Introducing n-Category Theory

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Abstract

In this expository paper we introduce the concept of an *n-category*, an extension of ordinary category theory where we allow for morphisms between other morphisms, and more generally allow for k-morphisms between other $(k-1)$-morphisms. We give two constructions; in the first part we introduce *strict n-categories*, largely following the construction as presented in [Lei04] using globular sets. In the second part, we give a definition of a *weak n-category* (one of many competing definitions that exist in the literature) first given in [BD98], based off using opetopic sets. We also present various examples showing how n-categories naturally arise in various branches of mathematics. This document is the author's broadening essay for the Oxford Part C course *C2.7 Category Theory*.

Contents

1 Introduction

An ordinary category consists of a collection of objects together with a collection of morphisms between the objects, with composition of morphisms governed by a couple of axioms. Informally, a 2-category adds in a collection of "2-morphisms" between the morphisms of the underlying category (or "1-category"), a 3-category adds in a collection of "3-morphisms" between the 2-morphisms of the underlying 2-category, and so on, with the idea that n-morphisms allow us to make sense of the of two $(n-1)$-morphisms being isomorphic in an analogous way in how morphisms allow us to make sense of two objects of a category being isomorphic (informally, "essentially the same"). This is best illustrated with a few examples.

Example 1.1. Let **Cat** be the category of small categories; the objects of **Cat** consist of categories $\mathcal{C}$ such that $\mathrm{ob}(\mathcal{C})$ and $\hom(\mathcal{C})$ are both sets, and the morphisms of **Cat** consist of functors between categories. **Cat** is naturally a (strict) 2-category. We can compose the 2-morphisms in two different directions, either along the objects (corresponding to horizontal composition of natural transformations) or along the morphisms (corresponding to vertical composition of natural transformations). In this context, an isomorphism between 1-morphisms is exactly the same as a natural isomorphism between functors. The term *strict* here refers to the fact that vertical and horizontal compositions are associative, for each functor F the corresponding identity natural transformation 1_F is an identity for both horizontal and vertical composition, and the two compositions are compatible in the following sense. Consider the following diagram in **Cat**:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, "F'"{name=M}] \arrow[r, bend right=50, "F''"'{name=L}] & \mathcal{D} \arrow[r, bend left=50, "G"{name=U2}] \arrow[r, "G'"{name=M2}] \arrow[r, bend right=50, "G''"'{name=L2}] & \mathcal{E}
\arrow[from=U, to=M, phantom, "\Downarrow\phi"] \arrow[from=M, to=L, phantom, "\Downarrow\phi'"]
\arrow[from=U2, to=M2, phantom, "\Downarrow\eta"] \arrow[from=M2, to=L2, phantom, "\Downarrow\eta'"]
\end{tikzcd} \tag{1.1}$$

Then, if $\phi \circ \phi'$ denotes vertical composition and $\phi * \eta$ denotes horizontal composition, we have the *interchange law*

$$(\phi \circ \phi') * (\eta \cdot \eta') = (\phi * \eta) \circ (\phi' * \eta'),$$

which means that (1.1) defines an unambiguous composite 2-morphism.

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Example 1.2. Let X be a topological space. We can form the *fundamental ω-groupoid* $\Pi_\omega(X)$ of X as follows. The objects of $\Pi_\omega(X)$ are the points of X. The 1-morphisms of $\Pi_\omega(X)$ are paths $I = [0,1] \to X$, the 2-morphisms are homotopies of paths relative to end-points, the 3-morphisms are homotopies of homotopies of paths relative to end-points, and inductively the n-morphisms are homotopies between $(n-1)$-morphisms relative to end-points. Composition is given by pasting together paths and homotopies using double-speed reparametrisations. For example, the composition of paths $u : x \to y$ and $v : y \to z$ is the path

$$(u * v)(t) = \begin{cases} u(2t) & 0 \le t \le \frac{1}{2}, \\ v(2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$

As defined, composition in $\Pi_\omega(X)$ is not associative nor unital; rather composition is only associative and unital *up to homotopy.* In this sense $\Pi_\omega(X)$ is a *weak* ω-category.[1] Were we to terminate our construction after n stages, we instead obtain a weak n-category $\Pi_n(X)$, the *fundamental n-groupoid* of X.

One can play a similar game to define a weak ω-category structure on the category **Top** of topological spaces, with 1-morphisms being continuous maps between topological spaces, 2-morphisms being homotopies between continuous maps, 3-morphisms being homotopies of homotopies, and so on.

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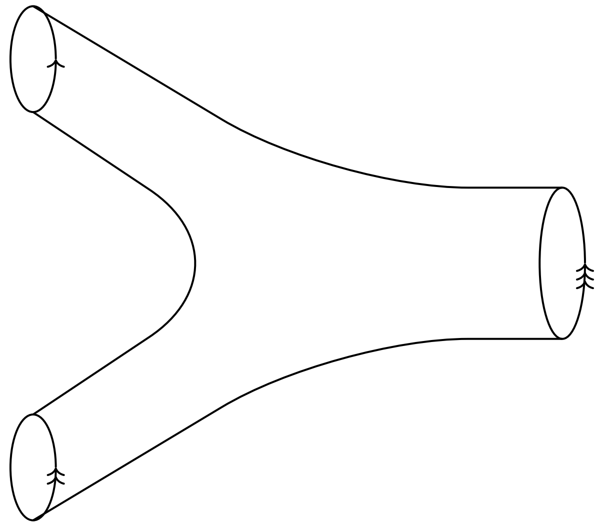
Example 1.3. We may define a weak ω-category of cobordisms, denoted **Cbd**, as follows. The objects of **Cbd** are 0-manifolds,[2] and for $n \ge 1$ the n-morphisms of **Cbd** consist of all n-manifolds with corners (we also allow the empty set). Composition is defined by gluing manifolds. For example, the cobordism

[1] ω denotes the first infinite ordinal.

