Reading Group on Toric Varieties

Exercise Sheet 1

Hilary Term 2022

Exercise 0: Complete all exercises, explicit or implicit, contained in the slides from this week.

Exercise 1: Recall the fan $\Sigma \subset \mathbb{R}^2$ obtained from $\mathbb{P}^2$. Show that applying the gluing construction to this fan recovers $\mathbb{P}^2$, that is $X_\Sigma \cong \mathbb{P}^2$. Construct a fan Σ' such that $X_{\Sigma'} \cong \mathbb{P}^n$.

Exercise 2: Show that any affine toric variety is defined by binomial equations. More precisely, suppose that S_σ is generated by elements $m_1, \dots, m_r$, so

$$\mathbb{C}[S_\sigma] = \mathbb{C}[\chi^{m_1}, \dots, \chi^{m_r}] = \mathbb{C}[X_1, \dots, X_r]/I.$$

Show that I is generated by polynomials of the form $X_1^{a_1} \cdots X_r^{a_r} - X_1^{b_1} \cdots X_r^{b_r}$, where the a_i and b_j are non-negative integers satisfying $\sum_i a_i m_i = \sum_i b_i m_i$. Conversely, if I is a prime ideal of $\mathbb{C}[X_1, \dots, X_r]$ generated by binomial polynomials, show that $\operatorname{Spec} \mathbb{C}[X_1, \dots, X_r]/I$ is an affine toric variety.

Exercise 3: Read up on the various valuative criteria for properties of morphisms of schemes (see for instance Hartshorne II.4). Show that X_Σ is separated using the valuative criterion for separatedness.