A rapid introduction to schemes

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January 14, 2022

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Example

X, Y affine varieties with coordinate rings $\mathcal{O}_X$ and $\mathcal{O}_Y$. A map of sets $f : X \to Y$ is a morphism of varieties iff the induced map $f^\sharp : \mathcal{O}_Y \to \mathcal{O}_X$ is an algebra morphism.

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Example

Let $X = Z(I) \subset \mathbb{C}^n$ be an affine variety with coordinate ring $\mathcal{O}_X = \mathbb{C}[x_1, \ldots, x_n]/I$. Can recover X completely from $\mathcal{O}_X$.

Definition

A commutative ring. **Maximal spectrum** of A:

$$\mathrm{Specm}(A) = \{\mathfrak{m} \subset A : \mathfrak{m} \text{ is a maximal ideal}\}.$$

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Can recover $X \subset \mathbb{C}^n$ from $\mathcal{O}_X$ as a **set** since $X \simeq \mathrm{Specm}\,\mathcal{O}_X$ via:

$$\phi : (a_1, \ldots, a_n) \mapsto (x_1 - a_1, \ldots, x_n - a_n) = \{f \in \mathcal{O}_X : f(a_1, \ldots, a_n) = 0\}.$$

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$$V(I) = \{\mathfrak{m} \in \mathrm{Specm}\,A : I \subset \mathfrak{m}\}.$$

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E.g. for $f \in \mathcal{O}_X$: $V((f)) = \{\mathfrak{m} \in \operatorname{Specm} \mathcal{O}_X : f(\phi^{-1}(\mathfrak{m})) = 0\}$.
With this topology, $\operatorname{Specm}(\mathcal{O}_X)$ is homeomorphic to X.

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Definition

For $I \subset A$ ideal define the closed set

$$V(I) = \{\mathfrak{p} \in \mathrm{Spec}\, A : I \subset \mathfrak{p}\} \text{ (all } f \in I \text{ vanish simultaneously).}$$

For $f \in A$ define the **distinguished open**

$$D_f = \{\mathfrak{p} \in \mathrm{Spec}\, A : f \notin \mathfrak{p}\} \text{ (points where } f(\mathfrak{p}) \neq 0).$$

The D_f form a basis of the topology defined by the $V(I)$.

Example

$X = \mathrm{Spec}\,\mathbb{C} = \{\star\}$, $Y = \mathrm{Spec}\,\mathbb{C}[x]/(x^2) = \{\star\}$. These spectra consist of a points to which a ring is attached. Topologically they are the same, but the ring distinguishes them!

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$X = \operatorname{Spec} \mathbb{C}[x, y]/(x^2 - y) = \operatorname{Specm} \mathbb{C}[x, y]/(x^2 - y) \cup \{(0)\}$. The point $\eta = (0)$ is called the **generic point**, since $\overline{\{\eta\}} = X$. It is "close to every point of X".

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Think of $(\operatorname{Spec} A, A)$ as a topological space together with a set of functions describing its geometry. General schemes will be covered by charts of this type.

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Definition

A **presheaf of rings** $\mathcal{F}$ on X is a functor:

$$\mathcal{F} : \mathrm{Op}(X)^{\mathrm{op}} \to \mathrm{Rng}\,.$$

Here $\mathrm{Op}(X)$ is the category of open sets on X with inclusions as morphisms. It thus assigns to each open subset U a ring $\mathcal{F}(U)$, called the **sections**, and prescribes **restriction maps** $r_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$. We write $r_{UV}(s) = s|_V$.

Definition

A presheaf of rings $\mathcal{F}$ is a **sheaf of rings** if the following hold for $\{U_i\}_{i \in I}$ open covering of open U:

- (gluing) If $s_i \in \mathcal{F}(U_i)$ and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \forall i, j \in I$ then $\exists s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.
- (locality) If $s, t \in \mathcal{F}(U)$ so that $s|_{U_i} = t|_{U_i} \forall i \in I$ then $s = t$.

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Example

- **Constant sheaf** $\mathbb{Z}$ sending $\emptyset \mapsto 0, U \mapsto \mathbb{Z}$.
- X topological/smooth manifold. Then $C^{(\infty)}(U)$ is a sheaf.

Example

A commutative ring. Define sheaf on $\operatorname{Spec} A$ via:

$$\mathcal{O}_X(D_f) = A[f^{-1}] = A_f.$$

The sheaf condition fixes the value of $\mathcal{O}_X$ for any other open set. The restriction maps are induced from the localisation maps. Think of $\mathcal{O}_X(U)$ as the rational functions on U.

Definition

$\mathcal{F}$ sheaf of rings over X. Let $x \in X$. The **stalk of $\mathcal{F}$ at x** is:

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Consider $\mathcal{O}_X$ over $X = \operatorname{Spec} A$. Then the stalks are given by:

$$\mathcal{O}_{X,\mathfrak{p}} = \varinjlim_{\mathfrak{p} \in D_f} A_f = \varinjlim_{f \notin \mathfrak{p}} A_f = A[(A \setminus \mathfrak{p})^{-1}] =: A_{\mathfrak{p}},$$

which is local by commutative algebra.

Definition

A **locally ringed space** is a topological space X together with a sheaf of rings $\mathcal{O}_X$ on it, which is called the **structure sheaf**. The stalk of $\mathcal{O}_X$ is a local ring at every point.

