

## ⑤ Divisors & Line bundles

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### ⑥ Plan I

- $\mathbb{A}^n$ -invariant Weil/Cartier divisors
  - How to compute them from fan
  - How to relate them to general divisors.
  - Divisor class groups  $Cl(X)$ ,  $CaCl(X)$
- Line bundles
  - Picard group  $Pic(X) = CaCl^T(X)$
  - Line bundles  $\leftrightarrow$  Piecewise linear function on  $N_{\mathbb{R}}$   $\psi$
  - Sections of line bundles  $\leftrightarrow$  Polytopes in  $N_{\mathbb{R}}$
  - Ampleness  $\leftrightarrow \psi$  is convex

### ① Divisors

Def • A Weil Divisor is a finite formal sum of codimension-1 irreducible subvarieties  $V_i \subseteq X$ :

$$D = \sum a_i V_i \quad a_i \in \mathbb{Z}$$

• A Cartier Divisor is given by

- affine covering  $X = \cup U_\alpha$

- non-zero rational functions  $f_\alpha$  on  $U_\alpha$

such that  $\frac{f_\alpha}{f_\beta}$  is nowhere vanishing and regular on  $U_\alpha \cap U_\beta$ .

(the locus of order 1 poles of  $f_\alpha$  cut out a codim-1 subvariety in nice cases)

We get the ideal sheaf  $\mathcal{O}(-D)$  which is the subsheaf of rational functions generated by  $f_\alpha$ .  
(functions which can only have poles of order 1 along  $D$  and nowhere else.)

The inverse sheaf  $\mathcal{O}(D)$  is generated by  $1/f_\alpha$ .

If  $X$  normal  $\Rightarrow$  local rings at generic points of codim-1 subvarieties are DVR

For  $D$  Cartier divisor we define a Weil divisor

$$[D] = \sum_{\substack{\text{codim}(V_i, X) = 1 \\ V_i \text{ irred.}}} \text{ord}_V(D) \cdot V$$

$\uparrow$  valuation of local ring  
 $\mathcal{O}_{X, \eta_V}$  - fraction field

$V$  irred.

valuation of local ring

$$\mathcal{O}_{x, \mathbb{A}^1} \hookrightarrow \text{fraction field} \\ \text{ord}_v : K(x) \rightarrow \mathbb{Z}$$

For  $f \in K(x)$ ,  $f \neq 0$  we can define a

- Cartier divisor:  $\{U_\alpha, f|_{U_\alpha}\} = \text{div}(f)$
- Weil divisor:  $[\text{div}(f)] = \sum \text{ord}_v(f) \cdot V$

Then are the principal divisors. We get the divisor class groups:

- $Cl(X) = \{\text{Weil-Divisors}\} / \{\text{princ. Weil-div.}\}$
- $CaCl(X) = \{\text{Ca-Div}\} / \{\text{princ. Ca-Div}\}$

Let's look at  $\mathbb{A}^n$ -invariant divisors.

### Weil | Orbit-fan correspondence

codim-1 irreducible subvarieties invariant under  $\mathbb{A}^n$ -action

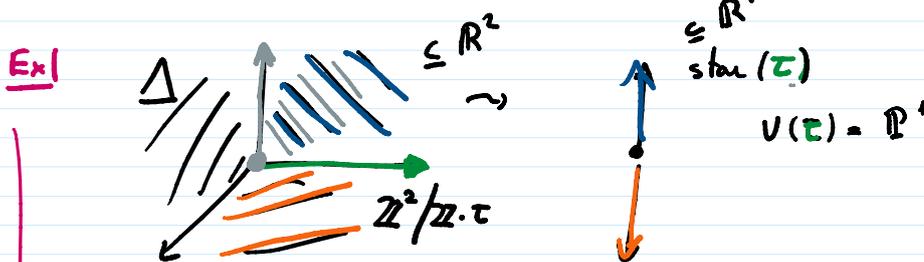
$$\updownarrow \\ 1\text{-dim sub-cones.} = \underline{\text{edges of } \Delta}$$

Denote edges of  $\Delta$  by  $\tau_1, \dots, \tau_n$

First lattice point of  $\tau_i$  by  $v_i$ .

