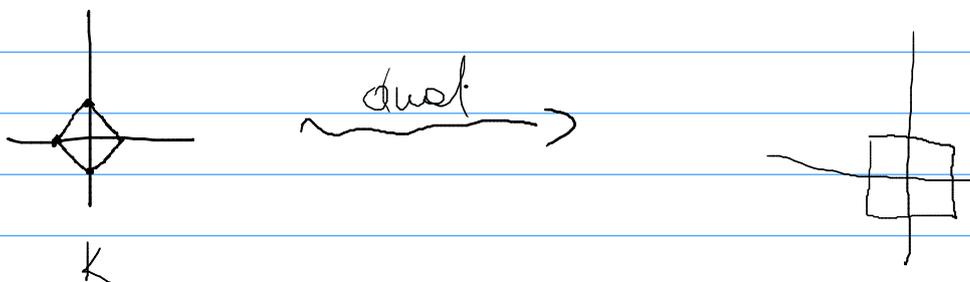


cones and flows \rightsquigarrow foric varieties

$K \subset E$ ^{f.d. v.s.} is a convex polytope
if it's a convex hull of $x_1, \dots, x_n \in E$



$$K^{\circ} = \left\{ u \in E^* \mid \underbrace{\langle u, v \rangle}_{\geq -1} \quad \forall v \in K \right\}$$

$\langle u, v \rangle = 0$

$K \hookrightarrow K \times 1 \subset E \times \mathbb{R} \rightsquigarrow$ take a cone through $K \times 1$

$\text{Def} \Rightarrow K^{\circ}$ is also a convex polytope

$$\underbrace{(K^{\circ})^{\circ}} = K \Rightarrow \underbrace{(K^{\circ})^{\circ}} = K$$

Face of $K \rightsquigarrow F^* = \{ u \in K^{\circ} \mid \langle u, v \rangle = -1 \quad \forall v \in F \}$

$F \xrightarrow{\quad} F^*$ is 1-1 & order reversing

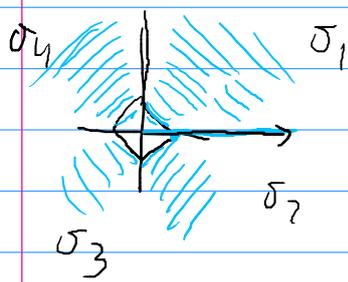
wrt $\mathbb{N} \subset E$ (lattice)

K rational \leftarrow means K has vertices in \mathbb{N} .

K rational wrt $\mathbb{N} \Rightarrow K^{\circ}$ is rational wrt M .

N -lattice $N_{\mathbb{R}} = N \otimes \mathbb{R}$

$0 \in K \subset N_{\mathbb{R}}$ interior $\rightsquigarrow \Delta$ through the faces of K .



$\rightsquigarrow \mathbb{Q}^2$

$\mathbb{C} \times \mathbb{C}^*$

$$(x^{-1}, y) \longleftrightarrow (x, y)$$

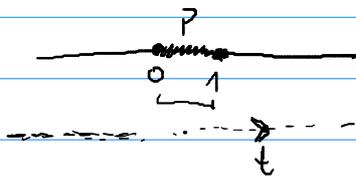
$\Rightarrow \mathbb{P}^1 \times \mathbb{P}^1$

$$\begin{array}{ccc} \uparrow & & \downarrow \\ (z^{-1}, y^{-1}) & \longleftrightarrow & (z, y^{-1}) \end{array}$$

We can start with a polytope $P \subset M_{\mathbb{R}} \rightsquigarrow \Delta_P$

for each face Q of P

$$\sigma_Q = \{ v \in N_{\mathbb{R}} \mid \langle u, v \rangle \in \underbrace{\langle u', v \rangle}_{\text{min}} \quad u \in Q \quad u' \in P \}$$