[2] Here "manifold" means a compact, smooth, oriented manifold.



corresponds to the following diagram:

$$\begin{tikzcd} L \arrow[rr, bend left, "M"{name=U}] \arrow[rr, bend right, "M'"'{name=D}] & & L \arrow[Rightarrow, from=U, to=D, "N"] \end{tikzcd}$$

Here L is the empty set, M is the disjoint union of the two copies of S^1 on the left hand side, M' is the copy of S^1 on the right hand side, and N is the entire "pair of pants".

The fact that **Cbd** is a *weak* n-category is due to there being no natural way to impose associativity on the nose on the operation of gluing together manifolds.

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2 Strict n-Categories

Strict n-categories, whilst encountered less often than their weak counterparts, are easier (and less ambiguous) to define. Following [Lei04, Sections 1.3-1.4], we will present two equivalent constructions, the first using *enrichment* of monoidal categories, and the second using *globular sets*.

2.1 Enriching Monoidal Categories

Throughout we fix a monoidal category $(\mathcal{V}, \otimes, I)$.

Definition 2.1. *A* $\mathcal{V}$-graph X *is a set* X_0 *together with a family* $(X(x,y))_{x,y\in X_0}$ *of objects of* $\mathcal{V}$*. A* map of $\mathcal{V}$-graphs $f : X \to Y$ *consists of a function* $f_0 : X \to Y$ *together with maps*

$$f_{x,y} : X(x,y) \to Y(f_0(x), f_0(y))$$

for all $x, y \in X_0$*. The category of* $\mathcal{V}$*-graphs is denoted* $\mathcal{V}$-**Gph**.

Example 2.2. A **Set**-graph can be viewed as an ordinary directed graph, where the set $X(x,y)$ is viewed as labelling the set of directed edges from the vertex labelled x to the vertex labelled y.

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Definition 2.3. *A $\mathcal{V}$-enriched category consists of a $\mathcal{V}$-graph A together with families of maps*

$$\Big(A(b,c) \otimes A(a,b) \overset{\circ_{a,b,c}}{\to} A(a,c)\Big)_{a,b,c \in A_0}, \quad \Big(I \overset{i_a}{\to} A(a,a)\Big)_{a \in A_0}$$

in $\mathcal{V}$, such that for all $a, b, c, d \in A_0$ the following diagrams commute:

$$\begin{tikzcd}
(A(c,d) \otimes A(b,c)) \otimes A(a,b) \arrow[rr] \arrow[d, "\circ_{b,c,d} \otimes \mathrm{id}_{A(a,b)}"] & & A(c,d) \otimes (A(b,c) \otimes A(a,b)) \arrow[d, "\mathrm{id}_{A(c,d)} \otimes \circ_{a,b,c}"'] \\
A(b,d) \otimes A(a,b) \arrow[r, "\circ_{a,b,d}"] & A(a,d) & A(c,d) \otimes A(a,c) \arrow[l, "\circ_{a,c,d}"']
\end{tikzcd} \tag{2.1}$$

and

$$\begin{tikzcd}
A(b,b) \otimes A(a,b) \arrow[r, "\circ_{a,b,b}"] & A(a,b) & A(a,b) \otimes A(a,a) \arrow[l, "\circ_{a,a,b}"'] \\
I \otimes A(a,b) \arrow[u, "i_b \otimes \mathrm{id}_{A(a,b)}"] \arrow[ur] & & A(a,b) \otimes I \arrow[u, "\mathrm{id}_{A(a,b)} \otimes i_a"'] \arrow[ul]
\end{tikzcd} \tag{2.2}$$

A $\mathcal{V}$-enriched functor $F : A \to B$ *is given by a map of the underlying $\mathcal{V}$-graphs which commutes with the structure maps i_a and $\circ_{a,b,c}$. The category of $\mathcal{V}$-enriched categories is denotes $\mathcal{V}$-**Cat**.*

Example 2.4. A $(\mathbf{Set}, \times, *)$-enriched category is an ordinary (small) category. An $(\mathbf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$-enriched category is a pre-additive category (in the sense of homological algebra). A **Top**-enriched category is the same as an ordinary category whose hom-sets are all topological spaces.

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We state the following useful proposition. Though we omit the proof, the proposition follows as a straightforward consequence of the definitions.

Proposition 2.5. *If $\mathcal{V}$ admits finite products then so does the category $\mathcal{V}$-**Cat**.*

2.2 Definition of a Strict n-Category

Definition 2.6. *[Lei04, Definition 1.4.1] We define a sequence $(\mathbf{Str}\text{-}n\text{-}\mathbf{Cat})_{n \in \mathbb{N}}$ of categories inductively by defining*
$\mathbf{Str}\text{-}0\text{-}\mathbf{Cat} = \mathbf{Set}$ *and* $\mathbf{Str}\text{-}(n+1)\text{-}\mathbf{Cat} = (\mathbf{Str}\text{-}n\text{-}\mathbf{Cat})\text{-}\mathbf{Cat}$.

A strict n-category *is an object of* $\mathbf{Str}\text{-}n\text{-}\mathbf{Cat}$*, and a* strict n-functor *is a map in* $\mathbf{Str}\text{-}n\text{-}\mathbf{Cat}$.

Remark. By Proposition 2.5 each $\mathbf{Str}\text{-}n\text{-}\mathbf{Cat}$ is monoidal, so our definition makes sense.

