Reading Group on Toric Varieties

Exercise Sheet 2

Hilary Term 2022

Recall that if X is a Noetherian scheme and if $\mathcal{E}$ is a locally free coherent sheaf on X of rank r, then $\mathbb{P}_X(\mathcal{E}) := \mathbf{Proj}_X(\mathcal{S})$, where $\mathcal{S} = \mathrm{Sym}(\mathcal{E})$ is the symmetric algebra of $\mathcal{E}$. Via the relative $\mathbf{Proj}$ constriction there is a natural morphism $\pi : \mathbb{P}_X(\mathcal{E}) \to X$, which is a $\mathbb{P}^{r-1}$-bundle over X.

Exercise 1: Let σ be a cone in $N_{\mathbb{R}}$. Show that if σ spans $N_{\mathbb{R}}$ then x_σ is the unique fixed point of the action of the torus T_N on U_σ. Conversely, show that if σ does not span $N_{\mathbb{R}}$ then there are no fixed points.

Exercise 2: Let a be a positive integer and let $\mathbb{F}_a$ be the corresponding Hirzebruch surface. Let $f : \mathbb{F}_a \to \mathbb{P}^1$ be the morphism induced by the lattice morphism $(x, y) \mapsto x$. Show that $\mathbb{F}_a$ is isomorphic to the the projective space bundle $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1})$, and that under this isomorphism the map f corresponds to the natural projection $\pi : \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(a)_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) \to \mathbb{P}^1$. Show that the map of lattices $x \mapsto (x, nx)$, where n is an integer, defines a section of f. Construct a polytope P such that $X_P \cong \mathbb{F}_a$.

Exercise 3: Let X_1, X_2 be toric varieties with tori $T_i \subset X_i$. A morphism $f : X_1 \to X_2$ is said to be a *toric morphism* if $f|_{T_1} : T_1 \to T_2$ is a group homomorphism.

1. Show that any toric morphism is equivariant with respect to the action of the tori.
2. Suppose $X_i = X_{\Sigma_i}$, with Σ_i a fan in $(N_i)_{\mathbb{R}}$. Suppose further f is induced by a morphism of lattices $\phi : N_1 \to N_2$ compatible with the fans Σ_i. Show that f is a toric morphism, with

$$f|_{T_1} = \phi \otimes 1 : N_1 \otimes_{\mathbb{Z}} \mathbb{C}^* \to N_2 \otimes_{\mathbb{Z}} \mathbb{C}^*.$$

3. Conversely, show that all toric morphisms $f : X_1 \to X_2$ arise via morphisms of lattices $\phi : N_1 \to N_2$.