$\subset M_{\mathbb{R}}$

min $\langle u', t \rangle$ will occur at $u' = 0$
 $u' \in P$

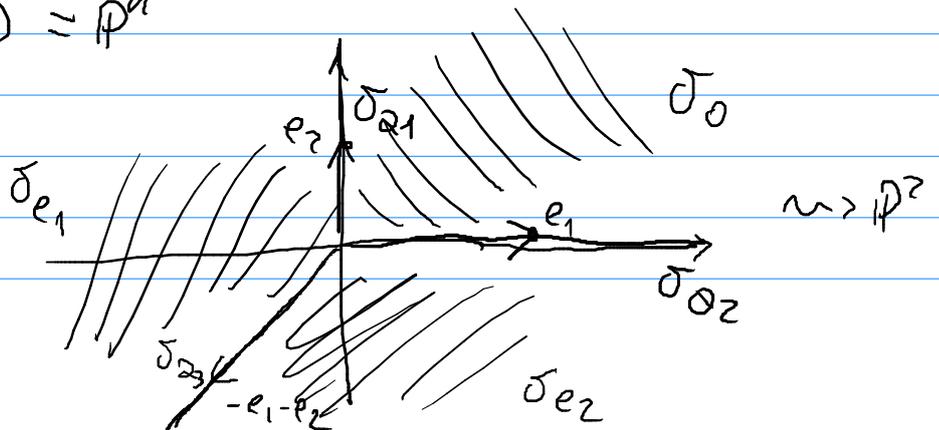
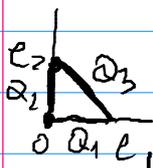
$$\sigma_0 = \text{ray from } 0 \text{ to } 1 \subset N_{\mathbb{R}}$$

$$\sigma_1 = \text{ray from } 1 \text{ to } 0 \subset N_{\mathbb{R}} \quad \Delta_P = \text{ray from } 0 \text{ to } 1 \subset N_{\mathbb{R}}$$

$$\sigma_P = \{0\} \quad X(\Delta_P) = \mathbb{P}^1$$

P is an n -simplex with vertices $0, e_1, \dots, e_n$

$$X(\Delta_P) = \mathbb{P}^n$$



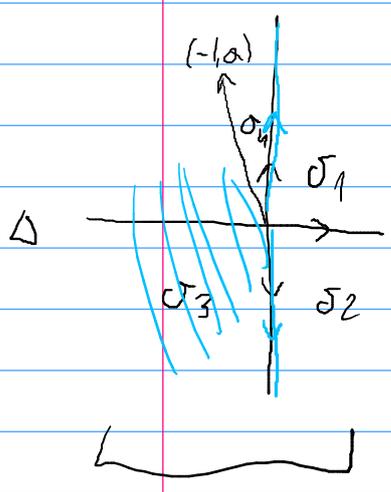
$$\begin{array}{c}
 N' \xrightarrow{\varphi} N \rightsquigarrow \text{morph of toric varieties} \\
 \left[\begin{array}{c}
 \tau \subset \sigma \\
 \tau^\vee \subset \sigma^\vee \rightsquigarrow S_\sigma \hookrightarrow S_{\tau^\vee} \\
 \rightsquigarrow \mathbb{C}S_\sigma \hookrightarrow \mathbb{C}S_{\tau^\vee} \\
 \rightsquigarrow U_\tau = \text{Spec } \mathbb{C}S_{\tau^\vee} \longrightarrow U_\sigma = \text{Spec } \mathbb{C}S_\sigma
 \end{array} \right.
 \end{array}$$

→ Prop: $U_\tau \hookrightarrow U_\sigma$ is an open embedding onto a basic open set.

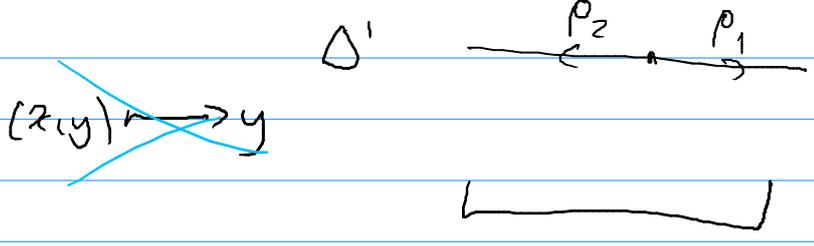
Prop: $N' \xrightarrow{\varphi} N$ is a morphism of lattices and (group homomorphism)
 Δ fan in N Δ' a fan in N'
 \forall cone $\sigma' \in \Delta'$ \exists cone $\sigma \in \Delta$ s.t. $\varphi(\sigma') \subseteq \sigma$

Hence φ induces $X(\Delta') \longrightarrow X(\Delta)$

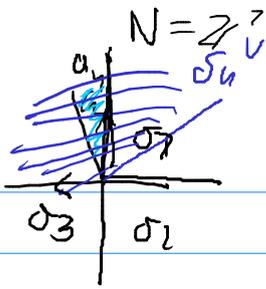
a) 0



$$\mathbb{Z}^2 \xrightarrow{\varphi} \mathbb{Z} \\
 (x, y) \mapsto x$$



F_a
" $X(\Delta)$



$N' = \mathbb{Z}$

$\mathbb{Z}^+ = \mathbb{Z} \geq 0$

$X(\Delta')$

$\varphi(\sigma_1) \subseteq \rho_1$

$\varphi: \mathbb{Z}^+ \rightarrow \mathbb{Z}$
· (0)

