STACKS AND HIGHER GEOMETRY WEEK 5: THE MODULI STACK OF SEMISTABLE VECTOR BUNDLES ON A CURVE

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Introduction

The aim of today's talk is to summarise some of the concepts we've seen so far regarding algebraic stacks in a concrete example. For simplicity, throughout we will work over an algebraically closed field $\mathbb{C}$ of characteristic 0.

Theorem 1. *Let C be a smooth connected projective curve. The moduli stack $\mathcal{B}^{ss}_{r,d}(C)$ of semistable vector bundles*[1] *of rank r and degree d over C is a smooth, irreducible, universally closed algebraic stack of dimension $r^2(g-1)$ which if $g \geq 2$ admits a projective good moduli space.*

Step 1 - Defining the Prestack

Whenever we have a tentative moduli stack, the first step is to write down what our stack is as a prestack (category fibred in groupoids). This requires us to first understand what objects we're actually dealing with. If we wish to end up with a quasi-compact stack, our objects should form a bounded collection. For vector bundles on a curve of a fixed rank and degree, boundedness does not hold, so we need to limit ourselves to a special class, the *semistable* bundles.

Example 2. Any bounded family (parametrised by a scheme *of finite type*, which in particular is Noetherian) of vector bundles on a curve will have bounded h^0, by the semi-continuity of sheaf cohomology in flat families. However if $n > 0$ then $\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$ is a rank 2 vector bundle of degree 0 with $h^0 = n + 1$. Hence the collection of rank 2 degree 0 vector bundles on $\mathbb{P}^1$ is unbounded.

Definition 3. *A vector bundle $\mathcal{F}$ is* (semi)stable *if for all non-zero proper subbundles $\mathcal{G} \subset \mathcal{F}$, $\mu(\mathcal{G}) < (\leq)\ \mu(\mathcal{F})$, where $\mu(\mathcal{F}) = \deg(\mathcal{F})/\operatorname{rank}(\mathcal{F})$.*

If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of vector bundles, then $\mu(\mathcal{F}') \leq \mu(\mathcal{F}) \leq \mu(\mathcal{F}'')$. In particular, a direct summand of a semistable vector bundle is a semistable vector bundle of the same slope. Any semistable vector bundle $\mathcal{F}$ admits a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F},$$

known as a *Jordan-Hölder filtration*, whose quotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ are all stable of the same slope as $\mathcal{F}$; the associated graded $\bigoplus_i \mathcal{F}_i/\mathcal{F}_{i-1}$ does not depend on the filtration (up to isomorphism).

The following lemma is needed to show that semistable vector bundles indeed form a bounded family.

Lemma 4. *Let $\mathcal{F}$ be a semistable vector bundle of degree $d > r(2g-1)$. Then*

(1) $H^1(C, \mathcal{F}) = 0$.
(2) *$\mathcal{F}$ is generated by its global sections.*

Sketch proof. If $H^1(C, \mathcal{F}) \neq 0$ then be Serre duality there's a non-zero homomorphism $f : \mathcal{F} \to \omega_C$; considering the kernel $\mathcal{K} \subset \mathcal{F}$ of f then contradicts $d > r(2g-1)$. For the second part, for any point $p \in C$ one can show that $H^1(C, \mathcal{F}(-p)) = 0$ by the same argument (twisting by a line bundle does not affect semistability), so the surjection $\mathcal{F} \to \mathcal{F} \otimes \mathbb{C}_p$ remains surjective when passing to global sections. □

[1] For me, vector bundle means a locally free $\mathcal{O}_C$-module.

We also need the following lemma on families of vector bundles.

Lemma 5. *Let $\mathcal{F} \to C \times_{\mathbb{C}} S$ be a flat family of vector bundles on C. Then there is an open subscheme $S' \subset S$ whose points are $\{s \in S : \mathcal{F}_s$ is semistable$\}$.*

Proof. See [HL10, Proposition 2.3.1]. □

Let us now define the prestack $\mathcal{B}^{ss}_{r,d}(C)$.