Then  $D_i := V(\tau_i) = X(\text{star}(\tau_i))$

Are the  $\mathbb{A}^n$ -invariant irred. subvarieties.



$\mathbb{P}^2$  has three torus invariant irred. subvar.,  
all  $\mathbb{P}^1$ 's "along  $x, y, z$ "

**Def |** A T-Weil divisor is a <sup>finite</sup> linear

$$\text{combination } \sum a_i D_i$$

Cartier | Require  $\text{div}(f_\alpha)$  on  $U_\alpha$  to be  $\mathbb{A}^n$ -invariant

For  $X = U_\sigma$  affine  $\dim_\sigma = n$

$f$  nonzero rational function

$\mathbb{T}^n$ -invariant  $\text{div}(f)$

$\Rightarrow$  no zeros or poles on  $\mathbb{T}^n \cong X$

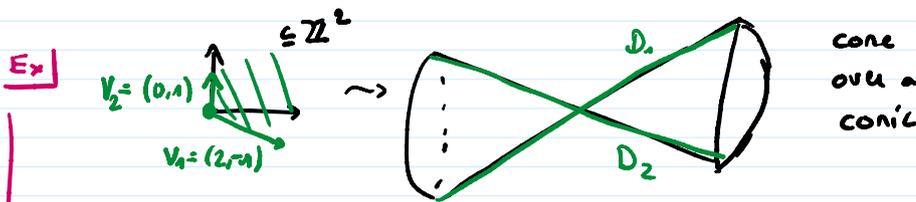
algebra  $\leadsto f = \lambda \cdot \chi^u \quad \lambda \in \mathbb{C}^\times, u \in M$

Thus any  $T$ -Cartier is  $\text{div}(\chi^u) \quad u \in M$

Prop  $u \in M \quad v \in N$  first lattice point of ray  $\tau$ .

Then  $\text{ord}_{v(\tau)}(\chi^u) = \langle u, v \rangle$

so  $[\text{div}(\chi^u)] = \sum_{i=1}^n \langle u, v_i \rangle D_i$



If  $u = (p,q) \in \mathbb{Z}^2$  then

$\text{div}(\chi^u) = (2p-q)D_1 + qD_2$

$\Rightarrow D_1$  is not Cartier but  $2D_1$  is

If  $\sigma$  full-dim  $\Rightarrow$  rays  $\tau_i$  generate  $N_\mathbb{R}$

$\Rightarrow [\text{div}(\chi^u)] = \sum \langle u, v_i \rangle D_i$

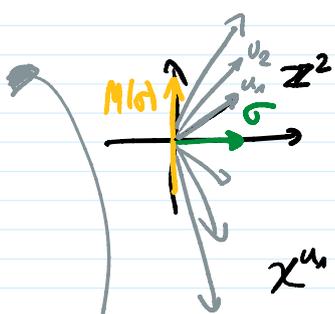
$\Rightarrow u$  uniquely determined.

If  $\sigma$  not full-dim

Then  $[\text{div}(\chi^u)] = [\text{div}(\chi^{u'})]$

$u - u' \in \sigma^\perp \cap M =: M(\sigma)$

Thus  $T$ -Cartier on  $U_\sigma$   
 $\Downarrow$   
 $M / M(\sigma)$



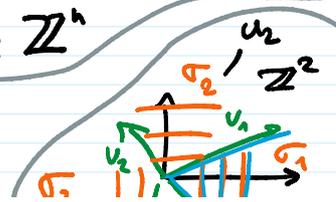
$\text{Spec}(\mathbb{C}[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}])$

