Algebraic Stacks Reading Group - Week 4
A Stacky Situation

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Today's Talk

Today: we'll cover the following things:

- we'll introduce algebraic spaces and stacks;
- we'll show that quotient stacks are always algebraic;
- we'll show that the stack of quiver representations is algebraic;
- we'll show that the diagonal of an algebraic stack is always representable; and
- we'll explain how to construct algebraic spaces and stacks from groupoids of schemes.

Conventions

Following Alper, we make the following conventions:

- All categories of schemes are assumed to be endowed with the étale topology (same as in Olsson; the Stacks Project uses the fppf topology).
- Prestacks are as defined in Alper, i.e. categories fibred in groupoids.
- `\mathcal` letters denote prestacks and stacks; schemes and algebraic spaces are denoted by usual letters.
- The definition of algebraicity for stacks we give is different (but equivalent) to that given in Olsson or the Stacks Project.

Strongly Representable Morphisms between Prestacks

Definition

A morphism $f : \mathcal{X} \to \mathcal{Y}$ of prestacks is *representable by schemes* or *strongly representable* if for every morphism $V \to \mathcal{Y}$ from a scheme V, the fibre product $\mathcal{X} \times_{\mathcal{Y}} V$ is a scheme.

Definition

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of prestacks representable by schemes and let $\mathcal{P}$ be a property of morphisms of schemes. f is said to have property $\mathcal{P}$ if for every morphism $V \to \mathcal{Y}$ from a scheme V, the morphism $\mathcal{X} \times_{\mathcal{Y}} V \to V$ has property $\mathcal{P}$.

Algebraic Spaces

Definition

An *algebraic space* is a sheaf X on the site Sch (or Sch/S for a base scheme S) which admits a morphism $U \to X$ from a scheme U which is representable by schemes, surjective and étale. $U \to X$ is said to be an *étale presentation* of X. Morphisms of algebraic spaces are given by morphisms of sheaves.

Example

Any scheme is an algebraic space; the identity morphism gives an étale presentation.

Algebraic spaces are similar to schemes, in that (intuitively) a scheme is formed by gluing together affine schemes using the Zariski topology, whereas an algebraic space is formed by gluing affine schemes using the étale topology.

Representable Morphisms between Prestacks

Definition

A morphism $f : \mathcal{X} \to \mathcal{Y}$ of prestacks is *representable* if for every morphism $V \to \mathcal{Y}$ from a scheme V, the fibre product $\mathcal{X} \times_{\mathcal{Y}} V$ is an algebraic space.

Definition

Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable morphism of prestacks and let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source. f is said to have property $\mathcal{P}$ if for every morphism $V \to \mathcal{Y}$ from a scheme V and for every étale presentation $U \to \mathcal{X} \times_{\mathcal{Y}} V$, the morphism $U \to V$ has property $\mathcal{P}$.

Examples of such $\mathcal{P}$: surjective, étale, smooth,...

Algebraic Stacks

Definition

An *algebraic stack* is a stack $\mathcal{X}$ over the site Sch (or Sch/S) which admits a morphism $U \to \mathcal{X}$ from a scheme U which is representable, surjective and smooth. $U \to \mathcal{X}$ is said to be a *smooth presentation* of $\mathcal{X}$. Morphisms of algebraic stacks are given by morphisms of stacks.

Definition (Deligne & Mumford, 1969)

A *Deligne-Mumford stack* or *DM stack* $\mathcal{X}$ is an algebraic stack which admits an *étale presentation*; i.e. the smooth presentation $U \to \mathcal{X}$ can be chosen to be étale.

Exercise

Show that the 2-categories of algebraic spaces, algebraic stacks and DM stacks are all closed under taking fibre products.

The Heirarchy of Stacks

$$
\begin{tikzcd}
\boxed{\text{Schemes}} \arrow[d, "\subset", "\text{functor of points}"'] \\
\boxed{\text{Algebraic spaces over } \mathcal{S} = (\mathbf{Sch}/S)_{\text{Ét}}} \arrow[d, "\subset", "\text{stack associated to a sheaf}"'] \\
\boxed{\text{Deligne-Mumford stacks over } \mathcal{S}} \arrow[d, "\subset", "\text{étale morphisms are smooth}"'] \\
\boxed{\text{Algebraic stacks over } \mathcal{S}}
\end{tikzcd}
$$

Recap: G-Torsors

Definition

Let $\mathcal{S}$ be a site and let G be a sheaf of groups on $\mathcal{S}$. A *G-torsor* on $\mathcal{S}$ is a sheaf of sets P on $\mathcal{S}$ together with a left action $\sigma : G \times P \to P$ such that:

1. for every object $T \in \mathcal{S}$ there is a covering $\{T_i \to T\}$ such that $P(T_i) \neq 0$ for all i; and
2. the action map $G \times P \to P \times P$ is an isomorphism.

