THE QUOTIENT SPACE $\mathcal{H}/PSL(2,\mathbb{Z})$

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Notation. Let $\mathcal{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the complex upper half-plane. Let $\Gamma = PSL(2,\mathbb{Z})$ act on $\mathcal{H}$ by Möbius transformations: $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = \frac{az+b}{cz+d}$.

$\mathcal{H}/\Gamma$ as a Riemann Surface. In this problem sheet you studied $\mathcal{H}/\Gamma$ as a topological space. Ideally, we would like to give $\mathcal{H}/\Gamma$ a complex structure inherited from the complex structure of $\mathcal{H}$. Away from points in the orbits of i and $e^{\pi i/3}$, the quotient map $q : \mathcal{H} \to \mathcal{H}/\Gamma$ is a local homeomorphism, so we may use the complex coordinate z on $\mathcal{H}$ to give local complex coordinates on $\mathcal{H}/\Gamma - \Gamma \cdot \{i, e^{\pi i/3}\}$. However, at points in the orbits of i and $e^{\pi i/3}$, the quotient map q is no-longer a local homeomorphism (near i, q is a ramified 2:1 cover; near $e^{\pi i/3}$, q is a ramified 3:1 cover). This means that we can't use the coordinate z to define local complex coordinates at the corresponding points of $\mathcal{H}/\Gamma$.[1]

What *does* turn out to work instead are the local coordinates $z^{1/2}$ and $z^{1/3}$ respectively. Suppose τ is one of the points i or $e^{\pi i/3}$. The matrix $\delta_\tau = \left(\begin{smallmatrix} 1 & -\tau \\ 1 & -\overline{\tau} \end{smallmatrix}\right)$ sends τ to 0 and $\overline{\tau}$ to ∞. The (conjugate of) the stabiliser of τ acts by Möbius transformations on $\mathbb{C}_\infty = \mathbb{CP}^1$ with fixed points 0 and ∞, so must consist of maps of the form $z \mapsto a_\tau z$, where a_τ is a second or third root of unity, as appropriate. Let ψ_τ be the map $\rho_\tau \circ \delta_\tau$, where $\rho_\tau : \mathbb{C} \to \mathbb{C}$ is the map $z \mapsto z^2$ or $z \mapsto z^3$ as appropriate. Then for an appropriate open neighbourhood U of τ *in* $\overline{D}$, the closure of the fundamental domain (see Figure 1), the image V of U under ψ_τ is a small open disc in $\mathbb{C}$ centred about 0, the image $q(U)$ is an open neighbourhood of $q(\tau)$ in $\mathcal{H}/\Gamma$, and there exists a diagram of the following form:

$$\begin{tikzcd} & U \arrow[dl, "q"'] \arrow[dr, "\psi_\tau"] & \\ \mathcal{H}/\Gamma \supset q(U) \arrow[rr, "\text{homeo.}"'] & & V \subset \mathbb{C} \end{tikzcd}$$

We may then use the homeomorphism $V \cong q(U)$ to define a complex coordinate on the open subset $q(U) \subset \mathcal{H}/\Gamma$. One may then check that these local coordinates define a

[1]This is suggesting that the quotient should really be viewed as the *orbifold* $[\mathcal{H}/\Gamma]$ - this latter space "remembers" that points in the orbits of i and $e^{\pi i/3}$ have non-trivial stabilisers. In fact $[\mathcal{H}/\Gamma]$ is non-singular as a complex orbifold, without needing to make any funny choices of coordinates.