Definition 6. *Define a prestack (category fibred in groupoids) $\mathcal{B}^{ss}_{r,d}(C)$ over $\mathbf{Sch}_{\text{Ét}}$ as follows:*

1. *If S is a scheme, the objects of $\mathcal{B}^{ss}_{r,d}(C)(S)$ consist of locally free sheaves $\mathcal{F}$ of rank r on $C_S = C \times_{\mathbb{C}} S$ which are flat over S and of relative degree d.*
2. *A morphism $(\mathcal{F}', S') \to (\mathcal{F}, S)$ consists of a map of schemes $f : S' \to S$ together with a map $\mathcal{F} \to (\mathrm{id}_C \times f)_* \mathcal{F}'$ whose adjoint is an isomorphism.*[2]

Step 2 - Descent

In order to know that $\mathcal{B}^{ss}_{r,d}(C)$ is a stack (as opposed to a prestack), we need to know that descent for objects and for morphisms holds. It does hold (over not only the étale topology but even in the fppf[3] and fpqc[4] topologies), and follows from descent for quasi-coherent sheaves together with the fact that being a vector bundle is an étale/fppf/fpqc-open condition for a quasi-coherent sheaf. I don't want to go into explaining why this descent result holds, as it's not particularly illuminating for our purposes.

Step 3 - Algebraicity

The next step is to show our stack $\mathcal{B}^{ss}_{r,d}(C)$ is an *algebraic* stack, in that there exists a smooth presentation from a scheme. In order to show that $\mathcal{B}^{ss}_{r,d}(C)$ admits a smooth presentation, we will show that $\mathcal{B}^{ss}_{r,d}(C)$ is isomorphic to a quotient stack $[X/G]$ and then appeal to the fact that $X \to [X/G]$ is always a smooth presentation. To do this, we will use Grothendieck *Quot schemes.* As a consequence of our approach we will also get boundedness for free.

Fix an ample line bundle $\mathcal{O}_C(1)$ on C (one exists as we're assuming C is projective). By Lemma 4 there exists an integer $m_0 > 0$ such that, for any $m \geq m_0$ and any semistable bundle $\mathcal{F}$ of rank r and degree d, $H^1(C, \mathcal{F}(m)) = 0$ and $\mathcal{F}(m) = \mathcal{F} \otimes \mathcal{O}_C(m)$ is globally generated. The latter statement implies that there is an exact sequence

$$H^0(C, \mathcal{F}(m)) \otimes \mathcal{O}_C(-m) \longrightarrow \mathcal{F} \longrightarrow 0.$$

As $H^1(C, \mathcal{F}(m)) = 0$, by Riemann-Roch we have $h^0(C, \mathcal{F}(m)) = \chi(C, \mathcal{F}(m)) = d + rm + r(1-g)$. Fixing an isomorphism $H^0(C, \mathcal{F}(m)) \cong V := \mathbb{C}^{d+rm+r(1-g)}$, we get a point

$$[V \otimes \mathcal{O}_C(-m) \to \mathcal{F}] \in Q := \mathrm{Quot}_C(V \otimes \mathcal{O}_C(-m), P), \quad P(m) = d + rm + r(1-g).$$

Let

$$V \otimes \mathcal{O}_{C \times Q}(-m) \longrightarrow \mathcal{U} \longrightarrow 0$$

be the universal quotient. One can check that the locus of points $q \in Q$ corresponding to the map $V \to H^0(C, \mathcal{U}_q(m))$ being an isomorphism forms an open subscheme (this follows by cohomology and base change, by looking at the locus where $H^1(C, \mathcal{U}_q(m)) \neq 0$). In turn, the properties of being locally free and being semistable are open in flat families, so there is an open subscheme $Q' \subset Q$ whose points are precisely those $q \in Q$ where $\mathcal{U}_q$ is a semistable vector bundle such that the induced map $V \to H^0(C, \mathcal{U}_q(m))$ is an isomorphism.

There is a morphism $f : Q' \to \mathcal{B}^{ss}_{r,d}(C)$ given by sending $[V \otimes \mathcal{O}_C(-m) \to \mathcal{U}_q \to 0] \mapsto \mathcal{U}_q$ The group $G = GL(V)$ acts naturally on Q' by precomposition, and with respect to

[2] That is, for every *choice* of pullback $(\mathrm{id}_C \times f)^* \mathcal{F}$, the adjoint map $(\mathrm{id}_C \times f)^* \mathcal{F} \to \mathcal{F}'$ is an isomorphism - this resolves any 2-categorical issues that may arise.