Example

Let G/S be a flat group scheme which is locally of finite presentation and let $P \to X$ be a principal G-bundle over an S-scheme X. Then we may consider $P \equiv h_P$ as a torsor on the site of X-schemes with the fppf topology for the group scheme $G_X = G \times_S X \to X$.

Quotient Stacks are Algebraic

Let U be an algebraic space over a base scheme S and let G/S be a smooth group scheme acting on U via the action $\sigma : G \times_S U \to U$. The stack $[U/G]$ is defined in one of the following two ways:

1. as the stack associated to the prestack $T \mapsto [U(T)/G(T)]$, with $[U(T)/G(T)]$ the quotient groupoid; or
2. as the stack whose objects over an S-scheme T are given by pairs of G_T-torsors $P \to T$ and G_T-equivariant morphisms of sheaves $P \to U_T$ (this is a stack by descent for G-torsors).

Exercise

Check that these two definitions are equivalent.

Quotient Stacks are Algebraic

Proposition

Let U be an algebraic space over a base scheme S and let G/S be a smooth group scheme acting on U. Then $[U/G]$ is an algebraic stack over S with smooth presentation $U' \to U \to [U/G]$ for any étale presentation $U' \to U$. Moreover $U \to [U/G]$ is a G-torsor, in the sense that for every morphism $T \to [U/G]$ from a scheme, $U \times_{[U/G]} T \to T$ is a G_T-torsor.

By Yoneda, we have a morphism $U \to [U/G]$ corresponding to the trivial torsor $G \times_S U$ over U. If $T \to [U/G]$ is a morphism from a scheme corresponding to the G_T-torsor P, one can check (exercise!) that $T \times_{[U/G]} U$ is isomorphic to P. If $U' \to U$ is an étale presentation, then $T \times_{[U/G]} U' = P \times_U U'$ is a sheaf which is étale-locally a scheme, as P is étale-locally on T a smooth group scheme; hence $T \times_{[U/G]} U'$ is an algebraic space. It also follows that $U' \to [U/G]$ is surjective and smooth.

Quiver Representations

Let $Q = (Q_0, Q_1, h, t)$ be a quiver. Fix a vector $d = (d_v) \in \mathbb{N}^{Q_0}$. We define a prestack $\mathcal{R} = \mathcal{R}ep(Q, d)$ as follows: for each scheme S, the objects of $\mathcal{R}(S)$ are given by collections $(\{\mathcal{E}_v\}, \{\phi_a\})$, where $\mathcal{E}_v$ is a locally free $\mathcal{O}_S$-module of rank d_v and where $\phi_a : \mathcal{E}_{ta} \to \mathcal{E}_{ha}$ is a morphism of $\mathcal{O}_S$-modules. A morphism $(\{\mathcal{E}'_v\}, \{\phi'_a\}) \to (\{\mathcal{E}_v\}, \{\phi_a\})$ over a morphism of schemes $f : S' \to S$ is given by isomorphisms $\alpha_v : f^*\mathcal{E}_v \to \mathcal{E}'_v$ such that for each $a \in Q_1$ we have $\phi'_a \circ \alpha_{ta} = \alpha_{ha} \circ f^*\phi_a$.

By using descent results in e.g. [Stacks, Tag 05AY] one can check that $\mathcal{R}$ is in fact a stack.

Quiver Representations

Form the affine scheme $R = \prod_{a \in Q_1} \underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathcal{O}_{\mathbb{Z}}^{\oplus d_{ta}}, \mathcal{O}_{\mathbb{Z}}^{\oplus d_{ha}})$. The group scheme $G = \prod_{v \in Q_0} GL(d_v)$ acts on R by conjugation:

$$(g \cdot \phi)_a = g_{ha} \circ \phi_a \circ g_{ta}^{-1}.$$

Proposition

There is an isomorphism of stacks $\mathcal{R}ep(Q, d) \cong [R/G]$. In particular, the stack $\mathcal{R}ep(Q, d)$ is algebraic.