Example 2.7. Taking $n = 2$ and unwinding the definition, a strict 2-category consists of a set A_0, a category $A(a,b)$ for each $a, b \in A_0$, an identity object i_a of $A(a,a)$ for each $a \in A_0$ and composition functors $\circ_{a,b,c}$ for all $a, b, c \in A_0$, all obeying the associativity and identity laws (2.1) and (2.2).

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Example 2.8. Let A_0 denote the class of topological spaces, and for spaces X and Y let $A(X, Y)$ denote the category whose objects are continuous maps $X \to Y$ and whose morphisms are homotopy classes of homotopies. This forms a (large) strict 2-category; the identity objects $1_X \in A(X, X)$ and the composition functors

$$A(Y, Z) \times A(X, Y) \to A(X, Z)$$

are the obvious ones.

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2.3 Strict n-Categories via Globular Sets

Though Definition 2.6 is reasonably concise, it is not at all explicit. Informally, a category can be thought of a directed graph with extra structure. Morally speaking then, a strict n-category ought to correspond to an n-dimensional analogue of a directed graph; this motivates the definition of a *globular set.*

Definition 2.9. *Let $n \in \mathbb{N}$. Let $\mathbb{G}_n$ be the category generated by objects and arrows*

$$\begin{tikzcd} n & n-1 \arrow[l, bend right, "\sigma_n"'] \arrow[l, bend left, "\tau_n"] & \cdots \arrow[l, bend right, "\sigma_{n-1}"'] \arrow[l, bend left, "\tau_{n-1}"] & 0 \arrow[l, bend right, "\sigma_1"'] \arrow[l, bend left, "\tau_1"] \end{tikzcd}$$

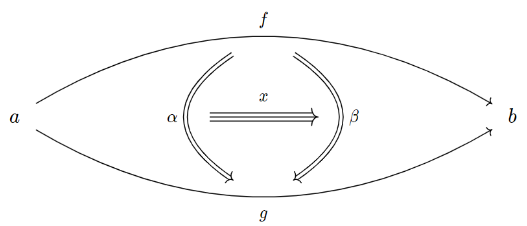
subject to the relations

$$\sigma_m \circ \sigma_{m-1} = \tau_m \circ \sigma_{m-1}, \quad \sigma_m \circ \tau_{m-1} = \tau_m \circ \tau_{m-1}, \quad m \in \{2, \dots, m\}.$$

*An n-*globular set *X is then a* **Set**-*valued presheaf on $\mathbb{G}_m^{\mathrm{op}}$. Elements of $X(m)$ are called m-*cells, *and* 0*-cells are called* objects.

Remark. When drawing n-globular sets, one thinks of the elements of $X(m)$ as labelling m-dimensional discs.[3] We will denote $s = X(\sigma) : X(m) \to X(m-1)$ and $t = X(\tau) : X(m) \to X(m-1)$, and call $s(x)$ the *source* of x and $t(x)$ the *target* of x.

Example 2.10. Let $x \in X(3)$ be a 3-cell of a globular set. Then $s(x) = \alpha$ and $t(x) = \beta$ are 2-cells with common source $f \in X(1)$ and common target $g \in X(1)$; moreover f and g have common source object a and common target object b. The 3-cell x corresponds to the following diagram:



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Given an n-globular set X and $0 \le p \le m \le n$, we denote

$$X(m) \times_{X(p)} X(m) = \{(x, y) \in X(m) \times X(m) : s^{m-p}(x) = t^{m-p}(y)\}.$$

The set $X(m) \times_{X(p)} X(m)$ is thought of as the set of all pairs of m-cells with the potential to be joined along some p-cell.

We now formulate a second definition of a strict n-category.

[3]Viewed in this way, an n-globular set is realised as an n-dimensional CW-complex, which explains the topological nomenclature.

Definition 2.11. *[Lei04, Definition 1.4.8] Let $n \in \mathbb{N}$. A* strict n-category *is an n-globular set A equipped with the following structure morphisms:*

- *For each $0 \leq p < m \leq n$ we have a composition map $\circ_p : A(m) \times A(m) \to A(m)$.*
- *For each $0 \leq p < n$ we have a map $i : A(p) \to A(p+1)$, $x \mapsto 1_x$.*

These maps satisfy the following axioms:

1. *For $x, y \in A(m)$ we have $s(x \circ_{m-1} y) = s(y)$, $t(x \circ_{m-1} y) = t(x)$, and for $p \leq m-2$ we have $s(x \circ_p y) = s(x) \circ_p s(y)$, $t(x \circ_p y) = t(x) \circ_p t(y)$.*
2. *If $p < n$ then $s(1_x) = t(1_x) = x$.*
3. *If $(x, y), (y, z) \in A(m) \times_{A(p)} A(m)$ then*
$$(x \circ_p y) \circ_p z = x \circ_p (y \circ_p z).$$
4. *If $0 \leq p < m \leq n$ and $x \in A(m)$ then*
$$i^{m-p}(t^{m-p}(x)) \circ_p x = x \circ_p i^{m-p}(s^{m-p}(x)) = x.$$
5. *(Binary Interchange) If $0 \leq q < p < m \leq n$ and $x, x', y, y' \in A(m)$ with*
$$(y', y), (x', x) \in A(m) \times_{A(p)} A(m), \quad (y', x'), (y, x) \in A(m) \times_{A(q)} A(m)$$
then
$$(y' \circ_p y) \circ_q (x' \circ x) = (y' \circ x') \circ_p (y \circ_q x).$$
6. *(Nullary Interchange) If $0 \leq q < p < n$ and $(x, y) \in A(p) \times_{A(q)} A(p)$ then*
$$1_x \circ_q 1_y = 1_{x \circ_q y}.$$

If A and B are strict n-categories then a strict n-functor *is a map $F : A \to B$ of the underlying globular sets which commutes with compositions and identities. The m-cells of a strict n-category A are also referred to as the* m-morphisms *of A, and the 0-cells are referred to as the* objects *of A.*

