STACKS AND HIGHER GEOMETRY WEEK 5: THE MODULI STACK OF SEMISTABLE VECTOR BUNDLES ON A CURVE

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INTRODUCTION

The aim of today's talk is to summarise some of the concepts we've seen so far regarding algebraic stacks in a concrete example. For simplicity, throughout we will work over an algebraically closed field \mathbb{C} of characteristic 0.

Theorem 1. Let C be a smooth connected projective curve. The moduli stack $\mathcal{B}_{r,d}^{ss}(C)$ of semistable vector bundles¹ of rank r and degree d over C is a smooth, irreducible, universally closed algebraic stack of dimension $r^2(g-1)$ which if $g \geq 2$ admits a projective good moduli space.

STEP 1 - DEFINING THE PRESTACK

Whenever we have a tentative moduli stack, the first step is to write down what our stack is as a prestack (category fibred in groupoids). This requires us to first understand what objects we're actually dealing with. If we wish to end up with a quasi-compact stack, our objects should form a bounded collection. For vector bundles on a curve of a fixed rank and degree, boundedness does not hold, so we need to limit ourselves to a special class, the *semistable* bundles.

Example 2. Any bounded family (parametrised by a scheme of finite type, which in particular is Noetherian) of vector bundles on a curve will have bounded h^0 , by the semicontinuity of sheaf cohomology in flat families. However if n > 0 then $\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$ is a rank 2 vector bundle of degree 0 with $h^0 = n + 1$. Hence the collection of rank 2 degree 0 vector bundles on \mathbb{P}^1 is unbounded.

Definition 3. A vector bundle \mathcal{F} is (semi)stable if for all non-zero proper subbundles $\mathcal{G} \subset \mathcal{F}, \ \mu(\mathcal{G}) < (\leq) \ \mu(\mathcal{F}), \ where \ \mu(\mathcal{F}) = \deg(\mathcal{F})/\operatorname{rank}(\mathcal{F}).$

If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of vector bundles, then $\mu(\mathcal{F}') \leq \mu(\mathcal{F}) \leq \mu(\mathcal{F}'')$. In particular, a direct summand of a semistable vector bundle is a semistable vector bundle of the same slope. Any semistable vector bundle \mathcal{F} admits a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F},$$

known as a Jordan-Hölder filtration, whose quotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ are all stable of the same slope as \mathcal{F} ; the associated graded $\bigoplus_i \mathcal{F}_i/\mathcal{F}_{i-1}$ does not depend on the filtration (up to isomorphism).

The following lemma is needed to show that semistable vector bundles indeed form a bounded family.

Lemma 4. Let \mathcal{F} be a semistable vector bundle of degree d > r(2g-1). Then

(1) $H^1(C, \mathcal{F}) = 0.$

(2) \mathcal{F} is generated by its global sections.

Sketch proof. If $H^1(C, \mathcal{F}) \neq 0$ then be Serre duality there's a non-zero homomorphism $f : \mathcal{F} \to \omega_C$; considering the kernel $\mathcal{K} \subset \mathcal{F}$ of f then contradicts d > r(2g-1). For the second part, for any point $p \in C$ one can show that $H^1(C, \mathcal{F}(-p)) = 0$ by the same argument (twisting by a line bundle does not affect semistability), so the surjection $\mathcal{F} \to \mathcal{F} \otimes \mathbb{C}_p$ remains surjective when passing to global sections. \Box

¹For me, vector bundle means a locally free \mathcal{O}_C -module.

We also need the following lemma on families of vector bundles.

Lemma 5. Let $\mathcal{F} \to C \times_{\mathbb{C}} S$ be a flat family of vector bundles on C. Then there is an open subscheme $S' \subset S$ whose points are $\{s \in S : \mathcal{F}_s \text{ is semistable}\}$.

Proof. See [HL10, Proposition 2.3.1].

Let us now define the prestack $\mathcal{B}_{r,d}^{ss}(C)$.