Quiver Representations

Proposition

There is an isomorphism of stacks $\mathcal{R}ep(Q, d) \cong [R/G]$. In particular, the stack $\mathcal{R}ep(Q, d)$ is algebraic.

The argument proceeds as follows:

1. There exists a fully faithful morphism of prestacks $\alpha : [R/G]^{\mathrm{pre}} \to \mathcal{R}$.
2. As $\mathcal{R}$ is a stack, the universal property of stackification gives a morphism of stacks $\Phi : [R/G] \to \mathcal{R}$, which is fully faithful (as α and stackification are fully faithful).
3. The morphism Φ is essentially surjective.

Quiver Representations

If S is a scheme then giving a morphism $S \to R$ is equivalent to giving an element of $\prod_{a \in Q_0} \mathrm{Hom}_S(\mathcal{O}_S^{\oplus d_{ta}}, \mathcal{O}_S^{\oplus d_{ha}})$, i.e. a free representation of Q over S with a fixed choice of basis. This defines the morphism $\alpha : [R/G]^{\mathrm{pre}} \to \mathcal{R}$.

Suppose we have an automorphism τ of a free representation $\mathcal{F} = (\{\mathcal{O}_S^{\oplus d_v}\}, \{\phi_a\}) \in \mathcal{R}(S)$. τ gives for each $v \in Q_0$ a uniquely determined element of $\mathrm{Aut}(\mathcal{O}_S^{\oplus d_v}) = GL(d_v)(S)$. Consequently we have a uniquely determined element of $G(S)$. It follows that α is fully faithful.

We now show that Φ is essentially surjective. For this, it is enough to show that for any representation $\mathcal{F} = (\{\mathcal{E}_v\}, \{\phi_a\}) \in \mathcal{R}(S)$ there exists an étale cover $\{S_i \to S\}$ such that each $\mathcal{F}|_{S_i}$ lies in the image of α. But we can choose a Zariski cover $S = \bigcup_i S_i$ of S such that each $\mathcal{E}_v$ is trivial over each S_i.

Quiver Representations

Given a stability parameter $\theta \in \mathbb{Z}^{Q_0}$ with $\sum_{v \in Q_0} \theta_v d_v = 0$, one may form the open substack $\mathcal{R}_{\mathbb{C}}^{\theta, ss} = [R_{\mathbb{C}}^{ss}(\chi_\theta)/G_{\mathbb{C}}]$ of $\mathcal{R}_{\mathbb{C}} = \mathcal{R} \times \operatorname{Spec} \mathbb{C}$ consisting of θ-semistable representations of Q. There is also the King moduli space $\mathcal{M}^{\theta, ss}(Q, d) \in \mathrm{Sch}/\mathbb{C}$ of θ-semistable representations of Q, formed by taking a GIT quotient of $R_{\mathbb{C}}$.

The natural morphism $\mathcal{R}_{\mathbb{C}}^{\theta, ss} \to \mathcal{M}^{\theta, ss}(Q, d) = R_{\mathbb{C}} /\!\!/_{\chi_\theta} \overline{G}_{\mathbb{C}}$ is an example of a *good moduli space* in the sense of Alper. Here $\overline{G}$ is the quotient of G by the kernel of the induced representation $G \to \mathrm{Aut}(R)$.

Quiver Representations

Every object of $\mathcal{R} = \mathcal{R}ep(Q, d)$ contains an embedded copy of $\mathbb{G}_m$ in its stabiliser, corresponding to the kernel of $G \to \mathrm{Aut}(R)$, compatible with pullbacks; consequently $\mathcal{R}$ has no hope of being DM. It is possible to produce a new algebraic stack $\mathcal{R}^{\mathbb{G}_m}$ which has the same objects as $\mathcal{R}$ but with $\mathrm{Aut}_{\mathcal{R}^{\mathbb{G}_m}} = \mathrm{Aut}_{\mathcal{R}}/\mathbb{G}_m$ for each geometric point $\operatorname{Spec} k \to \mathcal{R}^{\mathbb{G}_m}$, and which satisfies a certain universal property (amongst other things).

