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Toric Geometry Reading group

More on line bundles

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Recap

N lattice $(\cong \mathbb{Z}^n)$, M dual lattice.

From a fan Δ in $N_{\mathbb{R}}$ (collection of cones) we create a toric variety $X = X(\Delta)$ by gluing $U_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma]$, $\sigma \in \operatorname{Cone}(\Delta)$, where

$$S_\sigma = M \cap \sigma^\vee,$$

$\sigma = \{0\}$, $U_\sigma = T$ $\quad \sigma^\vee = \{ u \in M \mid \langle u, v \rangle \geq 0, \ \forall v \in \sigma \}$.

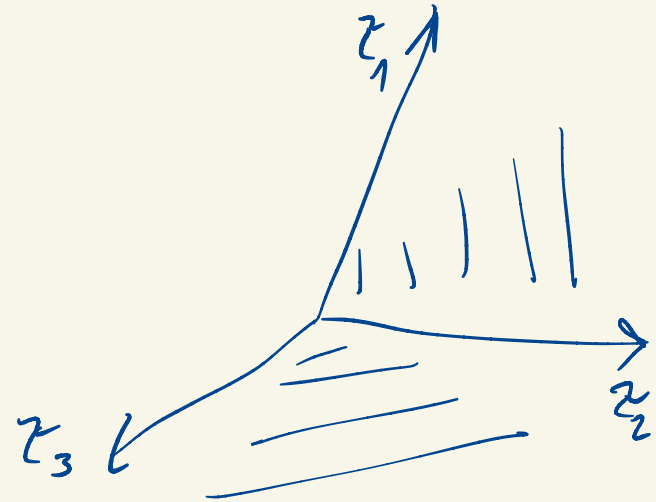
Any $u \in M$ defines a rational function x^u on X, as u gives

$$\begin{tikzcd} X \\ T \arrow[u, phantom, "\cap"] \arrow[r, "u"] \arrow[rr, bend right] & \mathbb{C}^\times \arrow[r] & \mathbb{C} \end{tikzcd}$$

functions on an open subset of X.

Divisors

Let τ_i be the edges of Δ.



Each τ_i determines an orbit closure $V(\tau_i) = D_i$ of codim 1.

T-Weil divisors $\overset{\sim}{\longleftrightarrow}$ sums $\sum a_i D_i$, $a_i \in \mathbb{Z}$.

T-Cartier divisors $D \overset{\sim}{\longleftrightarrow}$ piecewise linear functions ψ_D on $|\Delta|$ s.t. on $|\sigma|$, $\psi_D(v) = \langle u(\sigma), v \rangle$, $v \in |\sigma|$, for some $u(\sigma) \in M$ $\left(M/M(\sigma),\ M(\sigma) = M \cap \sigma^{\perp} \right)$.

<u>Properties:</u>

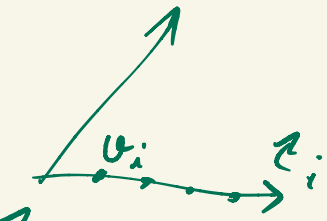
- $\psi_D + \psi_{D'} = \psi_{D+D'}$

- $\psi_{\mathrm{div}(\chi^u)} = -u$

associated Weil divisor

- $[D] = \sum_i -\psi_D(v_i) \cdot D_i$

v_i : first integral vector along τ_i



- Have SES

T-Cartier divisors

$$0 \to M \to \mathrm{Div}_T X \longrightarrow \mathrm{Pic}(X) \longrightarrow 0$$

$$u \longmapsto \mathrm{div}(\chi^u) = -u$$

- Can define the polytope of linear functions over ψ_D.

$$P_D = \{ u \in M_{\mathbb{R}} \mid u \geq \psi_D \text{ on } |\Delta| \},$$

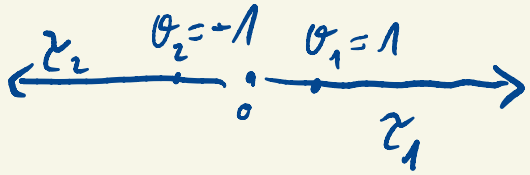
then

$$H^0(X, \mathcal{O}(D)) = \bigoplus_{u \in P_D \cap M} \mathbb{C} \cdot \chi^u .$$