Proposition 2.12. *[Lei04, Proposition 1.4.9] The definitions of* **Str**-n-**Cat** *in Definitions 2.6 and 2.11 are equivalent.*

Sketch proof. For each $n \in \mathbb{N}$ we define the category n-**Gph** of *n-graphs* by setting 0-**Gph** $=$ **Set** and $(n+1)$-**Gph** $=$ (n-**Gph**)-**Gph**. Then an $(n+1)$-globular set X corresponds to the graph $(X(a,b))_{a,b \in X(0)}$ of n-globular sets, where $X(a,b)$ is the n-globular set given by
$$X(a,b)(m) = \{x \in X(m+1) : s^{m+1}(x) = a,\ t^{m+1}(x) = b\}.$$
By induction on n it follows that n-**Gph** $\simeq \mathrm{Fun}(\mathbb{G}_n^{\mathrm{op}}, \mathbf{Set})$, so the underlying graph structures are equivalent.

We now compare the respective structure morphisms. Suppose A is a strict $(n+1)$-category in the sense of Definition 2.11. Then for any $a, b \in A(0)$ the maps $\circ_p$ and $i : A(p) \to A(p+1)$, taken over $0 \leq p < n$, give a strict n-category structure on $A(a,b)$, and hence $(A(a,b))_{a,b \in A(0)}$ is a graph of strict n-categories. We can use the maps $\circ_0$ and $i : A(0) \to A(1)$ to give this graph the structure of a category enriched over **Str**-n-**Cat**. With a little more work, one can show that every (**Str**-n-**Cat**)-enriched category arises in this way, which by induction is enough to establish the proposition. $\square$

Remark. Either Definition 2.6 or 2.11 can be used to define the category **Str-ω-Cat** of *strict ω-categories*, by omitting the upper limit of n.

Example 2.13. Consider again Example 1.1. In the terminology of Definition 2.11, **Cat** is a strict 2-category with categories as 0-cells, functors as 1-cells and natural transformations as 2-cells. The composition $\circ_1$ on 2-cells corresponds to horizontal composition of natural transformations and $\circ_0$ corresponds to vertical composition.

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Example 2.14. Let $\mathcal{C}$ = **Str**-2-**Cat**. The objects of $\mathcal{C}$ are strict 2-categories and the 1-morphisms are 2-functors between strict 2-categories. The 2-morphisms of $\mathcal{C}$ are given by *(strict) 2-natural transformations* $\eta : F \Rightarrow G$ between 2-functors. η sends objects of $\mathcal{A}$ to 1-morphisms in $\mathcal{B}$ and morphisms in $\mathcal{A}$ to 2-morphisms in $\mathcal{B}$, and satisfies the following commutativity condition for every 2-morphism $\alpha : f \Rightarrow g$ in $\mathcal{A}$:

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd}
\qquad
\begin{tikzcd} {} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & {} \end{tikzcd}
\qquad
\begin{tikzcd} F(x) \arrow[r, bend left, "F(f)"] \arrow[r, bend right, "F(g)"'] \arrow[r, phantom, "\Downarrow F(\alpha)"] \arrow[d, "\eta(x)"'] \arrow[dr, phantom, "\eta(g) \quad \eta(f)"] & F(y) \arrow[d, "\eta(y)"] \\ G(x) \arrow[r, bend left, "G(f)"] \arrow[r, bend right, "G(g)"'] \arrow[r, phantom, "\Downarrow G(\alpha)"] & G(y) \end{tikzcd}$$

In much the same way how **Cat** is naturally a strict 2-category, $\mathcal{C}$ is naturally a strict 3-category, whose 3-morphisms consist of *modifications* $\mu : \eta \Rrightarrow \xi$ between 2-natural transformations with the same source and target. Here μ sends each object of $\mathcal{A}$ to a 2-morphism $\mu(x) : \eta(x) \Rightarrow \xi(x)$, such that for every 1-morphism $f : x \to y$ in $\mathcal{A}$ the diagram

$$\begin{tikzcd} \eta(x) \arrow[r, Rightarrow, "\mu(x)"] \arrow[d, Rightarrow, "\eta(f)"'] & \xi(x) \arrow[d, Rightarrow, "\xi(f)"] \\ \eta(y) \arrow[r, Rightarrow, "\mu(y)"'] & \xi(y) \end{tikzcd}$$

commutes. More generally, the category **Str**-n-**Cat** is naturally a strict $(n+1)$-category for every $n \in \mathbb{N}$.

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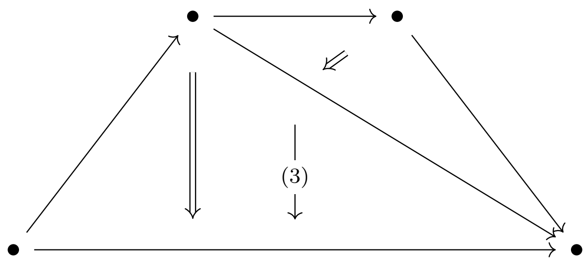
3 Weak n-Categories

Unlike with their strict counterparts, there are several proposed definitions in the literature of weak n-categories, not all of which are known to be equivalent to each other. A good overview of many of the proposed definitions can be found in Leinster's paper [Lei01a], and a more up-to-date list can be found on *nLab*.[4] We will give one construction due to Baez and Dolan [BD98] (see also [Bae97], which is an exposition based on [BD98]), using the algebra of *opetopes*. This gives a more flexible model of n-categories compared to the previous section; for example it allows us to consider diagrams of the following form:



The model also allows for composites to be unique only up to some form of equivalence, where the equivalence is given by a universal property. For a simple example explaining why such a notion is useful, consider sets X, Y and Z. In terms of the set-theoretic (von Neumann) description of Cartesian products, we have

$$(X \times Y) \times Z \neq X \times (Y \times Z),$$

and so the Cartesian product $\times$ is not naturally a product operation on **Set**. However, both $(X \times Y) \times Z$ and $X \times (Y \times Z)$ satisfy the universal property of the (category-theoretic) product of the sets X, Y and Z, and so should be considered as being "the same".