[3] Short for *fidèlement plat et de présentation fini.*

[4] Short for *fidèlement plat et quasi-compact.*

this action the morphism f is invariant, so f factors through the quotient $[Q'/G]$. This morphism is fully faithful, since any automorphism of a semistable vector bundle $\mathcal{G}$ over $C \times_{\mathbb{C}} S$ arising from a morphism $S \to Q'$ induces an automorphism of the free[5] sheaf $(\mathrm{pr}_S)_*(\mathcal{G}(m)) = \mathcal{O}_S^{P(m)}$, which gives an element of $GL(V)(S)$, which in turn acts on $V \otimes \mathcal{O}_{C\times S}(-m)$ in a way which preserves $\mathcal{G}$. By construction f is essentially surjective, and so is an isomorphic of stacks. This proves the following result.

Proposition 7. *$\mathcal{B}^{ss}_{r,d}(C)$ is a Noetherian[6] algebraic stack of finite type over $\mathbb{C}$.* □

Step 4 - Deformation Theory

We will make use of the following result.

Proposition 8 (Infinitesimal lifitng criterion for smoothness)**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a finite type morphism of Noetherian algebraic stacks. Consider 2-commutative diagrams*

$$\begin{tikzcd}
\operatorname{Spec} A_0 \arrow[r] \arrow[d, hook] & \mathcal{X} \arrow[d, "f"] \\
\operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & \mathcal{Y}
\end{tikzcd}$$

of solid arrows, where $\phi : A \to A_0$ is a surjection of Artinian local rings with residue field $\mathbb{C}$ such that $\ker\phi = \mathbb{C}$ and such that $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec} A_0 \to \mathcal{X}$ is a finite type point. Then f is smooth if and only if for every such diagram, there exists a lifting $\operatorname{Spec} A \to \mathcal{X}$.

Let $[\mathcal{F}] \in \mathcal{B}^{ss}_{r,d}(C)(\mathbb{C})$ and let ϕ be a surjection as above. To show that $\mathcal{B}^{ss}_{r,d}(C)$ is smooth, we need to show that every vector bundle $\mathcal{F}_0$ on C_{A_0} which restricts to $\mathcal{F}$ extends to a vector bundle $\mathcal{F}'$ on C_A. Results from deformation theory[7] give an obstruction class $\mathrm{ob}_{\mathcal{F}} \in \operatorname{Ext}^2_C(\mathcal{F},\mathcal{F})$ such that $\mathrm{ob}_{\mathcal{F}} = 0$ if and only if there is such an extension, but

$$\operatorname{Ext}^2_C(\mathcal{F},\mathcal{F}) = H^2(C, \mathcal{F}\otimes\mathcal{F}^\vee) = 0$$

by dimension reasons. Hence all deformations are unobstructed, and by the infinitesimal lifting criterion $\mathcal{B}^{ss}_{r,d}(C)$ is smooth over $\mathbb{C}$.

Results from deformation theory also give an identification

$$T_{[\mathcal{F}]}\mathcal{B}^{ss}_{r,d}(C) = \operatorname{Ext}^1_C(\mathcal{F},\mathcal{F}) = H^1(C, \mathcal{F}\otimes\mathcal{F}^\vee).$$

By Riemann-Roch

$$h^1(C, \mathcal{F}\otimes\mathcal{F}^\vee) = -\chi(C, \mathcal{F}\otimes\mathcal{F}^\vee) + h^0(C, \mathcal{F}\otimes\mathcal{F}^\vee) = r^2(g-1) + \hom_C(\mathcal{F},\mathcal{F}).$$

In turn $\dim \operatorname{Aut}_C(\mathcal{F}) = \hom_C(\mathcal{F},\mathcal{F})$, so $\dim T_{[\mathcal{F}]}\mathcal{B}^{ss}_{r,d}(C) = r^2(g-1) + \dim\operatorname{Aut}_C(\mathcal{F})$. It then follows from the following result that $\dim \mathcal{B}^{ss}_{r,d}(C) = r^2(g-1)$:

Proposition 9. *If $\mathcal{X}$ is a smooth Noetherian algebraic stack over $\mathbb{C}$ and if $x \in \mathcal{X}(\mathbb{C})$ has smooth stabiliser G_x, then*

$$\dim_x \mathcal{X} = \dim T_x\mathcal{X} - \dim G_x.$$

Here $\dim_x \mathcal{X} := \dim_u U - \dim_{e(u)} R_u$, where $U \to \mathcal{X}$ is a smooth presentation with corresponding smooth groupoid $U \rightrightarrows R$, and where $u \in U$ is a preimage of x.

