Reading Group on Toric Varieties

Exercise Sheet 0

Hilary Term 2022

Exercise 1: Let X be a scheme and A be a ring. Show that there are isomorphisms

$$\mathrm{Mor}_{\mathbf{Sch}}(X, \operatorname{Spec} A) \cong \mathrm{Hom}_{\mathbf{Ring}}(A, \Gamma(X, \mathcal{O}_X))$$

which are natural in both A and X; that is, Spec and $\Gamma(\cdot)$ are adjoint functors.[1] Deduce the following:

1. Any scheme X admits a unique morphism to $\operatorname{Spec} \mathbb{Z}$.
2. Any scheme X admits a natural morphism $X \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$.

Exercise 2: The *affine communication lemma*[2] states that following. Let X be a scheme and $\mathcal{P}$ is a property of affine open subsets of X which satisfies the following properties:

- if $\operatorname{Spec} A \subset X$ has $\mathcal{P}$ then for any $f \in A$, $\operatorname{Spec} A_f \subset X$ has $\mathcal{P}$ as well;
- if $A = (f_1, \dots, f_n)$ and if each $\operatorname{Spec} A_{f_i} \subset X$ has $\mathcal{P}$, then so does $\operatorname{Spec} A \subset X$.

Then, if $X = \bigcup_i \operatorname{Spec} A_i$ and if each $\operatorname{Spec} A_i$ has property $\mathcal{P}$, then any other open affine of X has property $\mathcal{P}$.

1. Prove the affine communication lemma. Hint: you may want to first show that if Y is any scheme and if $\operatorname{Spec} A$, $\operatorname{Spec} B \subset Y$ are open affines, then $\operatorname{Spec} A \cap \operatorname{Spec} B$ can be covered by open sets which are simultaneously distinguised open subsets of $\operatorname{Spec} A$ and $\operatorname{Spec} B$.
2. Let X be a scheme. Show that X admits a cover by open affines $\operatorname{Spec} A$ such that each ring A is Noetherian, if and only if for every open affine $\operatorname{Spec} A \subset X$, A is Noetherian. Prove the corresponding statement when "Noetherian" is replaced with "domain".
3. Let $f : X \to Y$ be a morphism of schemes. Show that there exists an open cover of Y by open affines $\operatorname{Spec} B$ such that each $f^{-1}(\operatorname{Spec} B)$ can be covered by open affines $\operatorname{Spec} A$, where A is a finitely-generated B-algebra, if and only if for every open affine $\operatorname{Spec} B \subset Y$, $f^{-1}(\operatorname{Spec} B)$ can be covered by open affines $\operatorname{Spec} A$, where A is a finitely-generated B-algebra. Show that there exists an open cover of Y by open affines $\operatorname{Spec} B$ such that each $f^{-1}(\operatorname{Spec} B)$ is affine, if and only if for every open affine $\operatorname{Spec} B \subset Y$, $f^{-1}(\operatorname{Spec} B)$ is affine.

[1] Modulo taking an opposite category somewhere.
[2] Terminology due to R. Vakil.