Definition 6. Define a prestack (category fibred in groupoids) $\mathcal{B}_{r,d}^{ss}(C)$ over $\mathbf{Sch}_{\acute{Et}}$ as follows:

- (1) If S is a scheme, the objects of $\mathcal{B}_{r,d}^{ss}(C)(S)$ consist of locally free sheaves \mathcal{F} of rank r on $C_S = C \times_{\mathbb{C}} S$ which are flat over S and of relative degree d.
- (2) A morphism $(\mathcal{F}', S') \to (\mathcal{F}, S)$ consists of a map of schemes $f : S' \to S$ together with a map $\mathcal{F} \to (\mathrm{id}_C \times f)_* \mathcal{F}'$ whose adjoint is an isomorphism.²

Step 2 - Descent

In order to know that $\mathcal{B}_{r,d}^{ss}(C)$ is a stack (as opposed to a prestack), we need to know that descent for objects and for morphisms holds. It does hold (over not only the étale topology but even in the fppf³ and fpqc⁴ topologies), and follows from descent for quasicoherent sheaves together with the fact that being a vector bundle is an étale/fppf/fpqcopen condition for a quasi-coherent sheaf. I don't want to go into explaining why this descent result holds, as it's not particularly illuminating for our purposes.

Step 3 - Algebraicity

The next step is to show our stack $\mathcal{B}_{r,d}^{ss}(C)$ is an *algebraic* stack, in that there exists a smooth presentation from a scheme. In order to show that $\mathcal{B}_{r,d}^{ss}(C)$ admits a smooth presentation, we will show that $\mathcal{B}_{r,d}^{ss}(C)$ is isomorphic to a quotient stack [X/G] and then appeal to the fact that $X \to [X/G]$ is always a smooth presentation. To do this, we will use Grothendieck *Quot schemes*. As a consequence of our approach we will also get boundedness for free.

Fix an ample line bundle $\mathcal{O}_C(1)$ on C (one exists as we're assuming C is projective). By Lemma 4 there exists an integer $m_0 > 0$ such that, for any $m \ge m_0$ and any semistable bundle \mathcal{F} of rank r and degree d, $H^1(C, \mathcal{F}(m)) = 0$ and $\mathcal{F}(m) = \mathcal{F} \otimes \mathcal{O}_C(m)$ is globally generated. The latter statement implies that there is an exact sequence

$$H^0(C, \mathcal{F}(m)) \otimes \mathcal{O}_C(-m) \longrightarrow \mathcal{F} \longrightarrow 0.$$

As $H^1(C, \mathcal{F}(m)) = 0$, by Riemann-Roch we have $h^0(C, \mathcal{F}(m)) = \chi(C, \mathcal{F}(m)) = d + rm + r(1-g)$. Fixing an isomorphism $H^0(C, \mathcal{F}(m)) \cong V := \mathbb{C}^{d+rm+r(1-g)}$, we get a point

$$[V \otimes \mathcal{O}_C(-m) \to \mathcal{F}] \in Q := \operatorname{Quot}_C(V \otimes \mathcal{O}_C(-m), P), \quad P(m) = d + rm + r(1-g).$$

Let

 $V \otimes \mathcal{O}_{C \times Q}(-m) \longrightarrow \mathcal{U} \longrightarrow 0$

be the universal quotient. One can check that the locus of points $q \in Q$ corresponding to the map $V \to H^0(C, \mathcal{U}_q(m))$ being an isomorphism forms an open subscheme (this follows by cohomology and base change, by looking at the locus where $H^1(C, \mathcal{U}_q(m)) \neq 0$). In turn, the properties of being locally free and being semistable are open in flat families, so there is an open subscheme $Q' \subset Q$ whose points are precisely those $q \in Q$ where \mathcal{U}_q is a semistable vector bundle such that the induced map $V \to H^0(C, \mathcal{U}_q(m))$ is an isomorphism.

There is a morphism $f: Q' \to \mathcal{B}_{r,d}^{ss}(C)$ given by sending $[V \otimes \mathcal{O}_C(-m) \to \mathcal{U}_q \to 0] \mapsto \mathcal{U}_q$ The group G = GL(V) acts naturally on Q' by precomposition, and with respect to

²That is, for every *choice* of pullback $(\mathrm{id}_C \times f)^* \mathcal{F}$, the adjoint map $(\mathrm{id}_C \times f)^* \mathcal{F} \to \mathcal{F}'$ is an isomorphism - this resolves any 2-categorical issues that may arise.