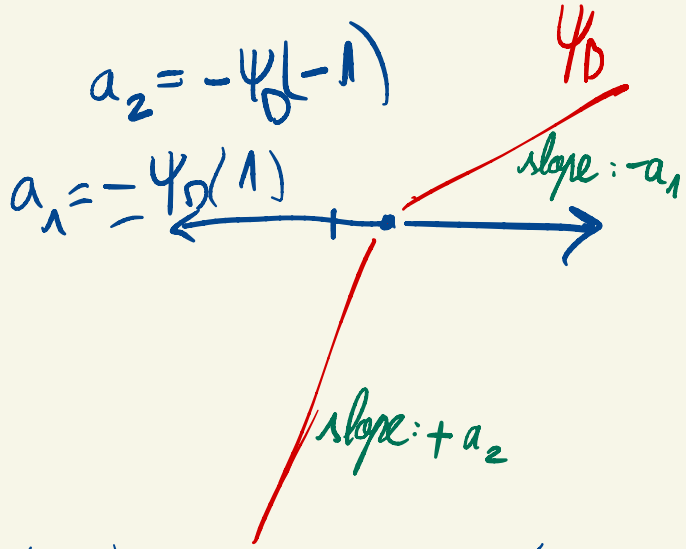
- If X is complete, then P_D is bounded.

- ψ_D and $\psi_{D'}$ determine the same line bundle $\iff$ $\psi_D - \psi_{D'}$ is linear.

Example: $\mathbb{P}^1$. Fan



Cartier divisors:



$$[D] = -\psi_D(v_1)\, D_1 - \psi_D(v_2)\, D_2 = a_1 D_1 + a_2 D_2$$

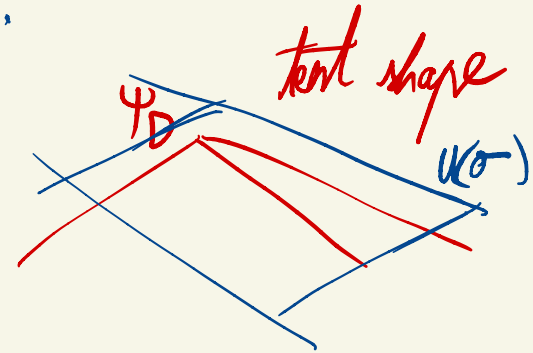
ψ_D concave : $a_2 \geq -a_1 \iff a_1 + a_2 \geq 0$.

$\mathcal{O}(D) = \mathcal{O}(a_1 + a_2) \iff$ generated by global sections.

φ_D strictly concave: $a_2 > -a_1 \iff \mathcal{O}(D)$ is ample.

Ampleness

DEF • ψ_D <u>concave</u> $\iff$ for all σ, $u(\sigma) \geq \psi_D$
(for some representative of $u(\sigma)$).



• Assume $|\Delta| = N_{\mathbb{R}}$, then

ψ_D <u>strictly concave</u> $\iff$ ψ_D concave and $u(\sigma) \neq u(\sigma')$
for $\sigma \neq \sigma'$ maximal faces.

Prop $\underbrace{\mathcal{O}(D) \text{ generated by global sections}}_{(1)} \iff \underline{\psi_D \text{ convex}}$.

Pf:

(1) $\iff$ $\forall \sigma$ cone in Δ, $\exists\, \omega(\sigma) \in M$ s.t.

i) $\omega(\sigma) \in \underline{P_D}$ ($\omega(\sigma)$ is global section)

ii) $\omega(\sigma)$ doesn't vanish on $\underline{U_\sigma}$.

i) $\iff \langle \omega(\sigma), v_i \rangle \geq -a_i \iff \underline{\omega(\sigma) \geq \psi_D}$.

ii) $\iff \underbrace{\langle \omega(\sigma), v_i \rangle}_{\text{order of vanishing of } \omega(\sigma) \text{ along } D_i} + a_i = 0$, $\forall i, \tau_i < \sigma \iff$

$\iff \omega(\sigma) = \psi_D$ on $|\sigma|$,

$\omega(\sigma) \neq u(\sigma)$, in $M/M(\sigma)$

$\square$

PROP X complete. $\mathcal{O}(D)$ ample $\iff$ ψ_D strictly convex.

Example: $\mathbb{P}^n$, $\underbrace{e_0, e_1, \dots, e_n}_{\text{basis}}$, $e_0 + e_1 + \dots + e_n = 0$.

Δ : fan whose cones are generated by proper subsets of $\{e_0, \dots, e_n\}$.