3.1 Operads and Opetopes

An *operad* is a category-theoretic gadget used to describe k-ary operations and their compositions, generalising many natural associativity properties found in many mathematical structures. A good introduction to the theory of operads can be found in [Lei04]; for our purposes we only need the basic definitions.

Definition 3.1. *Let S be a set. An S-*operad *O consists of the following data:*

1. *For any elements $x_1, \dots, x_k, x' \in S$ we have a set $O(x_1, \dots, x_k; x')$, whose elements are known as* operations*.*
2. *For any $f \in O(x_1, \dots, x_k; x')$ and any collection of elements $g_j \in O(x_{j1}, \dots, x_{j,i_j}; x_j)$, $j = 1, \dots, k$, there is an element*

$$f \circ (g_1, \dots, g_k) \in O(x_{11}, \dots, x_{1,i_1}, \dots, x_{k,i_k}; x').$$

3. *For all $x \in S$ there is an element $1_x \in O(x; x)$.*
4. *For any permutation $\sigma \in S_k = \mathrm{Sym}(\{1, \dots, k\})$ there is a map*

$$\sigma : O(x_1, \dots, x_k; x') \to O(x_{\sigma(1)}, \dots, x_{\sigma(k)}; x'), \quad f \mapsto f\sigma.$$

The above data is required to satisfy the following conditions:

(a) We have

$$f \circ (g_1 \circ (h_{11}, \dots, h_{1,i_1}), \dots, g_k \circ (h_{k,1}, \dots, h_{k,i_k})) = (f \circ (g_1, \dots, g_k)) \circ (h_{11}, \dots, h_{1,i_1}, \dots, h_{k,i_k})$$

whenever both sides make sense.

[4] See `https://ncatlab.org/nlab/show/n-category`.

(b) *For any* $f \in O(x_1, \dots, x_k; x')$ *we have*

$$f = 1_{x'} \circ f = f \circ (1_{x_1}, \dots, 1_{x_k}).$$

(c) *For any* $f \in O(x_1, \dots, x_k; x')$ *and any* $\sigma, \sigma' \in S_k$ *we have* $f(\sigma\sigma') = (f\sigma)\sigma'$*.*

(d) *For any* $f \in O(x_1, \dots, x_k; x')$*,* $\sigma \in S_k$ *and any collection of elements* $g_j \in O(x_{j1}, \dots, x_{j,i_j}; x_j)$*,* $j = 1, \dots, k$*, we have*

$$(f\sigma) \circ (g_{\sigma(1)}, \dots, g_{\sigma(k)}) = (f \circ (g_1, \dots, g_k))\rho(\sigma),$$

where ρ *is the obvious homomorphism* $S_k \to S_{i_1+\dots+i_k}$*.*

(e) *For any* $f \in O(x_1, \dots, x_k; x')$*, any collection of elements* $g_j \in O(x_{j1}, \dots, x_{j,i_j}; x_j)$*,* $j = 1, \dots, k$*, and any collection of permutations* $\sigma_j \in S_{i_j}$*,* $j = 1, \dots, k$*, we have*

$$f \circ (g_1\sigma_1, \dots, g_k\sigma_k) = (f \circ (g_1, \dots, g_k))\rho'(\sigma_1, \dots, \sigma_k),$$

where ρ' *is the obvious homomorphism* $\prod_{j=1}^k S_{i_j} \to S_{i_1+\dots+i_k}$*.*

The set S *is referred to as the set of* types *of* O*. We denote the set of all operations of* O *by* $\mathrm{elt}(O)$*.*

Operations in an operad are best visualised as trees, such as in the following diagram:

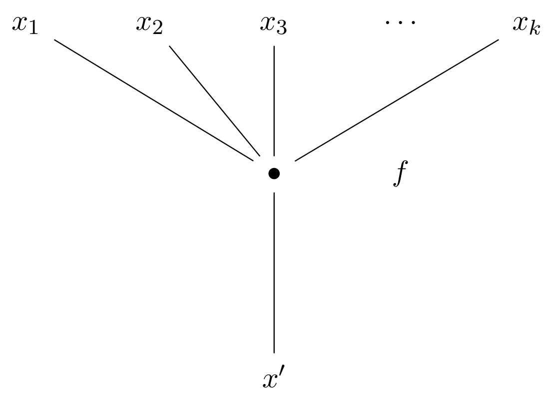


Figure 1: An operation $f \in O(x_1, \dots, x_k; x')$.

The operation $f \circ (g_1, \dots, g_k)$ is visualised in terms of trees as follows:

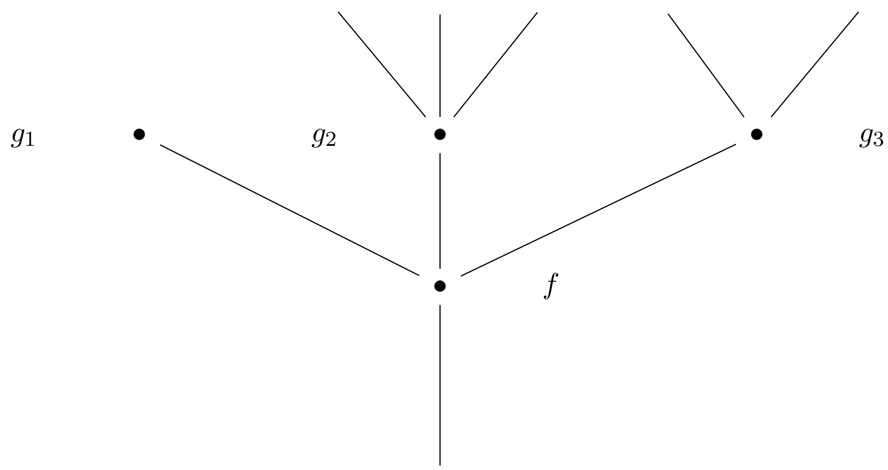


Figure 2: A composition $f \circ (g_1, g_2, g_3)$.

