Toric Varieties Reading Group - Week 1

George Cooper

Hilary Term 2022

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- Explain how to construct toric varieties from fans.
- Organise speakers for the following talks!

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- If X is a variety then a *point* of X always means a closed point, i.e. an element of $X(\mathbb{C})$.
- $\mathbb{A}^n = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]$ and $\mathbb{P}^n = \operatorname{Proj} \mathbb{C}[x_0, \ldots, x_n]$.

Algebraic Groups

Definition

An *(affine) algebraic group* G is an (affine) scheme G with morphisms $e : \operatorname{Spec}\mathbb{C} \to G$ (identity element), $m : G \times G \to G$ (group multiplication) and $\iota : G \to G$ (group inversion) satisfying the expected group axioms.

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Example

$G = \mathbb{G}_m := \operatorname{Spec}\mathbb{C}[t, t^{-1}]$, with

- $e^* : \mathbb{C}[t, t^{-1}] \to \mathbb{C}, t \mapsto 1$
- $m^* : \mathbb{C}[t, t^{-1}] \to \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}], t \mapsto t \otimes t$
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Can show: for any $\mathbb{C}$-algebra A, $\mathbb{G}_m(A)$ is a group and is naturally isomorphic to $A^\times$. In particular $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$.

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Example

The usual action of $\mathbb{G}_m^n = \operatorname{Spec} \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ on $\mathbb{A}^n$ given by

$$(t_1, \ldots, t_n) \cdot (x_1, \ldots, x_n) = (t_1 x_1, \ldots, t_n x_n)$$

is algebraic.

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Warning

The embedding $T \hookrightarrow X$ is part of the data of a toric variety!

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- Tori $\mathbb{G}_m^n$.
- Affine n-space $\mathbb{A}^n$.
- Products of toric varieties.
- Projective n-space $\mathbb{P}^n$: give $\mathbb{P}^n$ homogeneous coordinates $[x_0 : \cdots : x_n]$. A dense open torus is given by identifying

$$T = \{[1 : t_1 : \cdots : t_n] : \text{each } t_i \in \mathbb{G}_m\} \subset \mathbb{P}^n.$$

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- Chow groups of X; and
- the intersection product on X.

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There is a natural pairing $\langle \cdot, \cdot \rangle : N \times M \to \mathbb{Z}$; identifying M and N with $\mathbb{Z}^n$, this pairing is the usual Euclidean inner product on $\mathbb{Z}^n$.

Toric Varieties Give Fans

Suppose X is a toric variety with torus $T \equiv \mathbb{G}_m^n$ and lattices M and N. Suppose we are also given $\lambda \in N$, so a morphism $\lambda : \mathbb{G}_m \to T \subset X$.

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Question

Does $\lim_{t \to 0} \lambda(t)$ exist? If so, what is this limit?

Toric Varieties Give Fans

As an example, take $X = \mathbb{P}^2$ and $T = \{[1 : t_1 : t_2]\}$, so $N \equiv \mathbb{Z}^2$ via

$$((\lambda_1, \lambda_2) \in \mathbb{Z}^2) \leftrightarrow (t \mapsto [1 : t_1^{\lambda_1} : t_2^{\lambda_2}]).$$

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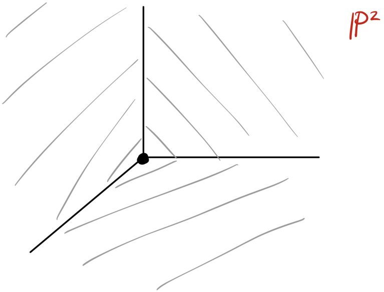
We have 7 cases:

1. $\lambda_1, \lambda_2 > 0$: limit is $[1 : 0 : 0]$.
2. $\lambda_1 > \lambda_2$ and $\lambda_2 < 0$: limit is $[0 : 0 : 1]$.
3. $\lambda_1 < \lambda_2$ and $\lambda_1 < 0$: limit is $[0 : 1 : 0]$.
4. $\lambda_1 = 0$ and $\lambda_2 > 0$: limit is $[1 : 1 : 0]$.
5. $\lambda_2 = 0$ and $\lambda_1 > 0$: limit is $[1 : 0 : 1]$.
6. $\lambda_1 = \lambda_2 < 0$: limit is $[0 : 1 : 1]$.
7. $\lambda_1 = \lambda_2 = 0$: limit is $[1 : 1 : 1]$.

