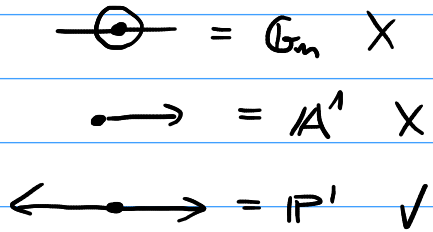
Toric varieties, 31.01.2022, Lecture 3

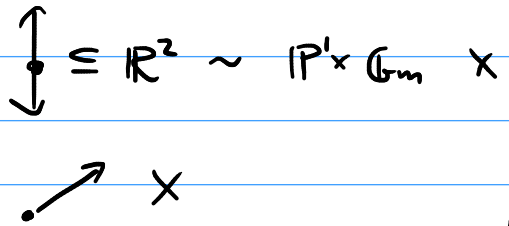
Properness of toric varieties

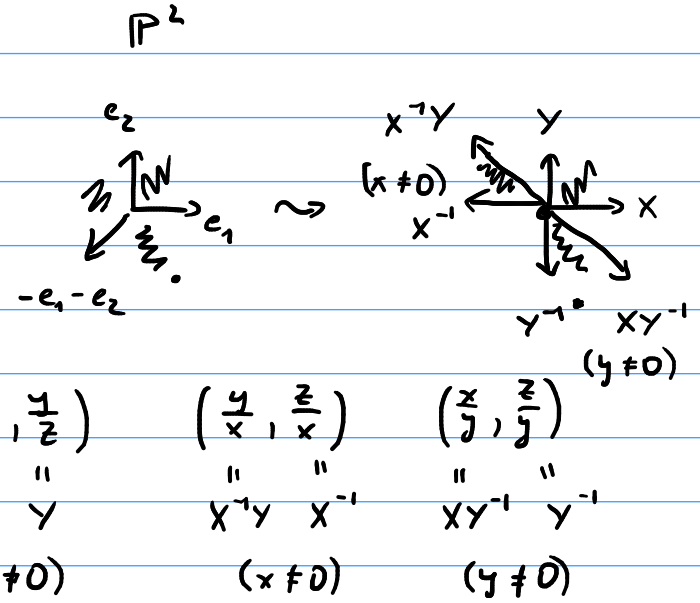
(remember to save during presentation!)

dim = 1

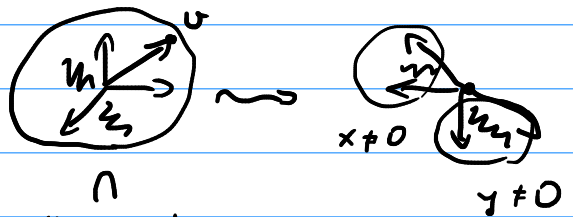


dim = 2





Throw out



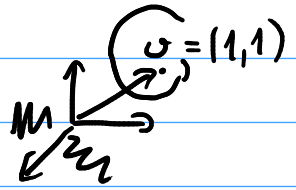
$\sim \mathbb{P}^2 \setminus [0:0:1]$ ← not proper!

$\cap$

"$\mathbb{P}^2$"

Idea: $\mathbb{G}_m \longrightarrow \mathbb{G}_m^n \subset X(\Delta)$ corresponding to such v will detect the missing point!

1-par subgr.



$$\begin{array}{ccc} \mathbb{A}^1 & & \\ \cup & & \\ \mathbb{G}_m & \longrightarrow & \mathbb{G}_m^2 \subset \mathbb{P}^2 \setminus [0:0:1] \end{array}$$

$z \longmapsto (z^1, z^1) = [z:z:1]$

$0 \longmapsto [0:0:1]$ – not possible!

Lemma. C : smooth curve, $c \in C$, X : proper (over $\operatorname{Spec}\mathbb{C}$).