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Example

$\operatorname{Spec} A$ together with the sheaf of rings $\mathcal{O}_{\operatorname{Spec} A}$ is a locally ringed space.

Definition

Let $\mathcal{F}$ be sheaf over X, and $f : X \to Y$ continuous. The **push-forward sheaf** is a sheaf on Y defined as:

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)) \text{ for } U \subset Y \text{ open.}$$

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A **morphism of locally ringed spaces** $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$ such that $f : X \to Y$ is continuous, $f^\sharp : \mathcal{O}_Y \to \mathcal{O}_X$ is a morphism of sheafs and the induced morphism on stalks $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a morphism of local rings.

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- $\operatorname{Spec} \mathbb{Z} = \{(0)\} \cup \bigcup_{p \text{ prime}} \{(p)\}$.
- $\mathbb{A}^n = \operatorname{Spec} A[x_1, \ldots, x_n]$ is the **affine plane**, where A comm. ring.
- $\operatorname{Spec} C^\infty(\mathbb{R}^n)$ is the affine scheme on which smooth manifolds are locally modeled.

A ring homomorphism $\phi : A \to B$ induces a map of locally ringed spaces between $\operatorname{Spec} B \to \operatorname{Spec} A$. Indeed:

Theorem

The functor Spec *defines an anti-equivalence between the category of affine schemes and the category of commutative rings.*

Definition

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Example

Every scheme admits a unique morphism to $\operatorname{Spec} \mathbb{Z}$. This is because there is a unique ring homomorphism from $\mathbb{Z}$ into any ring.

How to construct non-affine schemes? By gluing!

Theorem

$\{X_i\}_{i \in I}$ schemes and $\{U_{ij} \subset X_i\}$ open subsets with isomorphisms $\phi_{ij} : U_{ij} \to U_{ji}$. The ϕ_{ij} satisfy compatibility conditions:

- $\phi_{ij} = \phi_{ji}^{-1}$,
- $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$,
- $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$.

Then $\exists$ glued scheme X with open embedding $X_i \hookrightarrow X$ obtained from $\bigsqcup_{i \in I} X_i$ by identifying U_{ij} with U_{ji}.

Example (Projective line and line with double origin)

Let $X_1 = \operatorname{Spec} \mathbb{C}[t], X_2 = \operatorname{Spec} \mathbb{C}[s]$ be two affine lines. Let $U_{12} = \operatorname{Spec} \mathbb{C}[t, t^{-1}] = \{t \neq 0\} \subset X_1$ and $U_{21} = \operatorname{Spec} \mathbb{C}[s, s^{-1}]$. Thus U_{ij} are affine lines with the origin removed.

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- Via ϕ_{12} induced by $\operatorname{Spec} \mathbb{C}[t, t^{-1}] \to \operatorname{Spec} \mathbb{C}[s, s^{-1}]; t \mapsto s$. The glued scheme is a line with two origins.

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- Via ϕ_{12} induced by $\operatorname{Spec}\mathbb{C}[t, t^{-1}] \to \operatorname{Spec}\mathbb{C}[s, s^{-1}]; t \mapsto s^{-1}$. The result is the projective line $\mathbb{P}^1$

What is the product of schemes X and Y? We want $\mathbb{A}^1 \times \mathbb{A}^1 \simeq \mathbb{A}^2$ as schemes, but this is not even the case as topological spaces! Can define the product of topological spaces X and Y as the limit of the diagram:

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We can define the product of two schemes in the same way!

More generally we define the **fibre product** $X \times_S Y$ of two schemes X, Y over S as the limit of the diagram:

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\begin{tikzcd}
\square \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
Y \arrow[r, "g"] & S
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This depends on the morphisms f and g. If $S = \operatorname{Spec} \mathbb{Z}$ we get the product $X \times Y := X \times_{\operatorname{Spec} \mathbb{Z}} Y$.

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Theorem

The fibre product always exists in the category of schemes.

Definition

- A morphism $j : X \to Y$ is an **open immersion** if the underlying continuous map is a homeomorphism of X with an open subset U of Y such that the sheaves $j_*\mathcal{O}_X$ and $\mathcal{O}_Y|_U$ are isomorphic.

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- A morphism $i : X \to Y$ is a **closed immersion** if the underlying continuous map is a homeomorphism between X and a closed subset of Y and the sheaf homomorphism $i^\sharp : \mathcal{O}_Y \to \mathcal{O}_X$ is surjective (meaning surjective on stalks).

Definition

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Definition

A scheme $(X, \mathcal{O}_X)$ is **separated** if the diagonal $\Delta : X \to X \times X$ is a closed immersion.

This is analogous to the Hausdorff property ($\Leftrightarrow$ diagonal closed in product topology), but which schemes almost never have. The line with double origin is not separated!

Definition

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A **scheme of finite type over** $\mathbb{C}$ is a $\mathbb{C}$-scheme $X \to \operatorname{Spec} \mathbb{C}$ such that there is a covering $X = \bigcup \operatorname{Spec} A_i$, where the restrictions $\operatorname{Spec} A_i \to \operatorname{Spec} \mathbb{C}$ are given by morphisms $\mathbb{C} \to A_i$. These homomorphisms then should induce finitely generated $\mathbb{C}$-algebra structures on A_i.