³Short for fidèlement plat et de présentation fini.

⁴Short for *fidèlement plat et quasi-compact*.

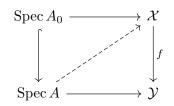
this action the morphism f is invariant, so f factors through the quotient [Q'/G]. This morphism is fully faithful, since any automorphism of a semistable vector bundle \mathcal{G} over $C \times_{\mathbb{C}} S$ arising from a morphism $S \to Q'$ induces an automorphism of the free⁵ sheaf $(\operatorname{pr}_S)_*(\mathcal{G}(m)) = \mathcal{O}_S^{P(m)}$, which gives an element of GL(V)(S), which in turn acts on $V \otimes \mathcal{O}_{C \times S}(-m)$ in a way which preserves \mathcal{G} . By construction f is essentially surjective, and so is an isomorphic of stacks. This proves the following result.

Proposition 7. $\mathcal{B}_{r,d}^{ss}(C)$ is a Noetherian⁶ algebraic stack of finite type over \mathbb{C} .

STEP 4 - DEFORMATION THEORY

We will make use of the following result.

Proposition 8 (Infinitesimal lifting criterion for smoothness). Let $f : \mathcal{X} \to \mathcal{Y}$ be a finite type morphism of Noetherian algebraic stacks. Consider 2-commutative diagrams



of solid arrows, where $\phi : A \to A_0$ is a surjection of Artinian local rings with residue field \mathbb{C} such that ker $\phi = \mathbb{C}$ and such that Spec $\mathbb{C} \to$ Spec $A_0 \to \mathcal{X}$ is a finite type point. Then f is smooth if and only if for every such diagram, there exists a lifting Spec $A \to \mathcal{X}$.

Let $[\mathcal{F}] \in \mathcal{B}^{ss}_{r,d}(C)(\mathbb{C})$ and let ϕ be a surjection as above. To show that $\mathcal{B}^{ss}_{r,d}(C)$ is smooth, we need to show that every vector bundle \mathcal{F}_0 on C_{A_0} which restricts to \mathcal{F} extends to a vector bundle \mathcal{F}' on C_A . Results from deformation theory⁷ give an obstruction class $\mathrm{ob}_{\mathcal{F}} \in \mathrm{Ext}^2_C(\mathcal{F}, \mathcal{F})$ such that $\mathrm{ob}_{\mathcal{F}} = 0$ if and only if there is such an extension, but

$$\operatorname{Ext}_{C}^{2}(\mathcal{F},\mathcal{F}) = H^{2}(C,\mathcal{F}\otimes\mathcal{F}^{\vee}) = 0$$

by dimension reasons. Hence all deformations are unobstructed, and by the infinitesimal lifting criterion $\mathcal{B}_{r,d}^{ss}(C)$ is smooth over \mathbb{C} .

Results from deformation theory also give an identification

$$T_{[\mathcal{F}]}\mathcal{B}^{ss}_{r,d}(C) = \operatorname{Ext}^{1}_{C}(\mathcal{F},\mathcal{F}) = H^{1}(C,\mathcal{F}\otimes\mathcal{F}^{\vee}).$$

By Riemann-Roch

$$h^{1}(C, \mathcal{F} \otimes \mathcal{F}^{\vee}) = -\chi(C, \mathcal{F} \otimes \mathcal{F}^{\vee}) + h^{0}(C, \mathcal{F} \otimes \mathcal{F}^{\vee}) = r^{2}(g-1) + \hom_{C}(\mathcal{F}, \mathcal{F}).$$

In turn dim $\operatorname{Aut}_{C}(\mathcal{F}) = \operatorname{hom}_{C}(\mathcal{F}, \mathcal{F})$, so dim $T_{[\mathcal{F}]}\mathcal{B}_{r,d}^{ss}(C) = r^{2}(g-1) + \operatorname{dim}\operatorname{Aut}_{C}(\mathcal{F})$. It then follows from the following result that dim $\mathcal{B}_{r,d}^{ss}(C) = r^{2}(g-1)$:

Proposition 9. If \mathcal{X} is a smooth Noetherian algebraic stack over \mathbb{C} and if $x \in \mathcal{X}(\mathbb{C})$ has smooth stabiliser G_x , then

$$\dim_x \mathcal{X} = \dim T_x \mathcal{X} - \dim G_x.$$

Here $\dim_x \mathcal{X} := \dim_u U - \dim_{e(u)} R_u$, where $U \to \mathcal{X}$ is a smooth presentation with corresponding smooth groupoid $U \rightrightarrows R$, and where $u \in U$ is a preimage of x.