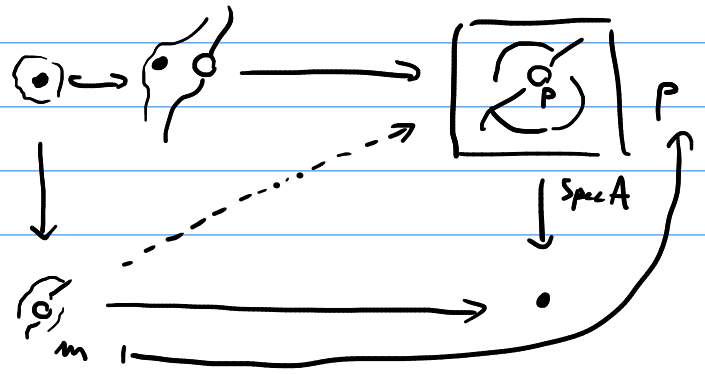
Then any morphism $C \setminus c \longrightarrow X$ has a (unique) ext. to $C \longrightarrow X$.

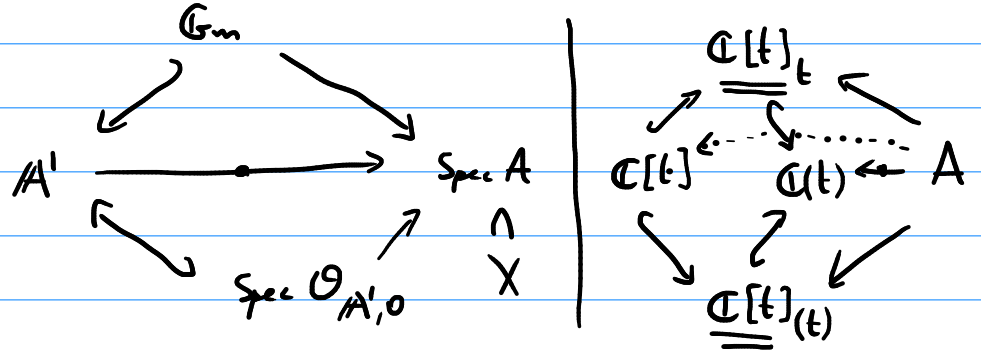
Proof. $(C = \mathbb{A}^1,\ c = 0)$ $\mathbb{A}^1 \setminus 0 \longrightarrow X$

$$\begin{array}{ccc} \operatorname{Spec} K = \operatorname{Spec} \mathbb{C}(\mathbb{A}^1) & \longrightarrow & X \\ \downarrow & \overset{\exists!}{\nearrow} & \downarrow \text{proper} \\ \operatorname{Spec} \mathcal{O} = \operatorname{Spec} \mathcal{O}_{\mathbb{A}^1,0} & \longrightarrow & \operatorname{Spec}\mathbb{C} \end{array}$$

The map $\nearrow$ exists by VCP.

Now, we can glue:





How to choose a good v (good to detect non-properness) ?

$$z \longmapsto (z^{v_1}, \dots, z^{v_n})$$

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{G}_m^n \\ \| & & \cap \| \\ \mathbb{A}^1 & \xrightarrow{\exists} & U_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma] \\ & & \cap \| \\ & & X(\Delta) \end{array}$$

$$\equiv$$

$$\begin{array}{ccccc} z^{\langle u,v\rangle} & \longleftarrow & z_1^{u_1} \cdots z_n^{u_n} & & \\ \| & & & & \\ \mathbb{C}[z^{\pm 1}] & \longleftarrow & \mathbb{C}[S_{\{0\}}] & & \\ \uparrow & & \uparrow & & \\ \mathbb{C}[z] & \longleftarrow & \mathbb{C}[S_\sigma] & \longleftarrow & \chi^u \\ z^{\langle u,v\rangle} & & \longleftarrow & & \end{array}$$

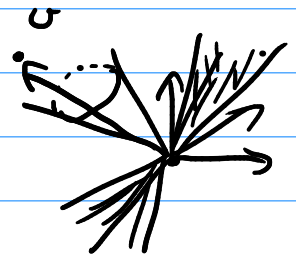
($z_1, \dots, z_n$: the coordinates on $\mathbb{G}_m^n$)

So the ext. exists iff v satisfies : $(\forall u \in \sigma^\vee \cap M)(\langle u, v\rangle \geq 0)$

$$\Updownarrow$$

$$v \in (\sigma^\vee)^\vee = \sigma$$

<u>Corollary.</u> If there is v outside all σ, then $X(\Delta)$ is not proper!