[5]Use cohomology and base change.

[6]Locally Noetherian, quasi-compact and quasi-separated - this follows as Q' is a Noetherian scheme.

[7]See last term's reading group.

Step 5 - The Valuative Criterion

In order to determine whether a stack is separated or universally closed, we have the valuative criterion.

Proposition 10 (The valuative criterion)**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a finite type morphism of Noetherian algebraic stacks with separated diagonals. Consider a 2-commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \arrow[dl, Rightarrow, "\alpha"] \\ \operatorname{Spec} R \arrow[r] & \mathcal{Y} \end{tikzcd}$$

where R is a DVR with fraction field K. Then:

1. *f is universally closed if and only if for every such diagram as above, there is an extension of DVR's $R \to R'$, with the map on fraction fields having finite transcendence degree, and a lifting*

$$\begin{tikzcd} \operatorname{Spec} K' \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} R' \arrow[r] \arrow[urr, dashed] & \operatorname{Spec} R \arrow[r] & \mathcal{Y} \end{tikzcd}$$

2. *f is separated if and only if any two liftings are uniquely isomorphic.*
3. *f is proper if and only if f is universally closed and separated.*

The result that $\mathcal{B}^{ss}_{r,d}(C)$ is universally closed then follows from the following result of S. Langton [Lan75].

Proposition 11 (Langton)**.** *Let R be a DVR with field of fractions K, and let $i : C_K \to C_R$ be the inclusion. If $\mathcal{F}_K$ is a semistable vector bundle on C_K, then there exists a subbundle $\mathcal{F}$ of $i_*\mathcal{F}_K$ whose restriction to C_K is $\mathcal{F}_K$ and whose restriction to the central fibre is semistable.*

Note that $\mathcal{B}^{ss}_{r,d}(C)$ cannot be proper if there are any strictly semistable sheaves. The reason why is because any strictly semistable sheaf $\mathcal{F}$ will admit a non-trivial Jordan-Hölder filtration by subsheaves whose associated graded $\mathrm{gr}(\mathcal{F})$ is also a semistable vector bundle of rank r and degree d (and $\mathrm{gr}(\mathcal{F}) \ncong \mathcal{F}$ if $\mathcal{F}$ is not polystable). On the other hand it is possible[8] to find a one-parameter degeneration from $\mathcal{F}$ to $\mathrm{gr}(\mathcal{F})$ whose general member is $\mathcal{F}$. This implies that $\mathcal{B}^{ss}_{r,d}(C)$ cannot be separated.

Conversely, if the only sheaves which appear are stable (for instance, if r and d are coprime) then, modulo taking a *rigidification* by the $\mathbb{G}_m$'s contained in the automorphism group of any vector bundle, the stack $\mathcal{B}^{ss}_{r,d}(C)^{\mathbb{G}_m}$ is actually a *projective scheme*! This is because $\mathrm{Aut}_C(\mathcal{F}) = \mathbb{G}_m$ for any stable vector bundle, so all objects of $\mathcal{B}^{ss}_{r,d}(C)^{\mathbb{G}_m}$ have trivial automorphisms and so $\mathcal{B}^{ss}_{r,d}(C)^{\mathbb{G}_m}$ is an algebraic space (informally, a stack without any stackiness). But this algebraic space admits a projective coarse moduli space (see below), which by the uniqueness of coarse moduli spaces must be canonically isomorphic to $\mathcal{B}^{ss}_{r,d}(C)^{\mathbb{G}_m}$. In particular, $\mathcal{B}^{ss}_{r,d}(C)^{\mathbb{G}_m}$ is a scheme-theoretic fine moduli space for the moduli functor parametrising families of vector bundles up to isomorphism.[9]

Step 6 - Existence of a Moduli Space

From now on, we will restrict attention to when C has genus $g \geq 2$. In this case, it turns out that $\mathcal{B}^{s}_{r,d}(C)$ is non-empty (see [NR69, Lemma 4.3]) and so is a dense open substack of $\mathcal{B}^{ss}_{r,d}(C)$.

[8]For instance, by using general theory coming from GIT.

[9]One of the functors introduced by Jakub the previous week.