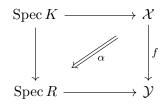
⁵Use cohomology and base change.

⁶Locally Noetherian, quasi-compact and quasi-separated - this follows as Q' is a Noetherian scheme. ⁷See last term's reading group.

STEP 5 - THE VALUATIVE CRITERION

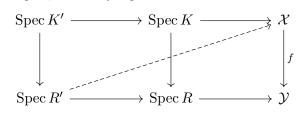
In order to determine whether a stack is separated or universally closed, we have the valuative criterion.

Proposition 10 (The valuative criterion). Let $f : \mathcal{X} \to \mathcal{Y}$ be a finite type morphism of Noetherian algebraic stacks with separated diagonals. Consider a 2-commutative diagram



where R is a DVR with fraction field K. Then:

(1) f is universally closed if and only if for every such diagram as above, there is an extension of DVR's $R \to R'$, with the map on fraction fields having finite transcendence degree, and a lifting



- (2) f is separated if and only if any two liftings are uniquely isomorphic.
- (3) f is proper if and only if f is universally closed and separated.

The result that $\mathcal{B}_{r,d}^{ss}(C)$ is universally closed then follows from the following result of S. Langton [Lan75].

Proposition 11 (Langton). Let R be a DVR with field of fractions K, and let $i: C_K \rightarrow C_R$ be the inclusion. If \mathcal{F}_K is a semistable vector bundle on C_K , then there exists a subbundle \mathcal{F} of $i_*\mathcal{F}_K$ whose restriction to C_K is \mathcal{F}_K and whose restriction to the central fibre is semistable.

Note that $\mathcal{B}_{r,d}^{ss}(C)$ cannot be proper if there are any strictly semistable sheaves. The reason why is because any strictly semistable sheaf \mathcal{F} will admit a non-trivial Jordan-Hölder filtration by subsheaves whose associated graded $\operatorname{gr}(\mathcal{F})$ is also a semistable vector bundle of rank r and degree d (and $\operatorname{gr}(\mathcal{F}) \ncong \mathcal{F}$ if \mathcal{F} is not polystable). On the other hand it is possible⁸ to find a one-parameter degeneration from \mathcal{F} to $\operatorname{gr}(\mathcal{F})$ whose general member is \mathcal{F} . This implies that $\mathcal{B}_{r,d}^{ss}(C)$ cannot be separated.

Conversely, if the only sheaves which appear are stable (for instance, if r and d are coprime) then, modulo taking a *rigidification* by the \mathbb{G}_m 's contained in the automorphism group of any vector bundle, the stack $\mathcal{B}_{r,d}^{ss}(C)^{\mathbb{G}_m}$ is actually a *projective scheme*! This is because $\operatorname{Aut}_C(\mathcal{F}) = \mathbb{G}_m$ for any stable vector bundle, so all objects of $\mathcal{B}_{r,d}^{ss}(C)^{\mathbb{G}_m}$ have trivial automorphisms and so $\mathcal{B}_{r,d}^{ss}(C)^{\mathbb{G}_m}$ is an algebraic space (informally, a stack without any stackiness). But this algebraic space admits a projective coarse moduli space (see below), which by the uniqueness of coarse moduli spaces must be canonically isomorphic to $\mathcal{B}_{r,d}^{ss}(C)^{\mathbb{G}_m}$. In particular, $\mathcal{B}_{r,d}^{ss}(C)^{\mathbb{G}_m}$ is a scheme-theoretic fine moduli space for the moduli functor parametrising families of vector bundles up to isomorphism.⁹

STEP 6 - EXISTENCE OF A MODULI SPACE

From now on, we will restrict attention to when C has genus $g \ge 2$. In this case, it turns out that $\mathcal{B}_{r,d}^s(C)$ is non-empty (see [NR69, Lemma 4.3]) and so is a dense open substack of $\mathcal{B}_{r,d}^{ss}(C)$.