The axioms governing operads can be summarised by saying that identity operations act as identities for composition, composition is compatible with permuting the arguments of an operation, and that composition is "associative" (in the expected sense).

Similarly to how abstract groups are best understood in terms of their representations, operads O are best understood in terms of their corresponding *O-algebras*, where the operations of the operad are represented as concrete functions.

Definition 3.2. *Let O be an S-operad. An O-*algebra *A consists of the following data:*

1. *For each $x \in S$ we have a set $A(x)$.*
2. *For each operation $f \in O(x_1, \ldots, x_k; x')$ there is a function $\alpha(f) : \prod_{j=1}^{k} A(x_j) \to A(x')$.*

The above data is required to satisfy the following conditions:

(a) *We have*

$$\alpha(f \circ (g_1, \ldots, g_k)) = \alpha(f) \circ (\alpha(g_1) \times \cdots \times \alpha(g_k))$$

whenever both sides make sense.

(b) *For all $x \in S$ we have $\alpha(1_x) = \mathrm{id}_{A(x)}$.*

(c) *For any $f \in O(x_1, \ldots, x_k; x')$ and any $\sigma \in S_k$ we have*

$$\alpha(f\sigma) = \alpha(f)\sigma,$$

where $\sigma \in S_k$ acts on the function $\alpha(f)$ on the right by permuting its arguments.

Opetopes were first introduced in [BD98] as part of the construction of weak n-categories given in that paper. In order to define opetopes, we need the following result.

Proposition 3.3. *Let O be an S-operad. Then there exists a unique* $\mathrm{elt}(O)$*-operad O^+, called the* slice operad *of O, such that an algebra of O^+ is precisely an operad over O, that is an S-operad equipped with an operad homomorphism to O.*

Proof. See Section 3 of [BD98]. □

It is possible to give a more explicit description of O^+, again due to Baez and Dolan:

- The types of O^+ are the operations of O.
- The operations of O^+ are the *reduction laws* of O, which are equations stating that some composite of operations in O (possibly with the arguments permuted) equals some other operation.
- The reduction laws of O^+ correspond to all possible ways of combining reduction laws of O to get other reduction laws of O.

We can also iterate the slice construction n-times to obtain the operad O^{n+}. For convenience, we denote $O^{0+} = O$.

Definition 3.4. *[Bae97, Definition 4] [BD98, Definition 22] Let O be an S-operad. An n-dimensional O-*opetope *is a type of O^{n+}; equivalently an O-opetope is an operation of $O^{(n-1)+}$ in the case $n \geq 1$.*

We will only need to concern ourselves with one particular family of O-opetopes, namely the family that arises when $O = I$ is the simplest possible operad.

Definition 3.5. *[Bae97, Definition 5] The* initial untyped operad I *is the operad whose set of types* $S = \{x\}$ *consists of a single element, whose set of operations consists only of the unital operation* $1_x \in O(x; x)$*, and consisting of all possible reduction laws.*

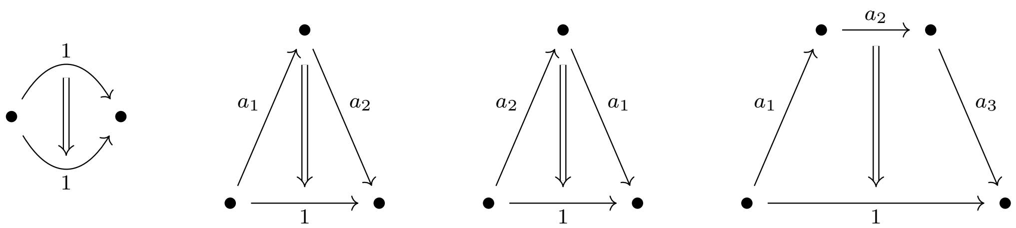
An n-dimensional opetope *is defined to be an n-dimensional I-opetope. A* morphism *between opetopes is given by a face inclusion of their corresponding directed simplicial realisations (see the following example). This defines a category denoted* **Ope**.

Example 3.6. We list some examples of n-dimensional opetopes for $n \leq 3$:

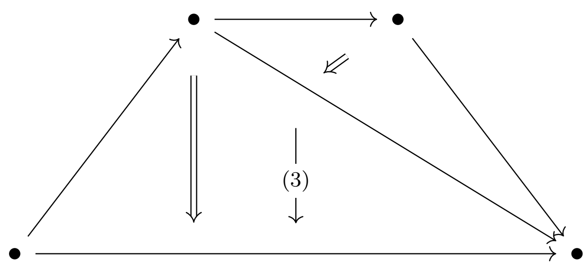
- There is only one 0-dimensional opetope, the set $\{x\}$, which we draw as $\bullet$.
- There is only one 1-dimensional opetope, corresponding to the single operation of I. We draw this as follows:

$$\bullet \xrightarrow{1} \bullet$$

- The 2-dimensional opetopes are the types of I^{++}, which correspond to the reduction laws of I. There are $k!$ distinct 2-dimensional opetopes with $k \geq 0$ infaces and one outface, since S_k acts freely on the set of k-ary operations of I^+. Some examples of distinct 2-dimensional opetopes are given below:



- The 3-dimensional opetopes are the types of I^{+++}, which describe all possible ways of composing 2-dimensional opetopes to get another 2-dimensional opetope. An example of a 3-dimensional opetope is given by the following diagram:



  Geometrically, the diagram depicts a 3-dimensional object whose front consists of two 3-sided, 2-dimensional "infaces" and one 4-sided, 2-dimensional "outface". The double arrows lie on the infaces whereas the arrow labelled "(3)" points from the union of the two infaces to the outface.