Theorem. Let $\varphi : N' \to N$ s.t. it maps Δ' to Δ so that it induces

$$\left[(\forall \sigma' \in \Delta')(\exists \sigma \in \Delta)(\varphi(\sigma') \subseteq \sigma) \right]$$

$\varphi_+ : X(\Delta') \to X(\Delta)$. Then φ_* is proper iff $\varphi^{-1}(|\Delta|) = |\Delta'|$.

Proof. $\Rightarrow$: skipped, done for $N = \mathbb{Z}^0 = \text{pt.} = \{0\}$.

$\Leftarrow$: We'll use the VCP in a strong form: fix $U \overset{\text{open}}{\subset} X$

$f : X \to Y$ between irreducible varieties is proper.

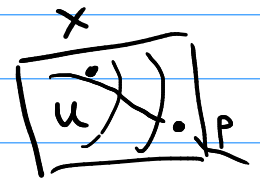
$$\iff \left(\begin{matrix} \forall \text{ diagrams with} \\ \operatorname{Spec} K \to U \subseteq X \end{matrix} \right) \left(\begin{matrix} \operatorname{Spec} K & \to & X \\ \downarrow & \exists! \nearrow & \downarrow \\ \operatorname{Spec} \mathcal{O} & \to & Y \end{matrix} \right)$$

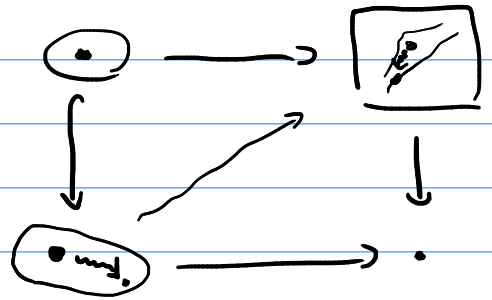
$\mathcal{O} \subset K$ is a DVR so

$K^* \xrightarrow{v} \mathbb{Z}$ with $\{v \geq 0\} = \mathcal{O}$

Nagata comp.

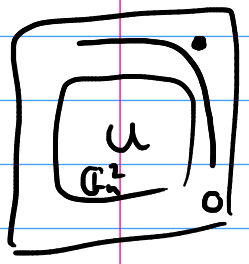
$$X \overset{\text{op}}{\hookrightarrow} \bar{X} \overset{\text{proper}}{\to} Y$$