$(\mathbb{C}^\times)^n$

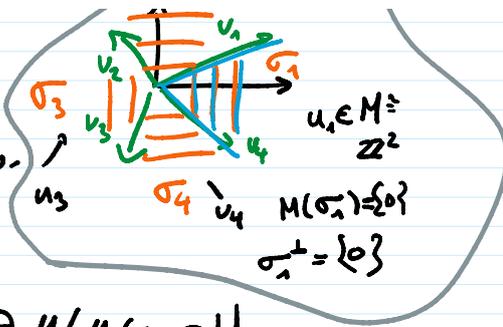
$\chi^u = X_1^{u_1} X_2^{u_2} \dots X_n^{u_n}$

$u \in \mathbb{Z}^n$

On general  $X(\Delta)$   $T$ -Cartier are  
 associated  $\sigma \mapsto u(\sigma) \in M / M(\sigma)$



On general  $X(\Delta)$  T- Cartier are assignments  $\sigma \mapsto u(\sigma) \in M/M(\sigma)$  which are compatible with intersections.



Prop  $\text{Div}_T^C(X) = \varprojlim M/M(\sigma)$

$$\left\{ \begin{array}{l} \text{[T-Cartier]} \\ \text{=} \ker \left( \bigoplus_i M/M(\sigma_i) \rightarrow \bigoplus_{i < j} M/M(\sigma_i \cap \sigma_j) \right) \\ \bigoplus_i [u_i] \mapsto \bigoplus_{i < j} (u_j - u_i) \end{array} \right.$$

Def Pic(X) group of line bundles / iso.

There is a map  $\Pi: \text{Div}^C(X) \rightarrow \text{Pic}(X)$   
 $D \mapsto \mathcal{O}(D)$

for X irreducible. In fact  $\text{Cl}(X) \cong \text{Pic}(X)$ .

Prop  $X = X(\Delta)$   $\Delta$  not contain in proper subspace of  $\mathbb{N}^n$ .

The following commutes with exact rows:

$$\begin{array}{ccccccc} 0 & \rightarrow & M & \xrightarrow{u \mapsto \text{div}(x^u)} & \text{Div}_T^C(X) & \xrightarrow{\Pi} & \text{Pic}(X) & \rightarrow & 0 \\ & & \parallel & & \downarrow d & & \downarrow & & \downarrow \mathcal{O}(D) \\ 0 & \rightarrow & M & \xrightarrow{u \mapsto \sum_{i=1}^d \langle u, v_i \rangle D_i} & \bigoplus_{i=1}^d \mathbb{Z} \cdot D_i & \rightarrow & \text{Cl}(X) & \xrightarrow{[\cdot]} & 0 \end{array}$$

$\text{Cl}(X) = \text{Divisors} / \text{princ. Div}$

Cor -  $\text{rank}(\text{Pic}(X)) \leq \text{rank}(\text{Cl}(X)) = d - n \geq 0$

Pic(X) free abelian

Proof Prop

•  $X = \bigcup_{i=1}^d D_i \cup \mathbb{T}^n$

All divisors on  $\mathbb{T}^n$  principal (since  $\mathbb{C}[x_1, \dots, x_n]$  UFD)

→ Exact sequence

$$\begin{array}{ccccc} \bigoplus_{i=1}^d \mathbb{Z} \cdot D_i & \rightarrow & \text{Cl}(X) & \rightarrow & \text{Cl}(\mathbb{T}^n) = 0 \\ \uparrow & & \downarrow & & \\ M & & & & \end{array}$$

Now if  $\sum a_i D_i = 0$  in  $\text{Cl}(X)$

Then  $a_i D_i = \text{div}(f)$  with  $f = \sum x^u$

$$[\text{div}(f)] = [\text{div}(x^u)] = \sum \langle u, v_i \rangle D_i$$

$u$  uniquely determined since  $v_i$  span  $\mathbb{N}^n$ .