Maximal cones : $\sigma_i = \mathbb{R}_{\geq 0}\langle e_0, \dots, \hat{e_i}, \dots, e_n \rangle$

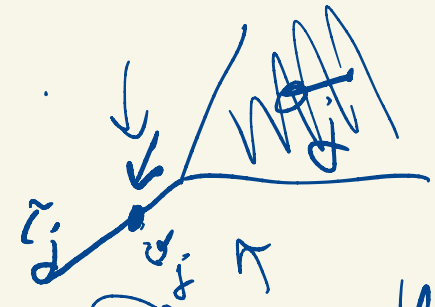
Edges : $\tau_i = \mathbb{R}_{\geq 0}\langle e_i \rangle$

Take a divisor $D = a_0 D_0 + \dots + a_n D_n$.

$$\varphi_D(e_i) = -a_i$$

On σ_j, $\quad \psi_D(v) = \langle u(\sigma_j), v \rangle, \quad v \in |\sigma_i|$,

$$\langle u(\sigma_j), e_i \rangle = -a_i, \quad i \neq j.$$



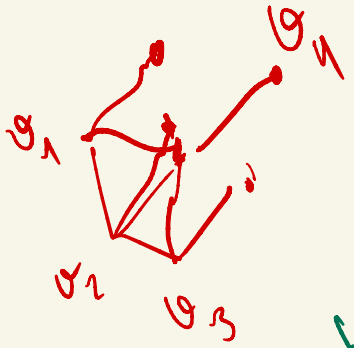
$\mathcal{O}(D)$ globally generated $\iff$ $\underbrace{\langle u(\sigma_j), e_j \rangle}_{\substack{\| \\ \langle u(\sigma_j), -\sum_{i \neq j} e_i \rangle \\ \| \\ \sum_{i \neq j} a_i}} \geq -a_j \quad \forall j$

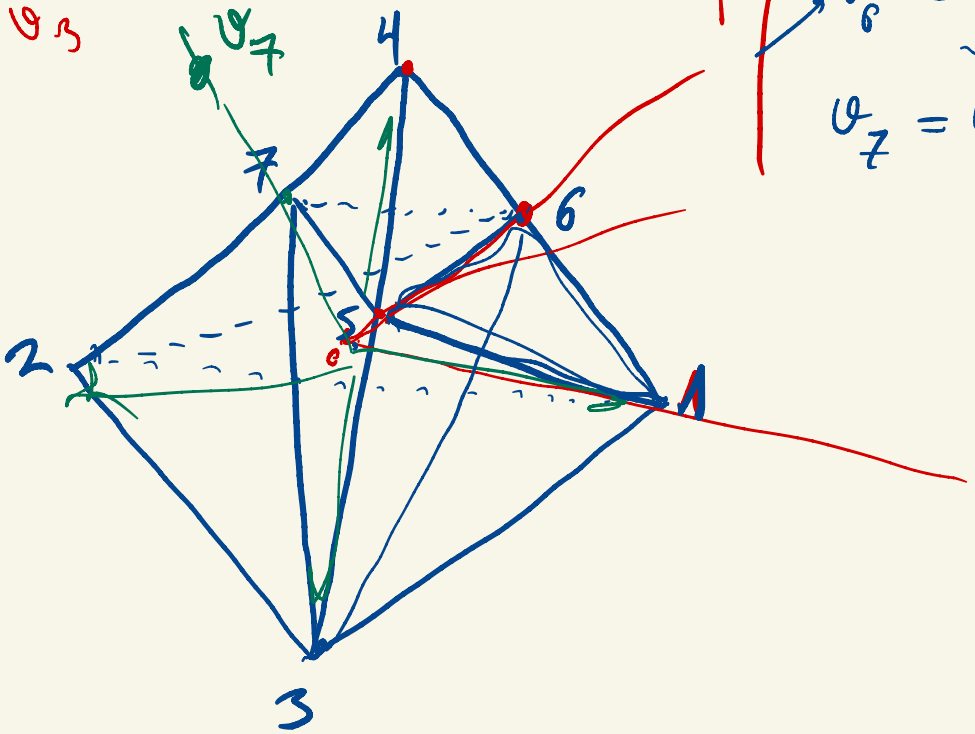
$\iff \sum a_i \geq 0.$ $\qquad \mathcal{O}(D) = \mathcal{O}\left(\sum_i a_i\right).$

Example Proper and non-projective variety.

in $\mathbb{R}^3 = N_{\mathbb{R}}$. e_1, e_2, e_3 basis,

$$\begin{cases} v_1 = -e_1 \\ v_2 = -e_2 \\ v_3 = -e_3 \\ v_4 = e_1 + e_2 + e_3 \\ v_5 = v_3 + v_4 \\ v_6 = v_1 + v_4 \\ v_7 = v_2 + v_4 \end{cases}$$





Δ

$X(\Delta)$ projective?