■

In general, an n-dimensional opetope has any number of $(n-1)$-dimensional infaces, glued together in a "tree-like" fashion, together with a single $(n-1)$-dimensional outface. The opetopes of dimension $n+1$ describe all possible compositions of n-dimensional opetopes.

3.2 Weak n-Categories as Opetopic Sets

According to the construction of Baez and Dolan, a weak n-category is an *opetopic set* satisfying certain properties.

Definition 3.7. *An* opetopic set *S is a* **Set***-valued presheaf on the category* **Ope**. *If s is an opetope, we call $S(s)$ the set of* cells *of S of shape s. If $\sigma \in S(s)$, $\tau \in S(t)$ are cells, then we call σ a* face *of τ if there exists a morphism $f : s \to t$ in* **Ope** *such that $\sigma = S(f)(\tau)$.*

If $j \geq 1$, we can represent a j-dimensional cell x of an opetopic set via the diagram

$$(a_1, \ldots, a_k) \xrightarrow{\quad x \quad} a'.$$

Here $a_1, \ldots, a_k$ are the infaces of x and a' is the outface of x; these are all cells of dimension $(j-1)$. If instead

$$(a_1, \ldots, a_k) \xrightarrow{\quad ? \quad} a'$$

is a configuration of $(j-1)$-cells, satisfying all of the boundary relations satisfied by the boundary of a cell, but with the corresponding j-cell missing, we call the configuration a j-dimensional *frame*. A *niche* is a frame whose outface is missing:

$$(a_1, \ldots, a_k) \xrightarrow{\quad ? \quad} ?$$

A *punctured niche* is a niche that is also missing one inface:

$$(a_1, \ldots, a_{i-1}, ?, a_{i+1} \ldots, a_k) \xrightarrow{\quad ? \quad} ?$$

If one of these configurations (frames, niches or punctured niches) can be extended to a cell x, we call x an *occupant* of that configuration. Occupants of the same frame are called *occupant-competitors*, and occupants of the same niche are called *niche-competitors*.

Example 3.8. Consider the following configuration of cells of an opetopic-set S:

$$\begin{tikzcd}
 & & c \arrow[dr, "h"] \arrow[dd, Rightarrow, "?"] & & \\
 & b \arrow[ur, "g"] & & d \arrow[dr, "k"] & \\
a \arrow[ur, "f"] \arrow[rrrr, "?"'] & & & & e
\end{tikzcd}$$

This is an example of a 2-dimensional niche, with corresponding cell-diagram

$$(f, g, h, k) \xrightarrow{\quad ? \quad} ?$$

■

We now define the notion of *universal* niche-occupants and *balanced* punctured niches. Whilst their definitions as stated call upon each other and themselves recursively, the reader can check that there is no logical circularity.

Definition 3.9. *Fix $n \in \mathbb{N}$. A j-dimensional niche-occupant*

$$(c_1, \ldots, c_k) \xrightarrow{\quad u \quad} d$$

*is said to be n-*universal *if any of the following conditions hold:*

- *$j > n$ and u is the only occupant if its niche.*
- *$j \leq n$ and for every frame-competitor d' of d, the $(j+1)$-dimensional punctured niches*

$$\begin{array}{ccc} ((c_1, \ldots, c_k) \overset{u}{\to} d, \quad d \overset{?}{\to} d') & \qquad & (d \overset{?}{\to} d', \quad (c_1, \ldots, c_k) \overset{u}{\to} d) \\ \Big\downarrow ? & & \Big\downarrow ? \\ (c_1, \ldots, c_k) \overset{?}{\to} d' & & (c_1, \ldots, c_k) \overset{?}{\to} d' \end{array}$$

  are n-balanced.

Definition 3.10. *Fix $n \in \mathbb{N}$. An m-dimensional punctured niche*

$$(a_1, \ldots, a_{i-1}, ?, a_{i+1} \ldots, a_k) \xrightarrow{\quad ? \quad} ?$$

*is said to be n-*balanced *if any of the following conditions hold:*

- *We have $m > n + 1$.*
- *The following conditions both hold:*
  1. *Any extension*

  $$(a_1, \ldots, a_{i-1}, ?, a_{i+1} \ldots, a_k) \xrightarrow{\quad ? \quad} b$$

  extends further to

  $$(a_1, \ldots, a_{i-1}, a_i, a_{i+1} \ldots, a_k) \xrightarrow{\quad u \quad} b$$

  with u n-universal in its niche.[5]

  2. *For any extension*

[5]This is best thought of as a generalisation of the notion of what it means for a functor to be essentially surjective.

$$(a_1, \ldots, a_{i-1}, a_i, a_{i+1} \ldots, a_k) \xrightarrow{\quad u \quad} b$$

with u n-universal in its niche, and for any frame-competitor a_i' of a_i, the $(m+1)$-dimensional punctured niches

$$\begin{array}{c} ((a_1, \ldots, a_{i-1}, a_i, a_{i+1} \ldots, a_k) \xrightarrow{u} b,\ a_i' \xrightarrow{?} a_i) \\ \Big\downarrow ? \\ (a_1, \ldots, a_{i-1}, a_i', a_{i+1} \ldots, a_k) \xrightarrow{?} b \end{array} \qquad \begin{array}{c} (a_i' \xrightarrow{?} a_i,\ (a_1, \ldots, a_{i-1}, a_i, a_{i+1} \ldots, a_k) \xrightarrow{u} b) \\ \Big\downarrow ? \\ (a_1, \ldots, a_{i-1}, a_i', a_{i+1} \ldots, a_k) \xrightarrow{?} b \end{array}$$

are n-balanced.[6]

Informally, a j-dimensional niche-occupant is n-universal if all of its niche-competitors uniquely factor through it, *up to equivalence*; this equivalence is made precise in [BD98, Proposition 55]. Universality allows us to define composition in opetopic sets.

