Moduli Spaces of Hyperplanar Admissible Flags in Projective Space

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The Moduli Problem

The following definition is motivated by Bertini's theorem.

Definition. *A* hyperplanar admissible flag *of subschemes of* $\mathbb{P}(V)$ *is the data of a chain of closed subschemes* $X^0 \subset X^1 \subset \cdots \subset X^n \subset \mathbb{P}(V)$ *along with a chain of subspaces* $Z^0 \subset Z^1 \subset \cdots \subset Z^n = V$ *such that:*

- $\dim X^i = i$.
- $\operatorname{codim}(Z^i \subset V) = n - i$.
- $X^i = X^n \cap \mathbb{P}(Z^i)$ *for all* i.

Such a hyperplanar admissible flag is said to be stable *if* $X^0 \subset \mathbb{P}(Z^0)$ *is a Chow stable subscheme.*

Aim. After fixing the projective space $\mathbb{P}(V)$ and all discrete invariants, we wish to construct (quasi-projective) moduli spaces of stable hyperplanar admissible flags of subschemes of $\mathbb{P}(V)$, considered up to projective automorphisms.

Result

Theorem (C., [3])**.** *Fix a projective space* $\mathbb{P}(V)$, $\dim V > n+1$. *For all but finitely-many choices of discrete invariants, there exists a quasi-projective coarse moduli space parametrising all* $\operatorname{Aut}(\mathbb{P}(V))$*-equivalence classes of stable hyperplanar admissible flags for which the* X^i $(i \geq 1)$ *are non-singular varieties and for which all embeddings* $X^i \subset \mathbb{P}(Z^i)$ *are non-degenerate.*

The proof makes use of *non-reductive* Geometric Invariant Theory.

Non-Reductive GIT

The state-of-the-art for forming quotients by non-reductive group actions is the *$\hat{U}$-theorem* of Bérczi–Doran–Hawes–Kirwan [1] [2]:

Ingredients.

- A linear algebraic group $H = U \rtimes L$, $U = R_u(H)$, with *internal grading* $\lambda : \mathbb{G}_m \to Z(L)$.
  - Meaning: $\lambda(\mathbb{G}_m)$ acts on $\operatorname{Lie}(U)$ with positive weights.
- An H-equivariant closed immersion $X \subset \mathbb{P}(W)$ of H-schemes (for simplicity, assumed non-degenerate).

Notation.

- $W_{\min} \subset W$ is the λ-minimal weight space, $Z_{\min} = X \cap \mathbb{P}(W_{\min})$.
- $X_{\min} = \{x \in X : \lim_{t\to 0} \lambda(t) \cdot x \in Z_{\min}\}$, with retraction $p_\lambda : X_{\min} \to Z_{\min}$, $p_\lambda(x) = \lim_{t \to 0} \lambda(t) \cdot x$.
- $\overline{L} = L/\lambda(\mathbb{G}_m)$, $\hat{U} = U \rtimes \lambda(\mathbb{G}_m)$.

Assumptions.

- $k = \overline{k}$, $\operatorname{char}(k) = 0$.
- The linearization $\mathcal{O}_X(1)$ is *well-adapted*; the λ-weights of W satisfy $-\varepsilon < \omega_{\min} = \omega_0 < 0 < \omega_1 < \cdots < \omega_{\max}$ for $0 < \varepsilon \ll 1$.
- Appropriate assumptions on the stabilizers of $z \in Z_{\min}$ hold.
  - In the basic setting ("stability coincides with semistability for $\hat{U}$"), we assume $\operatorname{Stab}_U(z) = \{e\}$ for all $z \in Z_{\min}$, and $\dim \operatorname{Stab}_{\overline{L}}(z) = \{e\}$ for all $z \in Z_{\min}^{\overline{L}-ss}$.

Conclusion (in the basic setting).

- For sufficiently divisible $c > 0$, the rings $\bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(cn))^{\hat{U}}$ and $\bigoplus_{n\geq 0} H^0(X, \mathcal{O}_X(cn))^H$ are finitely-generated.
- $X_{\min}$ admits a locally trivial U-quotient $X_{\min} \to X_{\min}/U$.
- $X_{\min} \setminus UZ_{\min}$ admits a projective geometric $\hat{U}$-quotient $X /\!\!/ \hat{U}$.
- $p_\lambda^{-1}(Z_{\min}^{\overline{L}-ss}) \setminus UZ_{\min}^{\overline{L}-ss}$ admits a projective geometric H-quotient $X /\!\!/ H$.