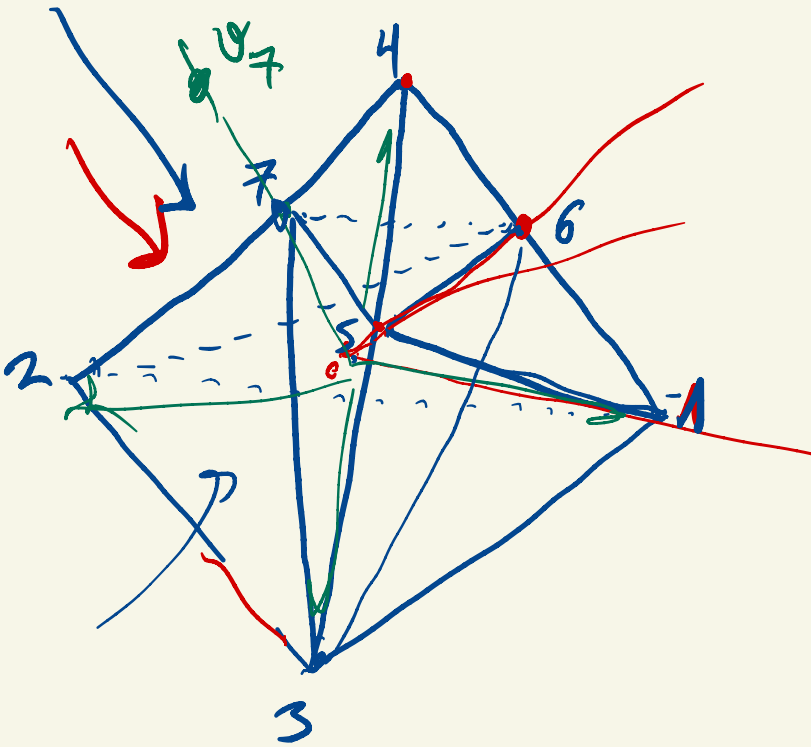
If so, $\exists\ \psi$ piecewise linear and strictly concave.

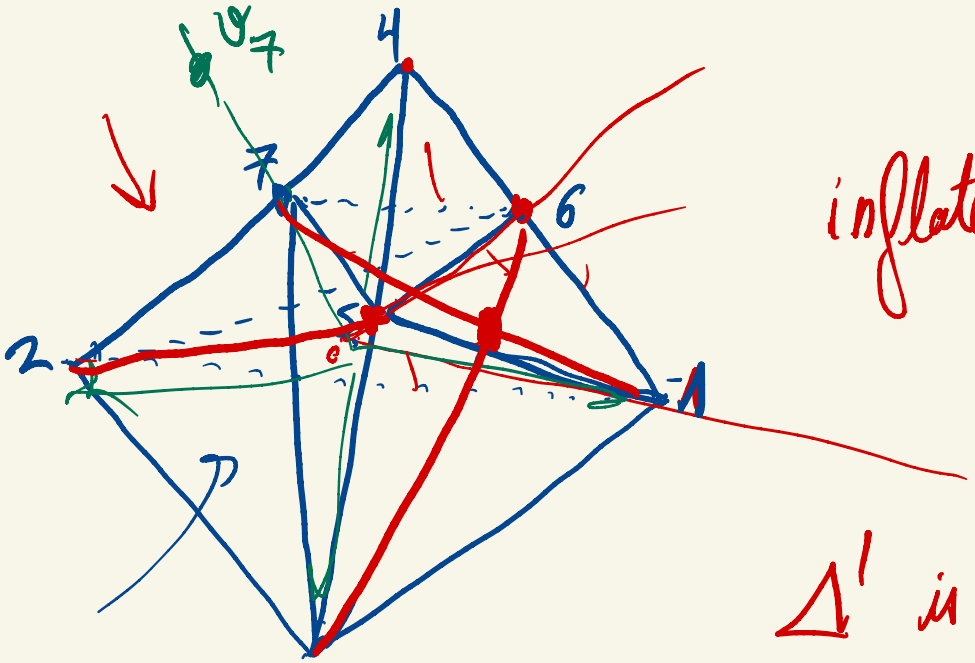
$$\underline{\psi(v_1) + \psi(v_5)} = \psi(v_1 + v_5) = \psi(v_3 + v_6) > \psi(v_3) + \psi(v_6)$$

$$\psi(v_3) + \psi(v_7) > \psi(v_2) + \psi(v_5)$$

$$\psi(v_2) + \psi(v_6) > \psi(v_7) + \psi(v_1)$$

Adding up, get a contradiction.