Recall that if $\mathcal{X}$ is an algebraic stack, a *good moduli space* of $\mathcal{X}$ is a quasi-compact morphism $q : \mathcal{X} \to X$ to an algebraic space X which is Stein and for which q_* is exact on quasi-coherent sheaves. Good moduli spaces are always unique up to unique isomorphism if they exist. Over $\mathbb{C}$, any coarse moduli space is a good moduli space.

The traditional way of both constructing the good moduli space $B^{ss}_{r,d}(C)$ of $\mathcal{B}^{ss}_{r,d}(C)$ and showing that it's a projective scheme involves using reductive GIT on the Quot scheme

$$Q = \mathrm{Quot}_C(V \otimes \mathcal{O}_C(-m), P)$$

with respect to the action of $SL(V)$; it turns out that GIT semistability (as characterised via the *Hilbert-Mumford criterion* in terms of one-parameter subgroups of $SL(V)$) essentially is the same as moduli semistability, and reductive GIT on projective schemes always produces good quotients which are projective.[10] A very accessible account of how the GIT story goes can be found in [Hos15]. Let us instead sketch how the *Beyond GIT* programme applies to $\mathcal{B}^{ss}_{r,d}(C)$, following the paper [ABB+22].

Θ-Reductivity. Recall that a Noetherian algebraic stack $\mathcal{X}$ is said to be Θ-*reductive* if for every DVR R, any diagram

$$\begin{tikzcd} \Theta_R \setminus \{0\} = \Theta_K \cup_K \operatorname{Spec} R \arrow[r] \arrow[d] & \mathcal{X} \\ \Theta_R \arrow[ur, dashed] & \end{tikzcd}$$

of solid arrows can be filled in. Here $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$, $\operatorname{Spec} R$ is $t \neq 0$ and Θ_K is $\varpi \neq 0$. If A is a finitely-generated $\mathbb{C}$-algebra, a morphism $\Theta_A \to \mathcal{B}^{ss}_{r,d}(C)$ is the same as a Θ_A-flat semistable vector bundle of rank r and degree d on $C \times \Theta_A$. By smooth descent, this corresponds to an $\mathbb{A}^1_A$-flat vector bundle $\mathcal{G}$ on $C \times \mathbb{A}^1_A$ with a $\mathbb{G}_m$-action, that is, a $\mathbb{Z}$-grading $\bigoplus_i \mathcal{G}_i$ which is compatible with multiplication by t: $t(\mathcal{G}_i) \subset \mathcal{G}_{i+1}$. Flatness implies that $\mathcal{G}$ is t-torsion-free, or in other words that $\times t : \mathcal{G} \to \mathcal{G}$ is injective. The fibre over $\operatorname{Spec} A \subset \Theta_A$ is

$$\begin{aligned} \mathcal{G} \otimes_{A[t]} A[t^{\pm 1}] &= \mathcal{G} \otimes_{A[t]} \operatorname{colim}(\cdots \to A \xrightarrow{t} A \to \cdots) \\ &= \operatorname{colim}(\cdots \to \mathcal{G} \xrightarrow{t} \mathcal{G} \to \cdots) \\ &= \bigoplus_{n \in \mathbb{Z}} \operatorname{colim}(\cdots \to \mathcal{G}_n \xrightarrow{t} \mathcal{G}_{n+1} \to \cdots), \end{aligned}$$

with $\mathbb{G}_m$-invariants $\mathcal{F} = \operatorname{colim}(\mathcal{G}_n)$. The fibre over 0 is $\mathcal{G}/t\mathcal{G} = \bigoplus_i \mathcal{G}_i/\mathcal{G}_{i-1}$ - this corresponds to a point of $\mathcal{B}^{ss}_{r,d}(C)$, so each non-zero $\mathcal{G}_i/\mathcal{G}_{i-1}$ is a semistable vector bundle on C_A of slope d/r. By finite-generation, only finitely-many of the $\mathcal{G}_i/\mathcal{G}_{i-1}$ are non-zero. It follows that giving a morphism $\Theta_A \to \mathcal{B}^{ss}_{r,d}(C)$ is the same as giving an A-flat semistable vector bundle $\mathcal{F}$ over C_A of rank r and degree d and a filtration

$$\cdots \subset \mathcal{F}_{n-1} \subset \mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \cdots$$

such that the quotients are 0 or A-flat semistable vector bundles of slope d/r, such that $\mathcal{F}_{\ll 0} = 0$ and such that $\mathcal{F}_{\gg 0} = \mathcal{F}$.