Summary of the Proof

- We wish to form a quotient of a subscheme $\mathcal{S}'$ (parametrizing the stable hyperplanar flags appearing in $\mathcal{S}$) of

$$\mathcal{S} = \prod_{i=0}^{n} \operatorname{Hilb}(\mathbb{P}(V), \Phi_i) \times \prod_{i=0}^{n} \operatorname{Gr}_{\operatorname{codim}=n-i}(V)$$

under the action of $SL(V)$. Here $\Phi_m(t) = \frac{d}{m!}t^m + O(t^{m-1})$.
- We wish to construct a geometric quotient $\mathcal{S}'/SL(V)$.
- The $SL(V)$-linearization on $\mathcal{S}$ is chosen such that the Chow linearization of $\operatorname{Hilb}(\mathbb{P}(V), \Phi_0)$ is dominant.

Issue. If $i > 1$, determining in practice when a point $[Y \subset \mathbb{P}(V)]$ of $\operatorname{Hilb}(\mathbb{P}(V), \Phi_i)$ is GIT (semi)stable under the $SL(V)$-action is a difficult problem, even when Y is a smooth subvariety.

Observations.

- By *fixing* the flag $Z^0 \subset \cdots \subset Z^n = V$ and considering all possible $X^0 \subset \cdots \subset X^n$ whose associated chain of subspaces is the fixed flag $Z^\bullet$, we obtain a closed subscheme $\mathcal{S}_0 \subset \mathcal{S}$ acted on by the parabolic subgroup $P \subset SL(V)$ preserving the flag $Z^\bullet$.
- The parabolic P consists of block upper-triangular matrices, where all but one of the diagonal blocks (the first block, corresponding to Z^0) are of size 1×1.
- The existence of a geometric quotient $\mathcal{S}'/SL(V)$ is equivalent to the existence of a geometric quotient $\mathcal{S}'_0/P$.

To quotient out by P, a *quotienting-in-stages* procedure is used, following a construction of Hoskins and Jackson [4], with a slight modification to take into account that not all points of the various $Z_{\min}$'s will have trivial unipotent stabilizers (at the cost of obtaining only a *quasi-projective* geometric P-quotient). The row filtration of P is used to construct a sequence of internally graded groups $\hat{H}_i = U_i \rtimes (R_i \times \lambda_i(\mathbb{G}_m))$ $(i = 0, \ldots, n)$, with λ_i the quotient of a 1PS $\lambda^{[i]} : \mathbb{G}_m \to P$ with only two weights.
After quotienting out by $\hat{H}_0, \ldots, \hat{H}_n$ in turn, we will have quotiented out by the action of P.

Observations.

- For all $i > 0$, $R_i = \{e\}$. The only non-trivial R_i, namely $R_0 = SL(Z^0)$, sees only the factor $\operatorname{Hilb}(\mathbb{P}(Z^0), \Phi_0) \subset \operatorname{Hilb}(\mathbb{P}(V), \Phi_0)$, where (reductive) GIT stability is completely understood.
- The only flat limits of subschemes of $\mathbb{P}(V)$ which need to be examined (to check the relevant NRGIT stabilizer conditions are satisfied) are those under 1PS's with only two weights, and these are given by taking joins of the form $J(Y \cap \mathbb{P}(U), \mathbb{P}(U'))$ for $Y \subset \mathbb{P}(U \oplus U')$ non-degenerate.

The idea is to show that the stable hyperplanar admissible flags in $\mathcal{S}$ survive all $n+1$ quotients, after suitable twists of the linearization are taken to ensure (well) adaptedness at each stage; this can be ensured (for all but a finite set of "bad" discrete invariants) as all $\lambda^{[i]}$-flat limits of such flags have the same $\lambda^{[i]}$-weight. The Chow stability condition on X^0 is used to show that stable hyperplanar admissible flags survive the first quotient, as the Chow stability of X^0 implies GIT stability under the action of R_0. The triviality of the unipotent stabilisers at each stage follows from elementary considerations for projective automorphisms of joins $J(Y \cap \mathbb{P}(U), \mathbb{P}(U'))$.

References

[1] G. Bérczi, B. Doran, T. Hawes, and F. Kirwan. Geometric Invariant Theory for Graded Unipotent Groups and Applications. *Journal of Topology*, 11(3):826–855, 2018.

[2] G. Bérczi and F. Kirwan. Moment Maps and Cohomology of Non-Reductive Quotients. *arXiv preprint arXiv:1909.11495*, 2019.

[3] G. Cooper. Moduli Spaces of Hyperplanar Admissible Flags in Projective Space. *arXiv preprint arXiv:2304.02453*, 2023.

[4] V. Hoskins and J. Jackson. Quotients by Parabolic Groups and Moduli Spaces of Unstable Objects. *arXiv preprint arXiv:2111.07429*, 2021.