⁸For instance, by using general theory coming from GIT.

⁹One of the functors introduced by Jakub the previous week.

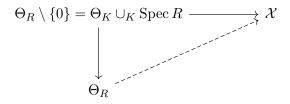
Recall that if \mathcal{X} is an algebraic stack, a good moduli space of \mathcal{X} is a quasi-compact morphism $q: \mathcal{X} \to X$ to an algebraic space X which is Stein and for which q_* is exact on quasi-coherent sheaves. Good moduli spaces are always unique up to unique isomorphism if they exist. Over \mathbb{C} , any coarse moduli space is a good moduli space.

The traditional way of both constructing the good moduli space $B_{r,d}^{ss}(C)$ of $\mathcal{B}_{r,d}^{ss}(C)$ and showing that it's a projective scheme involves using reductive GIT on the Quot scheme

$$Q = \operatorname{Quot}_C(V \otimes \mathcal{O}_C(-m), P)$$

with respect to the action of SL(V); it turns out that GIT semistability (as characterised via the *Hilbert-Mumford criterion* in terms of one-parameter subgroups of SL(V)) essentially is the same as moduli semistability, and reductive GIT on projective schemes always produces good quotients which are projective.¹⁰ A very accessible account of how the GIT story goes can be found in [Hos15]. Let us instead sketch how the *Beyond GIT* programme applies to $\mathcal{B}_{r,d}^{ss}(C)$, following the paper [ABB⁺22].

 Θ -Reductivity. Recall that a Noetherian algebraic stack \mathcal{X} is said to be Θ -reductive if for every DVR R, any diagram



of solid arrows can be filled in. Here $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$, Spec R is $t \neq 0$ and Θ_K is $\varpi \neq 0$. If A is a finitely-generated \mathbb{C} -algebra, a morphism $\Theta_A \to \mathcal{B}_{r,d}^{ss}(C)$ is the same as a Θ_A -flat semistable vector bundle of rank r and degree d on $C \times \Theta_A$. By smooth descent, this corresponds to an \mathbb{A}_A^1 -flat vector bundle \mathcal{G} on $C \times \mathbb{A}_A^1$ with a \mathbb{G}_m -action, that is, a \mathbb{Z} -grading $\bigoplus_i \mathcal{G}_i$ which is compatible with multiplication by $t: t(\mathcal{G}_i) \subset \mathcal{G}_{i+1}$. Flatness implies that \mathcal{G} is t-torsion-free, or in other words that $\times t: \mathcal{G} \to \mathcal{G}$ is injective. The fibre over Spec $A \subset \Theta_A$ is

$$\mathcal{G} \otimes_{A[t]} A[t^{\pm 1}] = \mathcal{G} \otimes_{A[t]} \operatorname{colim}(\dots \to A \xrightarrow{t} A \to \dots)$$
$$= \operatorname{colim}(\dots \to \mathcal{G} \xrightarrow{t} \mathcal{G} \to \dots)$$
$$= \bigoplus_{n \in \mathbb{Z}} \operatorname{colim}(\dots \to \mathcal{G}_n \xrightarrow{t} \mathcal{G}_{n+1} \to \dots),$$

with \mathbb{G}_m -invariants $\mathcal{F} = \operatorname{colim}(\mathcal{G}_n)$. The fibre over 0 is $\mathcal{G}/t\mathcal{G} = \bigoplus_i \mathcal{G}_i/\mathcal{G}_{i-1}$ - this corresponds to a point of $\mathcal{B}_{r,d}^{ss}(C)$, so each non-zero $\mathcal{G}_i/\mathcal{G}_{i-1}$ is a semistable vector bundle on C_A of slope d/r. By finite-generation, only finitely-many of the $\mathcal{G}_i/\mathcal{G}_{i-1}$ are non-zero. It follows that giving a morphism $\Theta_A \to \mathcal{B}_{r,d}^{ss}(C)$ is the same as giving an A-flat semistable vector bundle \mathcal{F} over C_A of rank r and degree d and a filtration

$$\cdots \subset \mathcal{F}_{n-1} \subset \mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \cdots$$

such that the quotients are 0 or A-flat semistable vector bundles of slope d/r, such that $\mathcal{F}_{\ll 0} = 0$ and such that $\mathcal{F}_{\gg 0} = \mathcal{F}$.