$\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}^2$
· (1 0)

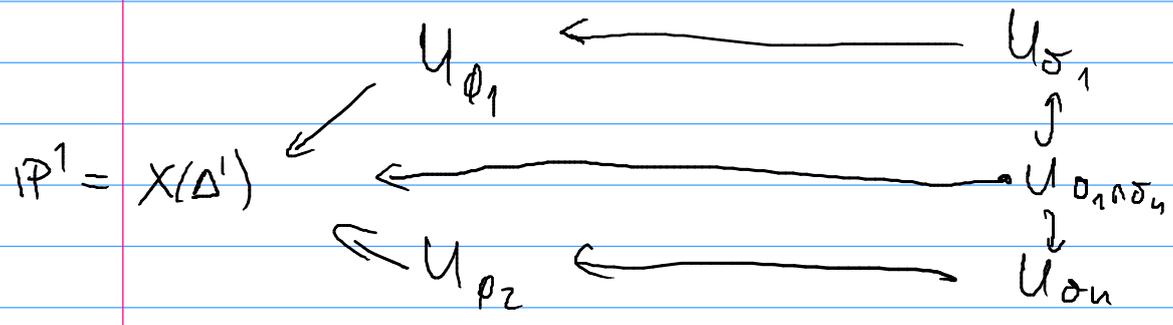
\mathbb{P}^1

$S_{\rho_1} = \mathbb{Z}^+ e_1^* \xrightarrow{\quad} \sigma_1^u \cap M' = \mathbb{Z}^+ e_1^* + \mathbb{Z}^+ e_2^*$
 $e_1^* \xrightarrow{\quad} e_1^*$

$\varphi(\sigma_u) \subseteq \rho_2$

$S_{\rho_2} = \mathbb{Z}^+ (e_2^*) \xrightarrow{\cdot (1 0)} S_{\sigma_u} = \mathbb{Z}^+ (e_1^*) + \mathbb{Z}^+ (ae_1^* + e_2^*)$
 $-e_1^* \xrightarrow{\quad} -e_1^*$

$\mathbb{C} S_{\rho_1} = \mathbb{C}[x] \xrightarrow{\quad} \mathbb{C}[x, y] = \mathbb{C} S_{\sigma_1}$
 $\downarrow \quad \downarrow$
 $\mathbb{C}[x, x^{-1}] \xrightarrow{\quad} \mathbb{C}[x, x^{-1}, y] = \mathbb{C} S_{\sigma_1 \cap \sigma_u}$
 $\uparrow \quad \uparrow$
 $\mathbb{C} S_{\rho_2} = \mathbb{C}[x^{-1}] \xrightarrow{\quad} \mathbb{C}[x^{-1}, x^a y] = \mathbb{C} S_{\sigma_u}$



Ex $\mathbb{Z} \xrightarrow{\quad} \mathbb{Z}^2$ induces sections $\mathbb{P}^1 \rightarrow X(\Delta)$
 $\downarrow \quad \downarrow$
 $1 \xrightarrow{\quad} (1, m)$

local properties of $X(\Delta)$

$\exists x_\sigma \in U_\sigma(\mathbb{C})$ U_σ sing $\Rightarrow x_\sigma$ sing
 x_σ smooth $\Rightarrow U_\sigma$ smooth.

$U_\sigma(\mathbb{C}) \leftrightarrow \{ \mathbb{C}[S_\sigma] \xrightarrow{\quad} \mathbb{C} \}$ of \mathbb{C} -algs

\leftarrow $S_\sigma \xrightarrow{\quad} \mathbb{C}$ of semigroup
 \uparrow additive \uparrow mult.
 1 is the mult. identity.

$$S_\sigma \xrightarrow{x_\sigma} \mathbb{C}$$

$$u \longmapsto \begin{cases} 1 \\ 0 \end{cases} \quad u \text{ has inverse} \Leftrightarrow u \in \sigma^\perp$$

$$S_\sigma = \sigma^\vee \cap \mathbb{N} \quad \sigma^\vee = \{ v \in \mathbb{N}_R \mid \langle u, v \rangle \geq 0 \text{ } \forall u \in \sigma \}$$

$$u \notin -u \in S_\sigma \Leftrightarrow \langle u, v \rangle \geq 0 \text{ \& } \langle -u, v \rangle \geq 0$$