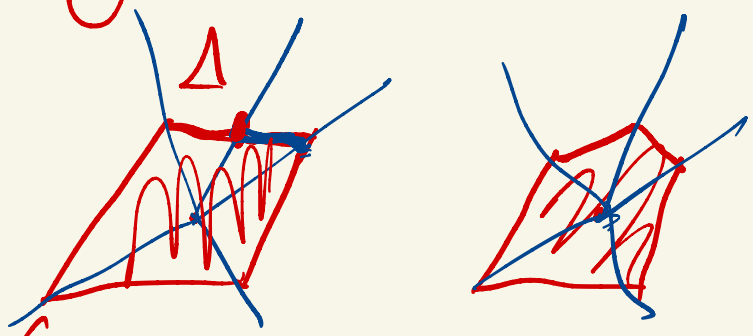




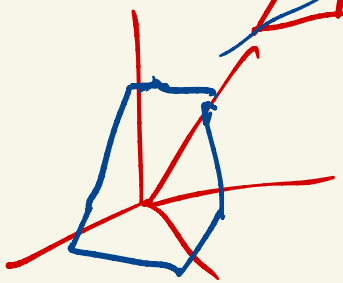
inflate it so that

Δ' is generated from convex polytope C containing the origin in it's interior.

$$\underline{\Psi(v)} = \min_{u \in C^0} \langle u, v \rangle$$



Ψ strictly convex.

Surfaces:  are projective!

Cohomology of line bundles:

$$H^0(X, \mathcal{O}(D)) = \bigoplus_{u \in M} H^0(X, \mathcal{O}(D))_u = \bigoplus_{u \in P_D \cap M} \mathbb{C} \cdot \chi^u$$

graded by M

So $$H^0(X, \mathcal{O}(D))_u = \begin{cases} \mathbb{C} \cdot \chi^u, & u \in P_D \\ 0, & u \notin P_D. \end{cases}$$

Now define, for $u \in M$, $Z(u) = \{ v \in |\Delta| \mid \langle u, v \rangle \geq \psi(v) \}$

$$H^0(X, \mathcal{O}(D))_u \neq 0 \iff Z(u) = |\Delta| \iff H^0_{Z(u)}(|\Delta|) \neq 0$$

$$H^0_{Z(u)}(|\Delta|) = H^0(|\Delta|, |\Delta| \setminus Z(u)) =$$

$$= \ker\left(\underline{H^0(|\Delta|) \longrightarrow H^0(\underline{|\Delta| \setminus Z(u)})} \right)$$

PROP For all $p \geq 0$,

$$H^p(X, \mathcal{O}(D)) \cong \bigoplus_{u \in M} H^p(X, \mathcal{O}(D))_u ;$$

$$H^p(X, \mathcal{O}(D))_u = H^p_{Z(u)}(|\Delta|).$$

LES.

$$H^{p-1}(|\Delta| \setminus Z(u)) \longrightarrow H^p_{Z(u)}(|\Delta|) \longrightarrow H^p(|\Delta|) \longrightarrow H^p(|\Delta| \setminus Z(u)) \longrightarrow H^{p+1}_{Z(u)}(|\Delta|)$$

Example X affine, $H^p(X, \mathcal{O}(D)) = 0$, $p > 0$.