 Θ -reductivity then translates to the statement that given an *R*-flat semistable vector bundle \mathcal{F} on C_R of rank *r* and degree *d* and a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}_K$$

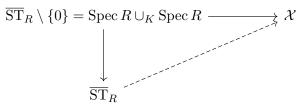
whose quotients are semistable vector bundles of slope d/r, the filtration extends to one over R for the vector bundle \mathcal{F} . The proof that such extensions exist boils down to the fact that relative Quot schemes are proper; see [ABB⁺22, Proposition 3.8].

¹⁰If G acts on a linearised projective scheme (X, L) then the GIT quotient $X^{ss}(L) /\!\!/ G$ is always a good moduli space for the quotient stack $[X^{ss}(L)/G]$.

S-Completeness. If R is a DVR with uniformiser ϖ and fraction field K, recall that \overline{ST}_R is defined to be the stack

$$\overline{\mathrm{ST}}_R = [(\operatorname{Spec} R[s, t]/(st - \varpi))/\mathbb{G}_m],$$

where \mathbb{G}_m acts with weight +1 on s and -1 on t. This stack has a unique closed point $0 = B\mathbb{G}_m$ given by the vanishing of s and t. The loci s = 0, t = 0 give copies of Θ_{κ} (where $\kappa = R/\varpi$ is the residue field of R) and the loci $s \neq 0, t \neq 0$ give copies of Spec R. Recall as well that a Noetherian algebraic stack \mathcal{X} is said to be *S-complete*¹¹ if for every DVR R, any diagram



of solid arrows can be filled in.

A morphism $\overline{\operatorname{ST}}_R \setminus \{0\} \to \mathcal{B}_{r,d}^{ss}(C)$ is the data of two *R*-flat semistable vector bundles on C_R (corresponding to $s \neq 0$ and $t \neq 0$ respectively) together with an isomorphism of their restrictions over C_K . Using the above description of morphisms $\Theta_R \to \mathcal{B}_{r,d}^{ss}(C)$, it can be checked (see [AHLH18, Corollary 7.13]) that a morphism $\overline{\operatorname{ST}}_R \to \mathcal{B}_{r,d}^{ss}(C)$ is equivalent to giving a diagram

$$\cdots \underbrace{\overbrace{t}^{s}}_{t} \mathcal{F}_{n} \underbrace{\overbrace{t}^{s}}_{t} \mathcal{F}_{n+1} \underbrace{\overbrace{t}^{s}}_{t} \cdots$$

with $st = ts = \varpi$, s, t being injections (and the same being true for $s : \mathcal{F}_{n-1}/t\mathcal{F}_n \to \mathcal{F}_n/t\mathcal{F}_{n+1}$), s being an isomorphism in large positive degrees, t being an isomorphism in large negative degrees and the two colimits

$$\mathcal{F}_s = \operatorname{colim}(\dots \to \mathcal{F}_n \xrightarrow{s} \mathcal{F}_{n+1} \to \dots), \quad \mathcal{F}_t = \operatorname{colim}(\dots \to \mathcal{F}_{n+1} \xrightarrow{t} \mathcal{F}_n \to \dots)$$

obtained by restricting to $s \neq 0$ and $t \neq 0$ respectively being *R*-flat slope semistable vector bundles on C_R of rank *r* and degree *d*. The point (s,t) = (1,0) corresponds to \mathcal{F}_s , (0,1)to \mathcal{F}_t and (0,0) to $\operatorname{gr}(\mathcal{F}_s) = \operatorname{gr}(\mathcal{F}_t)$.

A proof of S-completeness for $\mathcal{B}_{r,d}^{ss}(C)$ can be found in [AHLH18, Lemma 8.4]. Scompleteness for $\mathcal{B}_{r,d}^{ss}(C)$ implies the following statement ([AHLH18, Remark 3.38]): given two families of semistable vector bundles over C_R which are isomorphic over C_K , the bundles over the central fibre are *S*-equivalent, in the sense that they have isomorphic associated graded sheaves.