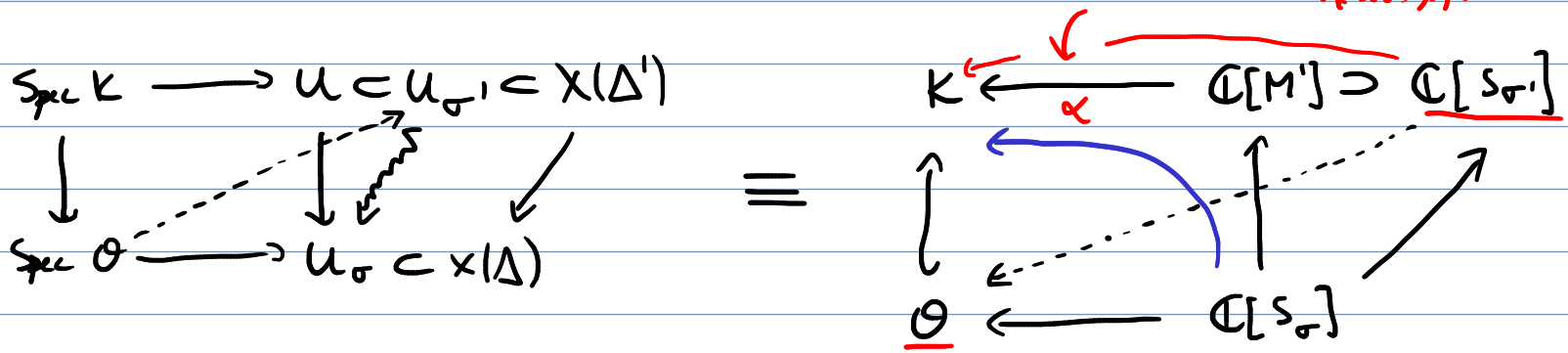


$U = \mathbb{G}_m^n$
$\cap$

X



Apply it for $X(\Delta') \longrightarrow X(\Delta)$:



makes sense for $\varphi(\sigma') \subseteq \sigma$

Note that : $\mathbb{C}[S_\sigma] \xrightarrow{\chi^{\varphi^*(u)}} \mathbb{C}[M'] \longrightarrow K$ it factors through $\mathcal{O}$ i.e.

$$M \xrightarrow{\varphi^*} M' \xrightarrow{\alpha} K^\times \xrightarrow{ord} \mathbb{Z}$$

$M \cong S_\sigma = \sigma^\vee \cap M$

$ord \circ \alpha$

$(ord \circ \alpha) \circ \varphi^* \in (\sigma^\vee)^\vee = \sigma$

$\Longrightarrow$ By ass., then is $\sigma' \in \Delta'$ with $\varphi(\sigma') \subset \sigma$ and $(ord \circ \alpha) \in \sigma' \in N'_{\mathbb{R}}$.

Recall : $\varphi^{-1}(|\Delta|) = |\Delta'|$

$\Downarrow$

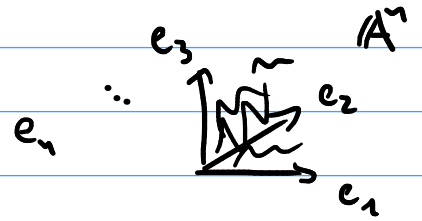
$(ord \circ \alpha)$

This implies that $(ord \circ \alpha) \in \sigma' = (\sigma'^\vee)^\vee$ is positive on $S_{\sigma'}$. $\square$

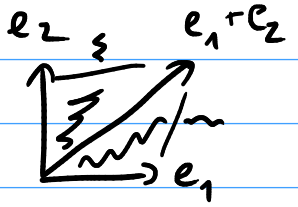
Blow-ups

Recall: $\mathrm{Bl}_{(0,\dots,0)} \mathbb{A}^n_{\mathbb{C}} = \bigcup_i \mathrm{Spec}\left(\frac{\mathbb{C}[x_1,\dots,x_n][y_{i1},\dots,y_{in}]}{(x_j - y_{ij} x_i)} \right)$, where u_i is the chart indexed by $(x_1,\dots,x_n)$.

$$I(x_1,\dots,x_n)\big|_{u_i} = I(x_i) \qquad [\text{here } y_{ij} = \tfrac{x_j}{x_i}]$$



Add a new vertex $e_0 = e_1 + \dots + e_n$:



Consider all subsets of $\{e_0, e_1, \dots, e_n\}$ not containing $\{e_1, \dots, e_n\}$ (i.e. not containing $e_1 + \dots + e_n$).