Θ-reductivity then translates to the statement that given an R-flat semistable vector bundle $\mathcal{F}$ on C_R of rank r and degree d and a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}_K$$

whose quotients are semistable vector bundles of slope d/r, the filtration extends to one over R for the vector bundle $\mathcal{F}$. The proof that such extensions exist boils down to the fact that relative Quot schemes are proper; see [ABB+22, Proposition 3.8].

[10] If G acts on a linearised projective scheme (X, L) then the GIT quotient $X^{ss}(L) /\!\!/ G$ is always a good moduli space for the quotient stack $[X^{ss}(L)/G]$.

S-Completeness. If R is a DVR with uniformiser ϖ and fraction field K, recall that $\overline{\mathrm{ST}}_R$ is defined to be the stack

$$\overline{\mathrm{ST}}_R = [(\operatorname{Spec} R[s,t]/(st-\varpi))/\mathbb{G}_m],$$

where $\mathbb{G}_m$ acts with weight $+1$ on s and -1 on t. This stack has a unique closed point $0 = B\mathbb{G}_m$ given by the vanishing of s and t. The loci $s=0$, $t=0$ give copies of Θ_κ (where $\kappa = R/\varpi$ is the residue field of R) and the loci $s\neq 0$, $t\neq 0$ give copies of $\operatorname{Spec} R$. Recall as well that a Noetherian algebraic stack $\mathcal{X}$ is said to be *S-complete*[11] if for every DVR R, any diagram

$$\begin{tikzcd}
\overline{\mathrm{ST}}_R \setminus \{0\} = \operatorname{Spec} R \cup_K \operatorname{Spec} R \arrow[r] \arrow[d] & \mathcal{X} \\
\overline{\mathrm{ST}}_R \arrow[ur, dashed] &
\end{tikzcd}$$

of solid arrows can be filled in.

A morphism $\overline{\mathrm{ST}}_R \setminus \{0\} \to \mathcal{B}^{ss}_{r,d}(C)$ is the data of two R-flat semistable vector bundles on C_R (corresponding to $s\neq 0$ and $t \neq 0$ respectively) together with an isomorphism of their restrictions over C_K. Using the above description of morphisms $\Theta_R \to \mathcal{B}^{ss}_{r,d}(C)$, it can be checked (see [AHLH18, Corollary 7.13]) that a morphism $\overline{\mathrm{ST}}_R \to \mathcal{B}^{ss}_{r,d}(C)$ is equivalent to giving a diagram

$$\begin{tikzcd}
\cdots \arrow[r, bend left, "s"] & \mathcal{F}_n \arrow[l, bend left, "t"] \arrow[r, bend left, "s"] & \mathcal{F}_{n+1} \arrow[l, bend left, "t"] \arrow[r, bend left, "s"] & \cdots \arrow[l, bend left, "t"]
\end{tikzcd}$$

with $st = ts = \varpi$, s,t being injections (and the same being true for $s : \mathcal{F}_{n-1}/t\mathcal{F}_n \to \mathcal{F}_n/t\mathcal{F}_{n+1}$), s being an isomorphism in large positive degrees, t being an isomorphism in large negative degrees and the two colimits

$$\mathcal{F}_s = \operatorname{colim}(\cdots \to \mathcal{F}_n \xrightarrow{s} \mathcal{F}_{n+1} \to \cdots), \quad \mathcal{F}_t = \operatorname{colim}(\cdots \to \mathcal{F}_{n+1} \xrightarrow{t} \mathcal{F}_n \to \cdots)$$

obtained by restricting to $s \neq 0$ and $t\neq 0$ respectively being R-flat slope semistable vector bundles on C_R of rank r and degree d. The point $(s,t) = (1,0)$ corresponds to $\mathcal{F}_s$, $(0,1)$ to $\mathcal{F}_t$ and $(0,0)$ to $\operatorname{gr}(\mathcal{F}_s) = \operatorname{gr}(\mathcal{F}_t)$.

A proof of S-completeness for $\mathcal{B}^{ss}_{r,d}(C)$ can be found in [AHLH18, Lemma 8.4]. S-completeness for $\mathcal{B}^{ss}_{r,d}(C)$ implies the following statement ([AHLH18, Remark 3.38]): given two families of semistable vector bundles over C_R which are isomorphic over C_K, the bundles over the central fibre are *S-equivalent*, in the sense that they have isomorphic associated graded sheaves.