$L_{\text{div}}(1,1) - L_{\text{div}}(2,1) = \sum u_i v_i / v_i$   
 $u$  uniquely determined since  $v_i$  span  $N_{\mathbb{R}}$ .

•  $\mathcal{L} = \mathcal{O}(E)$  any Cartier.

$\mathcal{L}|_{\mathbb{P}^n}$  trivial since  $C(\mathbb{P}^n) = 0$

$$\Rightarrow \mathcal{L}|_{\mathbb{P}^n} = \text{div}(f)$$

Then  $D = E - \text{div}(f) \sim E$  and  $D$  supported away from  $\mathbb{P}^1$ . Thus  $D$  T-Weil  $\Rightarrow$  T-Cartier.

Prop | All max cones n-dim □  
 $\Rightarrow \text{Pic}(X(\Delta)) \cong H^2(X(\Delta), \mathbb{Z})$

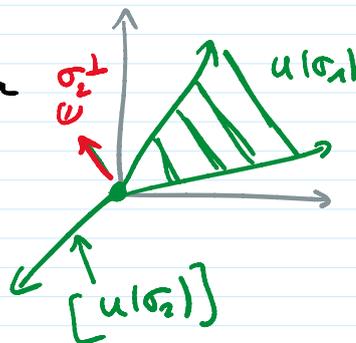
Pf  $\text{Div}_T^c = \text{Ker} \left( \bigoplus_i M/M(\sigma_i) \rightarrow \bigoplus_{i < j} M/M(\sigma_i \cap \sigma_j) \right)$   
 $H^2(X(\Delta), \mathbb{Z}) = \text{Ker} \left( \bigoplus_{i < j} H(\sigma_i \cap \sigma_j) \rightarrow \bigoplus_{i < j < k} H(\sigma_i \cap \sigma_j \cap \sigma_k) \right)$

$$\text{Div}_T \rightarrow H^2$$

$$\bigoplus u_i \mapsto \bigoplus_{i < j} (u_j - u_i)$$

D

The data  $D = \{u(\sigma) \in M/M(\sigma)\}$  defines a piecewise linear continuous function  $\psi_D$  on the support  $|\Delta|$ .



$$\psi_D|_{\sigma}(v) = \langle u(\sigma), v \rangle \quad v \in \sigma$$

Conversely, if  $\psi$  piecewise linear continuous and integral  $\Rightarrow \psi = \psi_D$  for some unique on each cone  $\langle u(\sigma), v \rangle$  T-Cartier  $D$

If  $D = \sum \alpha_i D_i \Rightarrow \psi_D$  determined by  $\psi_D(v_i) = -\alpha_i$

$$[D] = \sum -\psi_D(v_i) D_i$$

Prop | •  $\psi_{D+E} = \psi_D + \psi_E$   
 •  $\psi_{mD} = m \psi_D$   
 •  $\psi_{\text{div}(X^n)} = -u$

- $T_{\Delta D} = M \cdot \psi_D$
- $\psi_{\text{div}(X^u)} = -u$
- $D \sim E \Rightarrow \psi_E = A \circ \psi_D$   
 $\uparrow$   
 linear map  $M \rightarrow M$

T-convex  $D = \sum a_i D_i$  determines a rational convex polytope in  $M_{\mathbb{R}}$

$$\begin{aligned} P_D &= \{ u \in M_{\mathbb{R}} : \langle u, v_i \rangle \geq -a_i \text{ for all } i \} \\ &= \{ u \in M_{\mathbb{R}} : u \geq \psi_D \text{ on } |\Delta| \} \end{aligned}$$

Prop  $\Gamma(X, \mathcal{O}(D)) = \bigoplus_{u \in P_D \cap M} \mathbb{C} \cdot X^u$