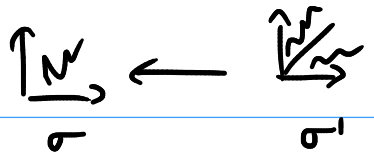
Let σ_i be the cone gen. by $e_0, \dots, \hat{e}_i, \dots, e_n$.

$$\sigma_i^\vee = \mathbb{N} e_i^* + \mathbb{N}(e_1^* - e_i^*) + \dots + \mathbb{N}(e_n^* - e_i^*)$$

$$\mathbb{C}[S_{\sigma_i}] = \mathbb{C}\left[x_i, \tfrac{x_1}{x_i}, \dots, \tfrac{x_n}{x_i}\right] = \mathbb{C}[x_i, y_{i1}, \dots, y_{in}]$$ – they glue together like the blow-up!

Moreover: $\Delta = \{\sigma\} \longleftarrow \Delta' = \{\sigma_i\}$

$\sigma = \sum \mathbb{R}^+ e_i$

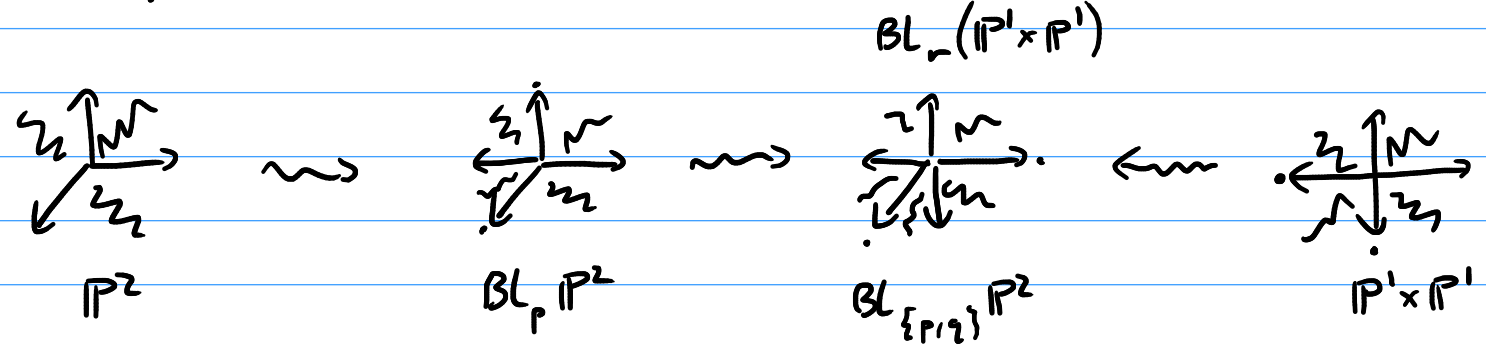


$X(\Delta) \longleftarrow X(\Delta')$

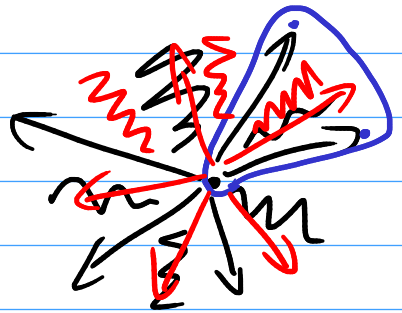
this is exactly the blow-up!

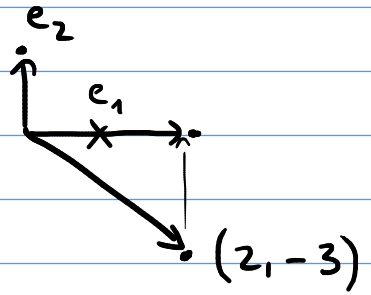
Sweet application:

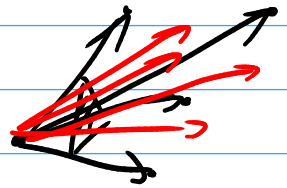
$$Bl_{\{p,q\}} \mathbb{P}^2 \simeq Bl_r(\mathbb{P}^1 \times \mathbb{P}^1)$$