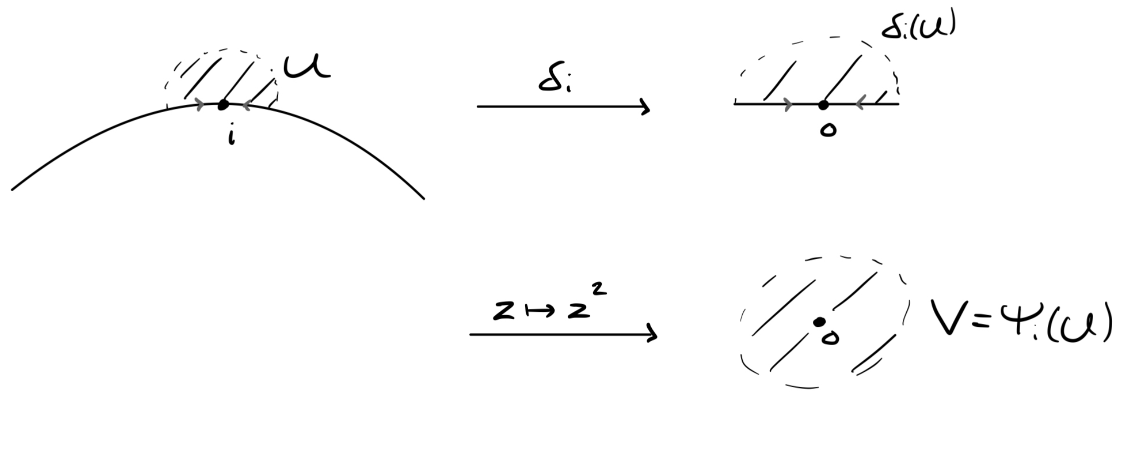

FIGURE 1. Constructing a local complex coordinate at the point $q(i) \in \mathcal{H}/\Gamma$.

holomorphic atlas on $\mathcal{H}/\Gamma$, making $\mathcal{H}/\Gamma$ into a Riemann surface which is biholomorphic to $\mathbb{C}$ (see [DS05, Section 2.2] for more details).

In fact, there is more to this story. It turns out that to any finite-index subgroup Γ' of $SL(2,\mathbb{Z})$, there is an associated *compact* Riemann surface $X_{\Gamma'}$ (known as the *modular curve* associated to Γ') which contains the quotient $\mathcal{H}/\Gamma'$ as a dense open subset. For example, we have $X_{SL(2,\mathbb{Z})} \cong \mathbb{CP}^1$, the Riemann sphere. The surfaces $X_{\Gamma'}$ play a key role in the theory of *modular forms*; indeed, modular forms $f \in M_k(\Gamma')$ of level Γ' and weight k are precisely the holomorphic differential k-forms on $X_{\Gamma'}$ (the holomorphic sections of the line bundle $\omega_{X_{\Gamma'}}^{\otimes k}$):

$$M_k(\Gamma') = H^0(X_{\Gamma'}, \omega_{X_{\Gamma'}}^{\otimes k}).$$

It is possible to compute the genus of $X_{\Gamma'}$, using the methods you've seen earlier in this course; that is, by taking a cellular decomposition of the closure of a fundamental domain and then computing the Euler characteristic. The *Riemann-Roch theorem* (which you'll come across if you go on to take *B3.3 Algebraic Curves*) can then be used to compute the dimension of the vector space $M_k(\Gamma')$, all without knowing any explicit modular forms beforehand! More on these ideas can be found in the course *C3.6 Modular Forms.*

The j-invariant, Briefly. We end by discussing the following remarkable function. Recall that if $\tau \in \mathcal{H}$ has associated Weierstrass $\wp$-function $\wp = \wp_\tau$, then we have the relation

$$(\wp')^2 = 4\wp^2 - g_2(\tau)\wp - g_3(\tau),$$

where

$$g_2(\tau) = 60 \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^4}, \quad g_3(\tau) = 140 \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^6}.$$

Definition 1. *The* j-invariant *is defined by*[2] $j(\tau) = 1728\frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2}$.

The j-invariant is a holomorphic map $j : \mathcal{H} \to \mathbb{C}$, which is $SL(2,\mathbb{Z})$-invariant. It turns out that j descends to give a (set-theoretic) bijection $\mathcal{H}/\Gamma \to \mathbb{C}$. In other words, the complex numbers $\mathbb{C}$ classify all possible genus 1 Riemann surfaces via the map j. Even more strikingly, we have the following theorem.

Theorem 2. *The j-invariant extends to give a biholomorphism of Riemann surfaces* $X_{SL(2,\mathbb{Z})} \xrightarrow{\cong} \mathbb{CP}^1$.

This result is proved by checking that j is holomorphic in local coordinates on $X_{SL(2,\mathbb{Z})}$ and $\mathbb{CP}^1$ respectively, then appealing to the result of Question 1 of this sheet, which states that a degree 1 map between compact connected Riemann surfaces is an isomorphism. For more on the basic properties of the j-invariant, see [DS05, Section 1.4].

In modern parlance, the j-invariant identifies $\mathbb{CP}^1$ with $\overline{M}_{1,1}$, the *coarse moduli space* of genus 1 stable curves with one marked point.

References

[DS05] Fred Diamond and Jerry Shurman, *A First Course in Modular Forms*, Graduate Texts in Mathematics, vol. 228, Springer, 2005.

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[2] The coefficient 1728 is there to normalise the Laurent series expansion $j(\tau) = \frac{1}{q} + \sum_{n=0}^{\infty} a_n q^n$, where $q = e^{2\pi i\tau}$.