$|\Delta|$ just one cone σ, ψ_D is linear, $\psi_D(v) = \langle u(\sigma), v \rangle$

$v \in Z(u) \iff \langle u - u(\sigma), v \rangle \geq 0$, cut out by hyperplane,

so $Z(u)$, $|\Delta| \setminus Z(u)$ convex.

$$\begin{tikzcd}
H^{p-1}(|\Delta| \setminus Z(u)) \arrow[r, "p \geq 2"] \arrow[d, equal] & H^p_{Z(u)}(|\Delta|) \arrow[r] \arrow[d, equal, "\Rightarrow 0"] & H^p(|\Delta|) \arrow[r] \arrow[d, equal] & H^p(|\Delta| \setminus Z(u)) \arrow[r] & H^{p+1}_{Z(u)}(|\Delta|) \\
0 & & 0 & & \\
H^0(\Delta) \arrow[r, "\sim"] & H^0(|\Delta| \setminus Z(u)) \arrow[r, "0"] & H^1_{Z(u)}(|\Delta|) \arrow[r] \arrow[uu, "0"] & 0 &
\end{tikzcd}$$

Corollary If $|\Delta|$ is convex and $\mathcal{O}(D)$ is generated by its sections, then

$$H^p(X, \mathcal{O}(D)) = 0, \quad \forall p > 0.$$

Pf:

ψ_D, concave.

$$Z(u) = \{ v \mid \langle u, v \rangle \geq \psi_D(v) \}$$

take $v, v' \in |\Delta| \setminus Z(u)$, $t \in [0,1]$,

$$\langle u, \underbrace{vt + (1-t)v'} \rangle < t\psi_D(v) + (1-t)\psi_D(v') \leq \psi_D(tv + (1-t)v')$$

$\Rightarrow |\Delta| \setminus Z(u)$ convex. Same argument.

Example: $\mathbb{P}^n$ again.

"Pf" of proposition: PROP For all $p \geq 0$,

$$H^p(X, \mathcal{O}(D)) \cong \bigoplus_{u \in M} H^p(X, \mathcal{O}(D))_u ;$$

$$H^p(X, \mathcal{O}(D))_u = H^p_{Z(u)}(|\Delta|).$$

$H^p(\mathcal{O}(D))$ can be computed from Čech complex. $(U_\sigma)_{\sigma \in \mathrm{cones}(\Delta)}$

$$C^p = \bigoplus_{\sigma_0, \dots, \sigma_p \in \mathrm{cones}(\Delta)} H^0(\mathcal{O}(D), U_{\sigma_0, \dots, \sigma_p}) =$$

$$= \bigoplus_{\sigma_0, \dots, \sigma_p \in \mathrm{cones}(\Delta)} H^0(\mathcal{O}(D), U_{\sigma_0 \cap \dots \cap \sigma_p}) =$$

$$= \bigoplus_{u \in M} \left(\bigoplus_{\sigma_0, \dots, \sigma_p \in \mathrm{cones}(\Delta)} H^0_{Z(u) \cap \sigma_0 \cap \dots \cap \sigma_p} \left(\sigma_0 \cap \dots \cap \sigma_p \right) \right)$$

topology

computer $H^p_{Z(u)}(|\Delta|)$

The Canonical Divisor

Prop X nonsingular toric variety. Then $D_i = V(\tau_i)$, τ_i edges of the fan.

$$\mathcal{O}\Big(-\sum_i D_i\Big) = \Omega^n_{X/\mathbb{C}}.$$

$$\mathcal{O}\Big(-\sum_{i=0}^{n} V(e_i)\Big) = \mathcal{O}(-n-1)$$

Pf: $e_1, \dots, e_n$ basis of N. $x_i = \chi^{e_i^*}$, rational functions on X. Set

$$\omega = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n},$$ rational section of Ω^n_X.

Key thing: If choose different basis, get $\pm\omega$.

Want to prove $\underline{\operatorname{div}(\omega)} = -\sum_i D_i$.

If U_σ open. U_σ smooth, can assume σ is generated by $\underbrace{e_1, \dots, e_k}$, then

$$U_\sigma = \operatorname{Spec}\left(\mathbb{C}\left[X_1, \dots, X_k, X_{k+1}, X_{k+1}^{-1}, \dots\right]\right),$$

$$\omega = \frac{\pm 1}{X_1 \cdots X_n} \underbrace{dX_1 \wedge \cdots \wedge dX_n}_{\text{nonvanishing}}.$$

$$\operatorname{div}(\omega)\big|_{U_\sigma} = \operatorname{div}\left(\frac{1}{X_1 \cdots X_n}\right) = \sum_i -D_i \Big|_{U_\sigma} \quad \square$$

$$\mathcal{F}$$

$$U \xrightarrow{j} X \xleftarrow{i} Z$$

$$0 \leftarrow i_* i^* \mathcal{F} \leftarrow \mathcal{F} \leftarrow j_! j^* \mathcal{F} \leftarrow 0 .$$