Pf on affines

$$\Gamma(U_{\sigma}, \mathcal{O}(D)) = \bigoplus_{P_D(\sigma) \cap M} \mathbb{C} \cdot X^u$$

$$P_D(\sigma) = \{ u \in M_{\mathbb{R}} : \langle u, v_i \rangle \geq -a_i \ \forall v_i \in \sigma \}$$

since sections need to have zeros along  $D_i$  of order  $\geq -a_i$ . By Lemma

$$\text{ord}_{D_i}(X^u) = \langle u, v_i \rangle$$

□

If  $X(\Delta)$  complete (i.e.  $|\Delta| = M_{\mathbb{R}}$ ), then cohomology of coherent sheaves such as  $\mathcal{O}(D)$  are f.d. ( $\Rightarrow P_D \cap M$  finite!)

Can see this directly!

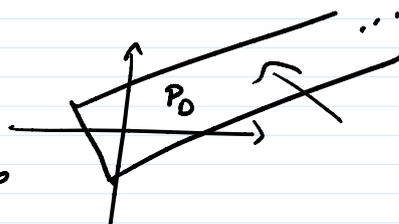
Pf  $\nearrow P_D$  unbounded

Find  $u_i \in P_D$  with  $|u_i| \rightarrow \infty$  as  $i \rightarrow \infty$ .

Then  $S^{n-1}$  cpt  $\Rightarrow$  WLOG  $u_i/|u_i| \rightarrow u \in S^{n-1}$ .

Now  $u_i \in P_D \Rightarrow \langle u_i, v_j \rangle \geq -a_j \Rightarrow \langle u, v_j \rangle \geq 0$ .

But  $v_j$  span  $M_{\mathbb{R}} \Rightarrow u \cdot 0 \notin S^{n-1}$   $\blacksquare$



now  $u, v \in \mathbb{R}^n \Rightarrow \langle u, v \rangle \geq -a_j \Rightarrow \langle u, v \rangle \geq 0$ .

But  $v_j$  span  $N_R \Rightarrow u \cdot 0 \notin S^{n-1}$  

□

## Ampleness

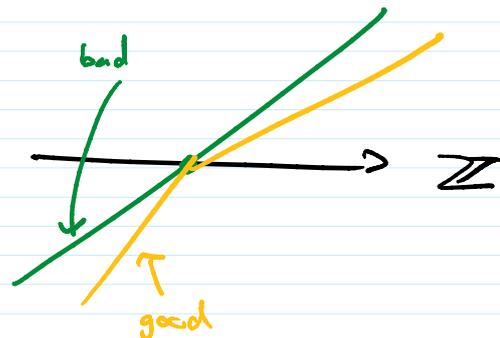
Def  $\psi: V \rightarrow \mathbb{R}$  is upper convex



if  $\psi(t \cdot v + (1-t) \cdot w) \geq t \cdot \psi(v) + (1-t) \cdot \psi(w)$

Def A convex  $\psi = \psi_D$  is strictly convex

if for any  $n$ -dim cones  $\sigma \neq \sigma'$  we have  $u(\sigma) \neq u(\sigma')$ .



Prop Assume all max. cones of  $\Delta$  are full dimensional.

$D$  a T-Cartier divisor.

Then  $\mathcal{O}(D)$  is generated by its sections

(For each  $x \in X \exists s \in \Gamma(X, \mathcal{O}(D))$   $s(x) \neq 0$ )

iff  $\psi_D$  is convex

Prop Suppose  $|\Delta| = N_{\mathbb{R}}$ .

•  $D$  T-Cartier very ample

$\Updownarrow$

$\psi_D$  strictly convex and for each

$n$ -dim. cone  $S_\sigma$  generated by

$\{u - u(\sigma) : u \in P_{0,n,M}\}$

•  $D$  ample  $\Leftrightarrow \psi_D$  strictly convex

## Applications

• Any complete two-dim toric variety is projective

• Can build non-projective varieties by

projective

- Can build non-projective varieties by creating a fan s.t. no function  $\psi$  can be strictly convex.  $\rightarrow ?$