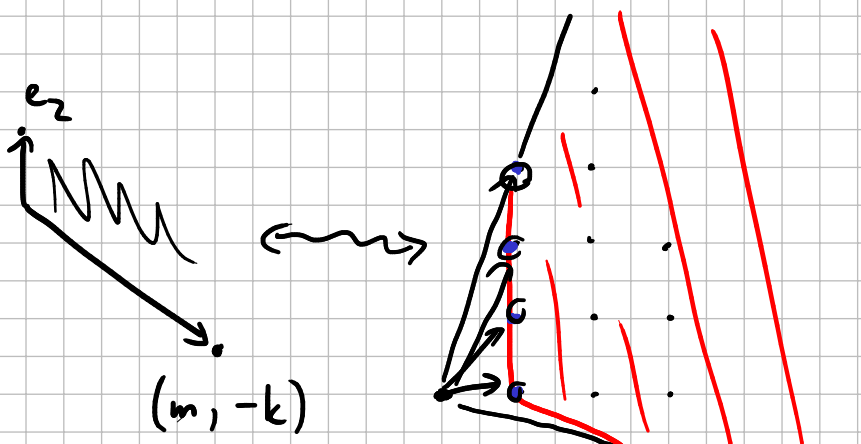
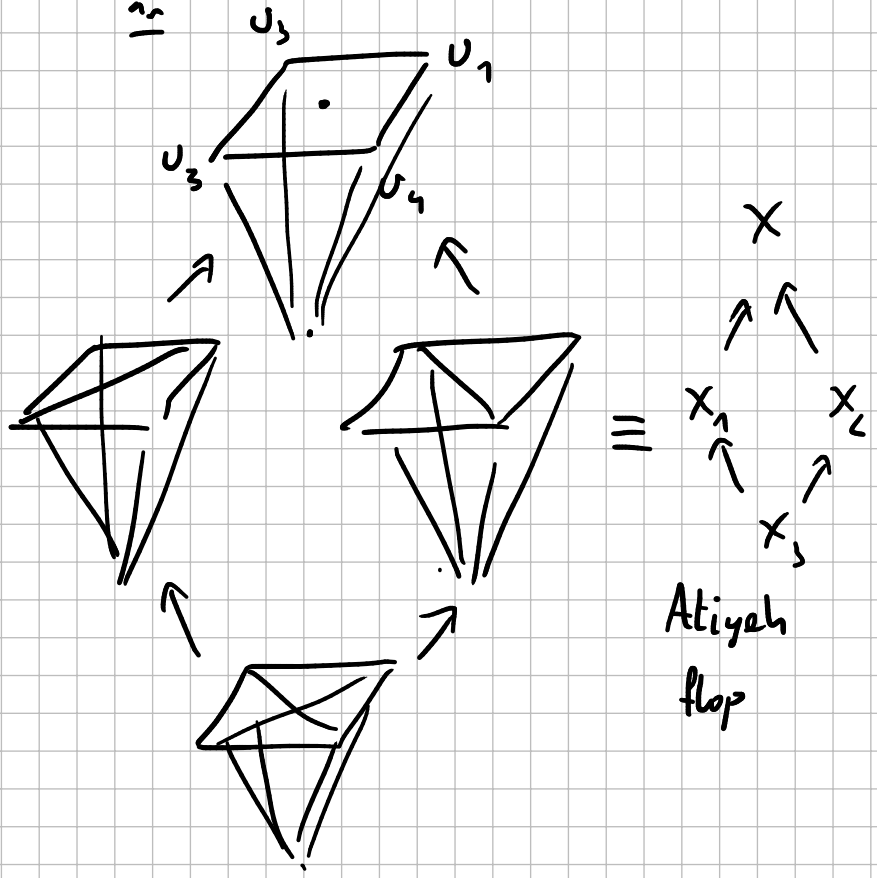


Note: $\Delta' \longrightarrow \Delta$ is a subdivision, then $X(\Delta') \longrightarrow X(\Delta)$ is proper (by the Theorem) and birational, as they both contain same dimension.




e_2
e_1
$(2, -3)$





$$\left(\frac{m}{k}\right) = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}}$$