By applying the machinery of the Beyond GIT programme, there exists a proper good moduli space $B_{r,d}^{ss}(C)$ in the category of algebraic spaces. As objects of $B_{r,d}^{ss}(C)$ have no automorphisms, the morphism $\mathcal{B}_{r,d}^{ss}(C) \to B_{r,d}^{ss}(C)$ factors through the rigidification $\mathcal{B}_{r,d}^{ss}(C) \to \mathcal{B}_{r,d}^{ss}(C)^{\mathbb{G}_m}$ (which is a \mathbb{G}_m -gerbe)¹² and so $B_{r,d}^{ss}(C)$ is also a good moduli space for $\mathcal{B}_{r,d}^{ss}(C)^{\mathbb{G}_m}$. In particular, the dimension increases by 1: dim $B_{r,d}^{ss}(C) = r^2(g-1) + 1$.

STEP 7 - PROJECTIVITY OF THE GOOD MODULI SPACE

Let \mathcal{U} be the universal vector bundle on $C \times \mathcal{B}_{r,d}^{ss}(C)$ (which is easily seen to tautologically exist, once the issue of interpreting coherent sheaves on algebraic stacks is dealt with). For a vector bundle \mathcal{V} on C, consider the determinantal line bundle

$$\mathcal{L}_{\mathcal{V}} = (\det \mathbf{R}(\mathrm{pr}_{\mathcal{B}})_* (\mathrm{pr}_{C}^* \mathcal{V} \otimes \mathcal{U}))^{\vee},$$

where if $\mathcal{E} \sim_{qis} [K^0 \to K^1]$ in the derived category $\mathbf{D}^b(\mathbf{Coh}(\mathcal{B}^{ss}_{r,d}(C)))$, with each K^i locally free, then det $\mathcal{E} := \det(K^0) \otimes \det(K^1)^{\vee}$. One can check that det \mathcal{E} is independent of the choice of resolution, and that such a resolution exists for $\mathcal{E} = \mathbf{R}(\mathrm{pr}_{\mathcal{B}})_*(\mathrm{pr}_{\mathcal{C}}^*\mathcal{V} \otimes \mathcal{U})$.

¹¹Short for Seshadri complete.

¹²Roughly speaking, a fibration whose fibres are $B\mathbb{G}_m = [*/\mathbb{G}_m]$; this is a stack of dimension -1.

- (1) For appropriate \mathcal{V} , the line bundle $\mathcal{L}_{\mathcal{V}}$ descends to the good moduli space, and is the pullback of a line bundle $L_{\mathcal{V}}$. This line bundle depends only on the rank and degree of \mathcal{V} .
- (2) For appropriate choices of invariants such that rank($\mathbf{R}(\mathrm{pr}_{\mathcal{B}})_*(\mathrm{pr}_{\mathcal{C}}^*\mathcal{V}\otimes\mathcal{U})$) = 0, there exists a tautological section $s_{\mathcal{V}}$ (which *does* depend on \mathcal{V} and not just its discrete invariants) of $\mathcal{L}_{\mathcal{V}}$, locally given by the determinant of $K^0 \to K^1$. This section also descends to a section of $L_{\mathcal{V}}$. The vanishing of this section at a particular vector bundle can be cohomologically characterised.
- (3) Enough sections $s_{\mathcal{V}}$ of $L_{\mathcal{V}}$ (given by varying \mathcal{V} whilst keeping its numerical invariants fixed in an appropriate way) can be found such that, corresponding to a suitably high power of $L_{\mathcal{V}}$, there is a *quasi-finite* morphism $B^{ss}_{r,d}(C) \to \mathbb{P}^N$.
- (4) As $B_{r,d}^{ss}(C)$ is proper, $B_{r,d}^{ss}(C) \to \mathbb{P}^N$ is proper, hence finite, hence affine, hence representable - this implies $B_{r,d}^{ss}(C)$ is a scheme. The pullback of $\mathcal{O}_{\mathbb{P}^N}(1)$ gives an ample line bundle on $B_{r,d}^{ss}(C)$, so this is a projective scheme.

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