$\mathcal{R}^{\mathbb{G}_m}$ is called the $\mathbb{G}_m$-*rigidification* (in the sense of Abramovich, Corti and Vistoli) of $\mathcal{R}$; in this case $\mathcal{R}^{\mathbb{G}_m} \simeq [R/\overline{G}]$.

Proposition

The algebraic stack $(\mathcal{R}^{\theta,ss})^{\mathbb{G}_m}$ is a DM-stack if and only if all θ-semistable representations of Q with dimension vector d are θ-stable.

Moduli Stack of Smooth Curves

Let $g \geq 2$ be an integer and let $\mathcal{M}_g$ be the stack parametrising families of smooth curves of genus g. Let $H = \mathrm{Hilb}(\mathbb{P}^{5g-6}_{\mathbb{Z}}, P)$ be the Hilbert scheme of genus g curves in $\mathbb{P}^{5g-6}_{\mathbb{Z}}$ of degree $6(g-1)$. Then, for an appropriate locally closed subscheme $H' \subset H$ parametrising smooth, tri-canonically embedded curves in $\mathbb{P}^{5g-6}_{\mathbb{Z}}$, there is an isomorphism

$$\mathcal{M}_g \cong [H'/PGL(5g-6)],$$

and there is a natural morphism $\mathcal{M}_g \to M_g$ to a scheme which is a (stack-theoretic) *coarse moduli space*. For more details see Alper §2.1.5 or Olsson §8.4.3.

Classifying Stacks

Definition

Let G/S be a smooth group scheme. The quotient stack $B_S G = [*/G]$ is known as the *classifying stack* of G; by definition it parametrises pairs (T, P) where T is an S-scheme and P is a G_T-torsor.

As quotient stacks, classifying stacks are always algebraic.

Example

Let k be a field.

- $B_k GL(n)$ parametrises k-schemes T together with locally free rank n sheaves $\mathcal{E} \in \mathrm{Vect}(T)$.
- $B_k \mu_n$ parametrises triples (T, L, α), where T is a k-scheme, L is a line bundle on T and $\alpha : \mathcal{O}_T \to L^{\otimes n}$ is an isomorphism.

The Diagonal

Proposition

The diagonal morphism of an algebraic stack is representable. The diagonal morphism of an algebraic space is representable by schemes.

As we will see later, the diagonal encodes the "stackiness" of $\mathcal{X}$!

The Diagonal

To prove the proposition for algebraic spaces, we need to show that for any algebraic space X and any morphism $T \to X \times X$ from a scheme, the sheaf $Q_T = X \times_{X \times X} T$ is a scheme. Pick an étale presentation $U \to X$ and consider the Cartesian cube

$$\begin{tikzcd}
& Q_{T'} \arrow[rr] \arrow[dl] \arrow[dd] & & T' \arrow[dl] \arrow[dd, "\text{surj+ét}"] & \\
R = U \times_X U \arrow[rr] \arrow[dd] & & U \times U \arrow[dd] & & \text{schemes} \\
& Q_T \arrow[rr, "\text{ét pres}"] \arrow[dl] & & T \arrow[dl, "\text{given}"] & \text{alg. spaces} \\
X \arrow[rr, "\Delta_X"'] & & X \times X & &
\end{tikzcd}$$

The Diagonal

Δ_X is a monomorphism (as it is a monomorphism at the level of sets), hence so is $R \to U \times U$; in particular $R \to U \times U$, hence $Q_{T'} \to T$, are separated and locally quasi-finite. By the following corollary of *effective descent for separated and locally quasi-finite morphisms* it follows that Q_T is a scheme:

Lemma (Alper Corollary B.3.6)

Let $f : X \to Y$ be a faithfully flat morphism of schemes which is either quasi-compact or locally of finite presentation. Let $Q \to Y$ be a morphism of algebraic spaces. Then, if $Q_X = Q \times_Y X$ is a scheme, so is Q.

The Diagonal

Now let $\mathcal{X}$ be an algebraic stack with smooth presentation $U \to \mathcal{X}$, and suppose we are given a morphism from a scheme $T \to \mathcal{X} \times \mathcal{X}$. Form the algebraic space $R = U \times_{\mathcal{X}} U$. Let T_1 be the algebraic space $T_1 = T \times_{\mathcal{X} \times \mathcal{X}} (U \times U)$, and pick an étale presentation $T_2 \to T_1$. $T_2 \to T$ is a surjective smooth morphism of schemes, so admits a section after an étale cover $T' \to T$ (see [Stacks, Tag 055U]). The composition $T' \to T_2 \to T_1 \to U \times U$ is then a lift of $T \to \mathcal{X} \times \mathcal{X}$.