Definition 3.11. *If*

$$(a_1, \ldots, a_k) \xrightarrow{\quad u \quad} b$$

is an n-universal occupant of a j-dimensional niche, we say that b is a composite *of $(a_1, \ldots, a_k)$.*

In Baez and Dolan's model of weak n-categories, the j-cells with $j \leq n$ play the role of j-morphisms, whereas the j-cells with $j > n$ give equations, equations between equations and so on. Universal occupants of a given niche are best understood as giving a "composition procedure" for that niche, with the outface corresponding to the composite of the infaces of the niche. Note that for $j \leq n$ there are possibly multiple n-universal occupants of a j-dimensional niche, though any n-universal occupant will be unique up to equivalence. However, as part of the definition of a weak n-category, for every $j > n$, every j-dimensional niche will have a unique universal occupant (which is automatically universal). We may then think of the universal occupant of a j-dimensional niche with $j > n$ as an equation stating that the composite of the equations corresponding to the infaces equals the equation corresponding to the outface.

We now arrive at Baez and Dolan's definition of a weak n-category.

Definition 3.12. *[Bae97, Definition 9] [BD98, Definition 41] A* weak n-category *is an opetopic set such that every niche has an n-universal occupant, and composites of n-universal cells are n-universal.*

Example 3.13. [BD98, Example 42] Let us see how the definition works in the case $n = 1$. Let $\mathcal{C}$ be a weak 1-category. Let $\mathcal{C}(0)$ (resp. $\mathcal{C}(1)$) denote the image under the presheaf $\mathcal{C}$ of the unique 0-dimensional (resp. 1-dimensional) opetope. In other words, $\mathcal{C}(0)$ is the set of all 0-cells and $\mathcal{C}(1)$ is the set of all 1-cells. Given 0-cells x and y, we denote the set of all frame-occupants of the frame

$$x \xrightarrow{\quad ? \quad} y$$

[6] This generalises the notion of what it means for a functor to be fully faithful.

as $\hom(x, y)$. The 2-dimensional niche

$$\begin{tikzcd} & {} \arrow[d, "?"] & \\ x \arrow[rr, "?"'] & {} & x \end{tikzcd}$$

has a unique occupant, which defines $1_x \in \hom(x, x)$:

$$\begin{tikzcd} & {} \arrow[d, "u"] & \\ x \arrow[rr, "1_x"'] & {} & x \end{tikzcd}$$

If z is a third 0-cell, the 2-dimensional niche

$$\begin{tikzcd} & (x \xrightarrow{f} y, \quad y \xrightarrow{g} z) \arrow[d, "?"] & \\ x \arrow[rr, "?"'] & {} & z \end{tikzcd}$$

has a unique occupant, which defines $f \circ g \in \hom(x, z)$:

$$\begin{tikzcd} & (x \xrightarrow{f} y, \quad y \xrightarrow{g} z) \arrow[d, "u"] & \\ x \arrow[rr, "f \circ g"'] & {} & z \end{tikzcd}$$

One can check using the 3-cells of $\mathcal{C}$ that the operations $x \mapsto 1_x$ and $\circ$ obey the usual axioms governing morphisms in a category, and so we have a (small) category $\mathcal{C}$ with $\mathrm{ob}(\mathcal{C}) = \mathcal{C}(0)$ and hom-sets $\hom(x, y)$.

∎

4 Conclusion and Further Reading

As mentioned in the previous section, in the literature there are several competing definitions of weak n-categories, and it not settled on which definition is the "correct" one. The classical definition of a weak 2-category, known as a *bicategory*, first appeared in a paper of Bénabou [Ben67]. Though we have not given a definition thus far (as to define this notion properly requires more technical work), it is possible to introduce *functors* between weak n-categories (in the sense of Baez and Dolan) as being a map $\phi : \mathcal{C} \to \mathcal{C}'$ between the underlying opetopic sets, sending each cell x of $\mathcal{C}$ to a cell $\phi(x)$ of $\mathcal{C}'$ of the same shape, with ϕ compatible with all face inclusions of the faces of the opetope corresponding to x. After modifying some of the details of the Baez-Dolan construction, in [Che03] Cheng proved that in the case $n = 2$ the Baez-Dolan-Cheng construction of weak n-categories, together with functors between weak n-categories, agrees with the category of bicategories together with lax functors.

Alternative opetopic-type constructions have been given by the work of Hermida-Makkai-Power [HMP00] and Leinster [Lei01b]; the Hermida-Makkai-Power approach also covers the case when $n = \omega$. Comparisons between these approaches are discussed in the papers [Che04a] and [Che04b]. As the reader may have noted, very little has been discussed in the way of other approached to n-categories, including the advantages and limitations the Baez-Dolan model has compared to other competing definitions. We leave it to the interested reader to make this journey for themselves. As noted earlier, the survey paper [Lei01a] is an excellent place to start for beginning to understand and compare the myriad definitions that currently exist in the literature.

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