indeed, one can check that the maps $F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$ are always bijective whenever F is pro-representable.

Schlessinger's Criterion

Theorem (Schlessinger, 1968)

F has a miniversal family if and only if:

(H0) *$F(k)$ has a single element.*

(H1) *For all small extensions $A'' \to A$ (and any $A' \to A$), the map $F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$ is surjective.*

(H2) *The map of (H1) is bijective whenever $A'' = D$ and $A = k$.*

(H3) *t_F is a finite-dimensional k-vector space.*

Moreover, F is pro-representable if and only if additionally:

(H4) *For every small extension $p : A'' \to A$ and every $\eta \in F(A)$ with $p^{-1}(\eta) \neq \emptyset$, t_F acts bijectively on $p^{-1}(\eta)$.*

Exercise

Show that (H4) implies that the map in (H1) is bijective. In particular, (H4) implies (H2).

Some Technical Results

Let A, A', A'' be rings with maps $A' \to A$, $A'' \to A$. Let M, M', M'' be modules over A, A', A'' respectively, with compatible maps $M' \to M$ and $M'' \to M$. Assume that $M' \otimes_{A'} A \xrightarrow{\simeq} M$ and $M'' \otimes_{A''} A \xrightarrow{\simeq} M$. Let $M^* = M' \times_M M''$, a module over $A^* = A' \times_A A''$.

Lemma

If $A'' \to A$ is surjective then $M^* \otimes_{A^*} A' \xrightarrow{\simeq} M'$.

Lemma

Assume $A'' \to A$ is surjective and has square-zero kernel J. Assume also M' and M'' are flat. Then M^* is flat over A^* and $M^* \otimes_{A^*} A'' \xrightarrow{\simeq} M''$.

Hilb is Pro-Representable

Let $X_0 \subset \mathbb{P}^n_k$ be a closed subscheme and let F be the corresponding Hilbert functor:

$$F(A) = \left\{ \begin{tikzcd} X \arrow[r, hook] \arrow[d, "\text{flat}"'] & \mathbb{P}^n_A \arrow[dl] \\ \operatorname{Spec} A & \end{tikzcd} : X \times_A k \cong X_0 \right\} \Big/ \simeq$$

Proposition

The functor F is pro-representable.

Sledgehammer proof.

Hilbert schemes are a thing! $\square$

Hilb is Pro-Representable, via Schlessinger

Proposition

The functor F is pro-representable.

(H0) $F(k) = \{X_0\}$ has one element.

(H1) Suppose we are given flat $X' \subset \mathbb{P}^n_{A'}$, $X'' \in \mathbb{P}^n_{A''}$, restricting to $X \subset \mathbb{P}^n_A$, where $A'' \to A$ is a small extension. Let X^* be X_0 with the sheaf of rings $\mathcal{O}_{X'} \times_{\mathcal{O}_X} \mathcal{O}_{X''}$. This is a closed subscheme of $\mathbb{P}^n_{A^*}$, where $A^* = A' \times_A A''$, is flat over A^*, and restricts to X', X'' respectively (by our earlier technical results).

(H3) $t_F \cong H^0(X, \mathcal{N}_{X_0/\mathbb{P}^n})$ is finite-dimensional.

Hilb is Pro-Representable, via Schlessinger

Proposition

The functor F is pro-representable.

(H4) is a consequence of the following result (Hartshorne Theorem 6.2):

Theorem

Given an extension $X \subset \mathbb{P}^n_A$ of X_0, the set of extensions of X in $\mathbb{P}^n_{A''}$ is a pseudotorsor under the action of the group $H^0(X_0, \mathcal{N}_{X_0/\mathbb{P}^n} \otimes_k J)$, where $J = \ker(A'' \to A)$.

Deformations of Schemes

Let X_0 be a scheme over k, and consider

$$F(A) = \left\{ (X, i) : \begin{tikzcd} X \arrow[d, "\text{flat}"'] & X_0 \arrow[l, hook', "i"'] \arrow[d] \\ \operatorname{Spec} A & \operatorname{Spec} k \arrow[l] \end{tikzcd}, i \otimes k : X_0 \xrightarrow{\simeq} X \otimes_A k \right\} \Big/ \sim$$

Proposition

F admits a miniversal family if X_0 is projective, or X_0 is affine with isolated singularities.

Warning

The functor F need not be pro-representable!

Deformations of Schemes

Proposition

F admits a miniversal family if X_0 is projective, or X_0 is affine with isolated singularities.

(H1) The argument is similar to that for the Hilbert functor.

(H2) This follows from the fact that if A is a local Artin k-algebra and if $f : X_1 \to X_2$ is a morphism of finite-type flat A-schemes which is an isomorphism over the central fibre, then f is an isomorphism.

Deformations of Schemes

Proposition

F admits a miniversal family if X_0 is projective, or X_0 is affine with isolated singularities.

(H3) We split into cases:

- if $X = \operatorname{Spec} B$ is affine, we have $t_F = T^1(B/k, B)$, and this is supported in the singular locus of X. Hence $\dim_k t_F < \infty$ if X additionally has isolated singularities.
- in general, there is an exact sequence

$$0 \to \underbrace{H^1(X_0, \mathcal{T}_{X_0})}_{\text{l.t. deformations}} \to \mathrm{Def}(X_0/k, D) \to H^0(X_0, \mathcal{T}^1(X_0/k, \mathcal{O}_{X_0}))$$

As X_0 is projective, it follows that $t_F = \mathrm{Def}(X_0/k, D)$ is of finite-dimension.

Deformations of Schemes

Theorem

With the same hypotheses as before, the functor F is pro-representable if and only if for each small extension $A' \to A$, and for each deformation X' over A' restricting to a deformation X over A, the natural map $\mathrm{Aut}(X'/X_0) \to \mathrm{Aut}(X/X_0)$ is surjective. In particular, if $H^0(X, \mathcal{T}_{X_0}) = 0$ then F is pro-representable.

Example

Let C be a non-singular curve of genus $g \geq 2$. Then $H^0(C, \mathcal{T}_C) = 0$ (as $\deg \mathcal{T}_C < 0$), so the corresponding functor F is pro-representable. As the obstruction space $H^2(C, \mathcal{T}_C)$ vanishes by dimension reasons, this implies that the associated formal moduli space (coming from pro-representability) is smooth.

Deformations of the Node Revisited

Let $X_0 = \{xy = 0\} \subset \mathbb{A}^2_k$, $X = \{xy - t = 0\} / \operatorname{Spec} k[t]/(t^2)$ and $X' = \{xy - t = 0\} / \operatorname{Spec} k[t]/(t^3)$. Automorphisms of X/X_0 all turn out to be of the form

$$x' = (1 + tf)x, \quad y' = (1 + tg)y, \qquad f, g \in k[x, y].$$

If this lifts to an automorphism of X'/X_0 then

$$x' = (1 + tf)x + f' t^2, \quad y' = (1 + tg) + g' t^2$$

for some $f', g' \in k[x, y]$, with $x'y' - t = u(xy - t)$ for some unit $u \in k[x, y, t]/(t^3)$. A calculation shows that this can only happen if $f + g \in (x, y)k[x, y]$. Taking $f = 1$, $g = 0$, we see that the associated automorphism doesn't lift, so F is not pro-representable.