As before, form the stacks $Q_T = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} T$ and $Q_{T'} = Q_T \times_T T' = R \times_{U \times U} T'$; $Q_{T'}$ is an algebraic space, as the fibre product of algebraic spaces. $T' \to T$ is étale and surjective, so $Q_{T'} \to Q_T$ is étale, surjective and representable by schemes. If $V \to Q_{T'}$ is an étale presentation, it follows that Q_T is an algebraic space with étale presentation $V \to Q_{T'} \to Q_T$.

The Diagonal

Corollary

Any morphism from a scheme to an algebraic stack (resp. space) is representable (resp. strongly representable).

To see this, let $T_1 \to \mathcal{X}$ and $T_2 \to \mathcal{X}$ be morphisms from schemes; then we have a Cartesian diagram

$$\begin{tikzcd} T_1 \times_{\mathcal{X}} T_2 \arrow[r] \arrow[d] & T_1 \times T_2 \arrow[d] \\ \mathcal{X} \arrow[r, "\Delta_{\mathcal{X}}"'] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

As $\Delta_{\mathcal{X}}$ is representable, $T_1 \times_{\mathcal{X}} T_2$ is an algebraic space.

The Diagonal

Proposition

1. If $\mathcal{X} \to \mathcal{Y}$ is a representable morphism of algebraic stacks then $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is strongly representable.
2. If $\mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks then $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable.

We recover our previous representability results by taking $\mathcal{Y} = \operatorname{Spec} \mathbb{Z}$.

Stabilisers of Points

Suppose T is a scheme and $x, y \in \mathcal{X}(T)$ are objects. Then there is a Cartesian diagram

$$\begin{tikzcd}
\underline{\mathrm{Isom}}_{\mathcal{X}(T)}(x,y) \arrow[r] \arrow[d] & T \arrow[d, "{(x,y)}"] \\
\mathcal{X} \arrow[r, "\Delta_{\mathcal{X}}"'] & \mathcal{X} \times \mathcal{X}
\end{tikzcd}$$

where $\underline{\mathrm{Isom}}_{\mathcal{X}(T)}(x,y) : (T' \xrightarrow{f} T) \mapsto \mathrm{Mor}_{\mathcal{X}(T')}(f^*x, f^*y)$. The representability of $\Delta_{\mathcal{X}}$ implies that $\underline{\mathrm{Isom}}_{\mathcal{X}(T)}(x,y)$ is always an algebraic space. Taking a field-valued point $x \in \mathcal{X}(k)$, we define $G_x = \underline{\mathrm{Isom}}_{\mathcal{X}(k)}(x,x)$. G_x is a priori a group algebraic space, however it turns out that G_x is always a group scheme if $\mathcal{X}$ has quasi-separated diagonal.

Characterising DM Stacks and Algebraic Spaces

Theorem

Let $\mathcal{X}$ be a Noetherian algebraic stack. The following are equivalent:

- *$\mathcal{X}$ is Deligne-Mumford;*
- *the diagonal $\Delta_{\mathcal{X}}$ is unramified; and*
- *every geometric point of $\mathcal{X}$ has a finite and reduced stabiliser group.*

Theorem

Let $\mathcal{X}$ be a Noetherian algebraic stack. The following are equivalent:

- *$\mathcal{X}$ is an algebraic space;*
- *the diagonal $\Delta_{\mathcal{X}}$ is a monomorphism; and*
- *every geometric point of $\mathcal{X}$ has a trivial stabiliser group.*

An Algebraic Space that isn't a Scheme

Let $\mathbb{Z}$ act on $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[x]$ by translation. The resulting quotient stack is an algebraic space $X = \mathbb{A}^1/\mathbb{Z}$ (X is an example of an algebraic space defined by an étale equivalence relation).

Claim

X is not a scheme.

To see this, if X were a scheme then it would have a non-empty affine open $U = \operatorname{Spec} A \subset X$. As the étale cover $\mathbb{A}^1 \to X$ is then a categorical quotient of schemes, pulling back along this map gives an inclusion $A \hookrightarrow \mathbb{C}(\mathbb{A}^1)^{\mathbb{Z}} = \mathbb{C}(x)^{\mathbb{Z}}$. But the elements of $\mathbb{C}(x)^{\mathbb{Z}}$ are precisely the constants (any non-constant $\mathbb{Z}$-invariant rational function would have infinitely many zeros and poles), so $A = \mathbb{C}$. This contradicts the fact there is an étale cover $\mathbb{A}^1 \to X$.