Toric Varieties Give Fans

These 7 cases give the following picture in $N \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^2$, which is an example of a fan:



Toric Varieties Give Fans

Some more pictures:

$\mathbb{P}^1$

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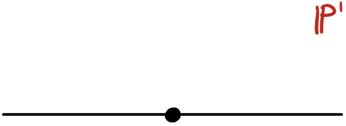

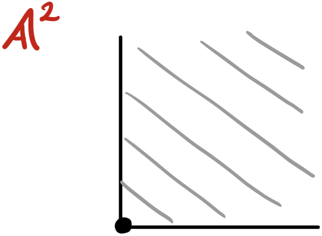

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Proposition

Any toric variety X determines a fan $\Sigma \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ in this way.

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There are two questions to answer here:

1. What do we actually mean by a fan?
2. Can this construction be reversed?

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Let $\sigma \subset N_{\mathbb{R}}$ be a cone. The *dual cone* is the subset of $M_{\mathbb{R}}$ given by $\sigma^{\vee} = \{m \in M_{\mathbb{R}} : \langle v, m \rangle \geq 0 \ \forall v \in \sigma\}$. A *face* of σ is any subset of the form $\tau = \sigma \cap u^{\perp} = \{v \in \sigma : \langle v, u \rangle = 0\}$ for some $u \in \sigma^{\vee}$. A *facet* of σ is a face of σ of codimension one.

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Definition

A *fan* in $N_\mathbb{R}$ is a collection Σ of cones in $N_\mathbb{R}$ that is closed under taking faces of cones and intersections, such that the intersection of any two cones $\sigma, \sigma' \in \Sigma$ is a face of each.

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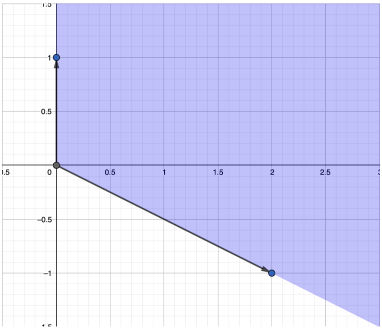
Consequently the "group algebra" $\mathbb{C}[S_\sigma]$ is a finitely generated $\mathbb{C}$-algebra, so $U_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma]$ is an affine scheme of finite type over $\mathbb{C}$. As a complex vector space $\mathbb{C}[S_\sigma]$ has basis χ^m, where m ranges over S_σ, and has multiplication $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$.

Affine Toric Varieties from Cones

Example

Take $\sigma \subset \mathbb{R}^2$ to be the cone spanned by e_2 and $2e_1 - e_2$. The semigroup S_σ is generated by e_1^*, $e_1^* + e_2^*$ and $e_1^* + 2e_2^*$, so $\mathbb{C}[S_\sigma] = \mathbb{C}[a, ab, ab^2] = \mathbb{C}[x, y, z]/(y^2 - xz)$.



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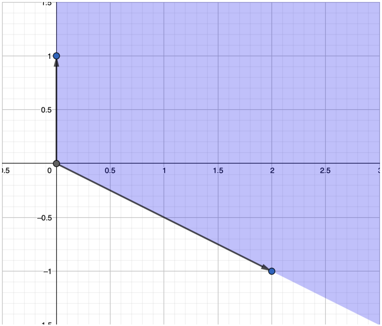

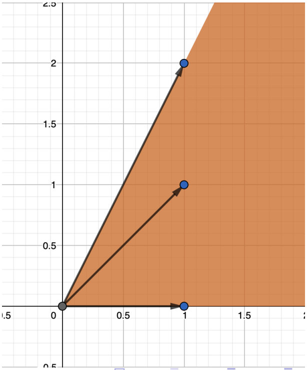

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Proof.

As $S_\sigma \hookrightarrow M = S_{\{0\}}$ then $\mathbb{C}[S_\sigma] \hookrightarrow \mathbb{C}[M]$.

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Upshot: $\mathbb{C}[S_\sigma] \hookrightarrow \mathbb{C}[S_\tau]$ corresponds to inverting χ^m, so U_τ is a basic affine open of U_σ. Moreover, $\mathbb{G}_m^n = \operatorname{Spec} \mathbb{C}[M] \hookrightarrow U_\sigma$ factors through the inclusion $U_\tau \subset U_\sigma$.

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Suppose Σ is a fan in $N_{\mathbb{R}}$. We construct the scheme X_Σ by gluing the affine schemes U_σ, $U_{\sigma'}$ along the common basic open U_τ, whenever τ is a face of both σ and σ'.

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Being a closed immersion is affine-local on the target and $X_\Sigma \times_{\mathbb{C}} X_\Sigma = \bigcup_{\sigma,\tau \in \Sigma} U_\sigma \times_{\mathbb{C}} U_\tau$, so it suffices to show that $\Delta : U_\tau \to U_{\sigma_1} \times_{\mathbb{C}} U_{\sigma_2}$ is a closed immersion whenever $\tau = \sigma_1 \cap \sigma_2$.

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