$$\Leftrightarrow \langle u, v \rangle = 0$$

To understand whether x_σ is a smooth point

we investigate $T_{x_\sigma}^* U_\sigma = \mathfrak{m} / \mathfrak{m}^2$

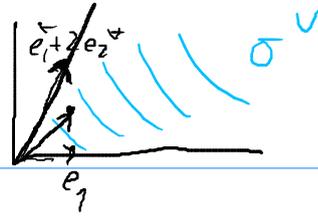
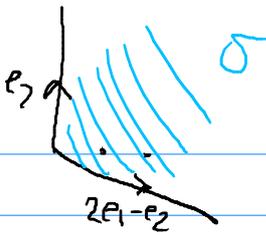
Say $\sigma^\perp = \langle \rho \rangle$

$$\mathbb{C}[S_\sigma] \xrightarrow{x_\sigma} \mathbb{C}$$

$$\mathfrak{m} = \ker x_\sigma = \langle u \in S_\sigma \setminus \langle \rho \rangle \rangle$$

$$\mathfrak{m}^2 = \langle s \in S_\sigma \setminus \langle \rho \rangle \mid s \text{ is a sum of } s_1, s_2 \in S_\sigma \setminus \langle \rho \rangle \rangle$$

$\mathfrak{m} / \mathfrak{m}^2$ has a basis $\{ s \in S_\sigma \setminus \langle \rho \rangle \mid \text{not a sum} \}$



$$\dim M / m^2 = 3 > 2 = \dim U_\sigma$$

$$\sigma^\perp = \ker \gamma \Rightarrow \geq n \text{ edges} \Rightarrow \sigma^\vee = \geq n \text{ edges}$$

Fact S_σ generates M as a group.

x_σ is a smooth point $\Rightarrow S_\sigma$ is generated by a basis for M .

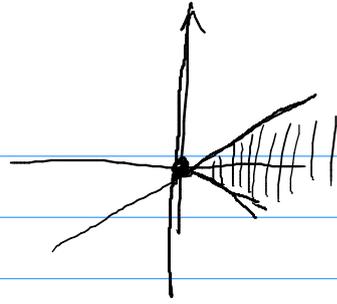


$$\Rightarrow U_\sigma \cong \mathbb{C}^n$$

$$\rightsquigarrow U_\sigma = \mathbb{C}^2$$

$$\sigma = \mathbb{R}^+ e_1 + \dots + \mathbb{R}^+ e_n \rightsquigarrow U_\sigma = \mathbb{C}^n$$

$$\sigma^\perp \neq \{0\}$$



$$N_\sigma = N \cap \sigma + N \cap (-\sigma) \subset N$$

• $N = N_\sigma \oplus N'$
 ↖ redundant dimensions.

we can view $\sigma \xrightarrow{\text{wave inside}} \tilde{\sigma} \subset N_\sigma$

$$\tilde{\sigma}^\perp = \{0\}$$

$$M = M_\sigma \oplus M'$$

\cup
 σ^\perp

$$S_\sigma = (\underbrace{\tilde{\sigma}^\perp \cap M_\sigma}_{S_{\tilde{\sigma}}}) \oplus M' \quad \uparrow S_{\text{ho}}'$$

$$\Rightarrow U_\sigma = U_{\tilde{\sigma}} \times T_{N'}$$

\uparrow \uparrow
 $\tilde{\sigma}$ is smooth (\Rightarrow) $\tilde{\sigma}^\perp$ is smooth

U_σ is non-singular $\Leftrightarrow \sigma$ is generated by part of a basis for N .

In this case $U_\sigma \cong \underbrace{\mathbb{C}^k} \times \underbrace{(\mathbb{C}^x)^{n-k}}$

$$U_\sigma(\mathbb{C}) = \text{Hom}(\mathbb{C}S_\sigma, \mathbb{C}) = \text{Hom}_{\text{smooth}}(S_\sigma, \mathbb{C})$$

$$T_N = S_{\text{ho}}$$

$$\sigma = \{0\}$$

$$\sigma \neq \{0\}$$

$$\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

$$U_{\text{ho}}(\mathbb{C}) = \text{Hom}_{\text{smooth}}(M, \mathbb{C}^x)$$

$$t \in T_N(\mathbb{C}) \quad \forall t \in U_\sigma(\mathbb{C})$$

$$t \cdot x : S_\sigma \longrightarrow \mathbb{C}$$

$$u \longmapsto x(u) t(u)$$

$$S_\sigma \xrightarrow{x_\sigma} \mathbb{C}$$

$$u \longmapsto \begin{cases} 1 & \text{if } \sigma = 0 \\ 0 & \text{otherwise} \end{cases}$$