By applying the machinery of the *Beyond GIT* programme, there exists a proper good moduli space $B^{ss}_{r,d}(C)$ *in the category of algebraic spaces.* As objects of $B^{ss}_{r,d}(C)$ have no automorphisms, the morphism $\mathcal{B}^{ss}_{r,d}(C) \to B^{ss}_{r,d}(C)$ factors through the rigidification $\mathcal{B}^{ss}_{r,d}(C) \to \mathcal{B}^{ss}_{r,d}(C)^{\mathbb{G}_m}$ (which is a $\mathbb{G}_m$-gerbe)[12] and so $B^{ss}_{r,d}(C)$ is also a good moduli space for $\mathcal{B}^{ss}_{r,d}(C)^{\mathbb{G}_m}$. In particular, the dimension *increases* by 1: $\dim B^{ss}_{r,d}(C) = r^2(g-1)+1$.

Step 7 - Projectivity of the Good Moduli Space

Let $\mathcal{U}$ be the universal vector bundle on $C \times \mathcal{B}^{ss}_{r,d}(C)$ (which is easily seen to tautologically exist, once the issue of interpreting coherent sheaves on algebraic stacks is dealt with). For a vector bundle $\mathcal{V}$ on C, consider the determinantal line bundle

$$\mathcal{L}_{\mathcal{V}} = (\det \mathbf{R}(\mathrm{pr}_{\mathcal{B}})_*(\mathrm{pr}_C^* \mathcal{V} \otimes \mathcal{U}))^\vee,$$

where if $\mathcal{E} \sim_{\mathrm{qis}} [K^0 \to K^1]$ in the derived category $\mathbf{D}^b(\mathbf{Coh}(\mathcal{B}^{ss}_{r,d}(C)))$, with each K^i locally free, then $\det \mathcal{E} := \det(K^0) \otimes \det(K^1)^\vee$. One can check that $\det\mathcal{E}$ is independent of the choice of resolution, and that such a resolution exists for $\mathcal{E} = \mathbf{R}(\mathrm{pr}_{\mathcal{B}})_*(\mathrm{pr}_C^*\mathcal{V} \otimes \mathcal{U})$.

[11] Short for *Seshadri complete.*

[12] Roughly speaking, a fibration whose fibres are $B\mathbb{G}_m = [*/\mathbb{G}_m]$; this is a stack of dimension -1.

The strategy for establishing projectivity ([ABB$^+$22, Theorem 5.1]) can be briefly summarised as follows:

(1) For appropriate $\mathcal{V}$, the line bundle $\mathcal{L}_{\mathcal{V}}$ descends to the good moduli space, and is the pullback of a line bundle $L_{\mathcal{V}}$. This line bundle depends only on the rank and degree of $\mathcal{V}$.

(2) For appropriate choices of invariants such that $\mathrm{rank}(\mathbf{R}(\mathrm{pr}_{\mathcal{B}})_*(\mathrm{pr}_C^*\mathcal{V}\otimes\mathcal{U})) = 0$, there exists a tautological section $s_{\mathcal{V}}$ (which *does* depend on $\mathcal{V}$ and not just its discrete invariants) of $\mathcal{L}_{\mathcal{V}}$, locally given by the determinant of $K^0 \to K^1$. This section also descends to a section of $L_{\mathcal{V}}$. The vanishing of this section at a particular vector bundle can be cohomologically characterised.

(3) Enough sections $s_{\mathcal{V}}$ of $L_{\mathcal{V}}$ (given by varying $\mathcal{V}$ whilst keeping its numerical invariants fixed in an appropriate way) can be found such that, corresponding to a suitably high power of $L_{\mathcal{V}}$, there is a *quasi-finite* morphism $B_{r,d}^{ss}(C) \to \mathbb{P}^N$.

(4) As $B_{r,d}^{ss}(C)$ is proper, $B_{r,d}^{ss}(C) \to \mathbb{P}^N$ is proper, hence finite, hence affine, hence representable - this implies $B_{r,d}^{ss}(C)$ is a scheme. The pullback of $\mathcal{O}_{\mathbb{P}^N}(1)$ gives an ample line bundle on $B_{r,d}^{ss}(C)$, so this is a projective scheme.

References

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