Equivalence Relations and Groupoids

Definition

An *étale (resp. smooth) groupoid of schemes* is a pair of schemes U and R together with étale (resp. smooth) morphisms $s : R \to U$ and $t : R \to U$ (the *source* and *target* morphisms) and a *composition* morphism $c : R \times_{t,U,s} R \to R$ satisfying the following properties:

1. composition is associative;
2. there is an *identity* morphism $e : U \to R$; and
3. there is an *inverse* morphism $i : R \to R$.

If $(s, t) : R \rightrightarrows U \times U$ is a monomorphism, we say that $s, t : R \rightrightarrows U$ is an *étale (resp. smooth) equivalence relation*.

Equivalence Relations and Groupoids

(1) (associativity) the following diagram commutes

$$\begin{tikzcd}
R \times_{t,U,s} R \times_{t,U,s} R \arrow[r, "c\times \mathrm{id}"] \arrow[d, "\mathrm{id}\times c"] & R \times_{t,U,s} R \arrow[d, "c"] \\
R \times_{t,U,s} R \arrow[r, "c"] & R,
\end{tikzcd}$$

(2) (identity) there exists a morphism $e\colon U \to R$ (called the *identity*) such that the following diagrams commute

$$\begin{tikzcd}
 & U \arrow[dl, "\mathrm{id}"'] \arrow[d, "e"] \arrow[dr, "\mathrm{id}"] & \\
U & R \arrow[l, "s"'] \arrow[r, "t"] & U
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "e\circ s,\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & R \times_{t,U,s} R \arrow[d, "c"] & R \arrow[l, "e\circ t,\mathrm{id}"'] \arrow[dl, "\mathrm{id}"] \\
 & R, &
\end{tikzcd}$$

(3) (inverse) there exists a morphism $i\colon R \to R$ (called the *inverse*) such that the following diagrams commute

$$\begin{tikzcd}
R \arrow[r, "i"] \arrow[dr, "s"'] & R \arrow[r, "i"] \arrow[d, "t"] & R \arrow[dl, "s"] \\
 & U &
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "s"] \arrow[d, "(\mathrm{id},i)"] & U \arrow[d, "e"] \\
R \times_{t,U,s} R \arrow[r, "c"] & R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "t"] \arrow[d, "(i,\mathrm{id})"] & U \arrow[d, "e"] \\
R \times_{t,U,s} R \arrow[r, "c"] & R.
\end{tikzcd}$$

Equivalence Relations and Groupoids

Example

Suppose G/S is an étale (resp. smooth) group scheme acting on a scheme U/S via $\sigma : G \times_S U \to U$. Taking $R = G \times_S U$, $s = \sigma$, $t = p_U$ and composition

$$c : ((g, u), (g', u')) \mapsto (gg', u')$$

defines an étale (resp. smooth) groupoid of schemes.

Groupoid Quotients

Let $R \rightrightarrows U$ be a smooth groupoid. We define a prestack $[U/R]^{\mathrm{pre}}$ as follows:

- Objects: morphisms $T \to U$ with T a scheme.
- Morphisms $(S \xrightarrow{a} U) \to (T \xrightarrow{b} U)$: are given by pairs (f, r), where $f : S \to T$ and $r : S \to R$, with $s(r) = a$ and $t(r) = b \circ f$.

We let $[U/R]$ denote the stackification of $[U/R]^{\mathrm{pre}}$.

Groupoid Quotients

Proposition

1. If $R \rightrightarrows U$ is a smooth (resp. étale) groupoid of schemes then $[U/R]$ is an algebraic (resp. DM) stack with smooth (resp. étale) presentation $U \to [U/R]$.
2. If $R \rightrightarrows U$ is an étale equivalence relation of schemes then $[U/R]$ is equivalent to an algebraic space U/R with étale presentation $U \to U/R$.

The proof combines ideas from the proofs of the algebraicity of quotient stacks and from the representability of the diagonal; we will omit it.

Corollary

If G is a discrete group acting freely on a scheme X, then the sheaf quotient X/G is an algebraic space.