ORBIT-CONE CORRESPONDENCE

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AIMS

To show there is one T_N-orbit of X_Δ for each cone $\sigma \in \Delta$.

To describe the closure of an orbit as its own toric variety.

Classify the open and closed T_N-stable subschemes of X_Δ.

ORBITS OF $\mathbb{C}^n$

The torus $(\mathbb{C}^*)^n$ acts on the toric variety $\mathbb{C}^n$ by multiplication on coordinates as we expect.

Clearly the orbits are in bijection with subsets $I \subset \{1, \dots, n\}$:

$$O_I = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_i = 0 \text{ if } i \in I, z_i \neq 0 \text{ if } i \notin I\}.$$

Each $O_I \cong (\mathbb{C}^*)^{n-|I|}$ via its components that are non-zero.

Notice that O_J is contained in the closure of O_I if and only if $I \subset J$.

ORBITS OF $\mathbb{C}^n$

We can realise $\mathbb{C}^n$ as the toric variety U_σ with σ generated by $e_1, \dots, e_n$ in $\mathbb{Z}^n$.

Then a subset $I \subset \{1, \dots, n\}$ corresponds to a face of σ, namely τ_I generated by the e_i with $i \in I$.

Note $U_{\tau_\emptyset} = U_{\{0\}} = (\mathbb{C}^*)^n$ and $U_{\tau_{\{1,\dots,n\}}} = U_\sigma = \mathbb{C}^n$.

In general we get U_{τ_J} contained in $\overline{U_{\tau_I}}$ whenever $I \subset J$.

BACK TO BASICS

Consider an affine toric variety $U_\sigma = Spec(\mathbb{C}[S_\sigma])$.

A point $x \in U_\sigma$ is given by a semigroup homomorphism

$$x : S_\sigma \to \mathbb{C}, \qquad x(0) = 1.$$

The torus $T_N = Hom_{sg}(M, \mathbb{C}^*)$ acts on U_σ by

$$t \cdot x : \ S_\sigma \to \mathbb{C}, \qquad u \mapsto t(u)x(u).$$

If Δ is a fan N, then T_N acts on the toric variety X_Δ by acting on the covering opens $\{U_\sigma\}_{\sigma \in \Delta}$ as above.

POINTS OF U_σ

If $x : S_\sigma \to \mathbb{C}$ defines a point in U_σ then we have

$$S_\sigma = x^{-1}(0) \ \sqcup \ x^{-1}(\mathbb{C}^*).$$

We always have $x^{-1}(\mathbb{C}^*) = \tau^\perp \cap S_\sigma$ for some face $\tau \prec \sigma$[1,§5.3].

Note also that $t \cdot x$ and x give the same decomposition. So we get a T_N-stable decomposition

$$U_\sigma = Hom_{sg}(S_\sigma, \mathbb{C}) \equiv \coprod_{\tau \prec \sigma} Hom_{sg}(\tau^\perp \cap S_\sigma, \mathbb{C}^*)$$

ORBITS OF U_σ (AND X_Δ)

Let $\tau \prec \sigma$ and $O_\tau = Hom_{sg}(\tau^\perp \cap S_\sigma, \mathbb{C}^*) \subset U_\sigma$. Notice

$$Hom_{sg}(\tau^\perp \cap S_\sigma, \mathbb{C}^*) \equiv Hom(\tau^\perp \cap M, \mathbb{C}^*).$$

This is just the torus associated with the lattice N/N_τ where N_τ is the sublattice generated by τ.

Consider $x_\tau \in O_\tau$ defined by sending all $\tau^\perp \cap M$ to $1 \in \mathbb{C}^*$.

Given $x \in O_\tau$, the surjection $N \to N/N_\tau$ implies we can find $t \in T_N$ with $t \cdot x_\tau = x$. We therefore see that O_τ is an orbit of U_σ and X_Δ.

STABLE AFFINE OPENS OF X_σ

We have shown that there is bijection between cones in Δ and orbits of X_Δ via $\tau \mapsto O_\tau$.

The orbit decomposition of U_σ is

$$U_\sigma = \coprod_{\tau \prec \sigma} Hom_{sg}(\tau^\perp \cap S_\sigma, \mathbb{C}^*) = \coprod_{\tau \prec \sigma} O_\tau \,.$$

One can show all T_N-stable affine opens have this form[1,§5.8]. In particular X_Δ is affine if and only it has a unique maximal cone, i.e. Δ consists of a cone σ and all its faces.

THE ORBIT CLOSURE $V(\tau)$

Let τ be a cone in Δ. We saw that O_τ can be identified with the torus T_{N/N_τ}.

In fact for each $\sigma \in \Delta$ with $\tau \prec \sigma$, let $\bar{\sigma}$ be its image in N/N_τ. Then $Star(\tau) = \{\bar{\sigma} : \tau \prec \sigma\}$ is a fan in N/N_τ.

Let $V(\tau)$ be the toric variety associated to this fan. Then the torus associated to the zero cone is O_τ and is dense in $V(\tau)$.

EMBEDDING $V(\tau)$ IN X_Δ

Note that $V(\tau)$ is covered by affine opens

$$\{Spec(\mathbb{C}[S_{\bar{\sigma}}])\}_{\tau\prec\sigma} \equiv \{Spec(\mathbb{C}[\tau^\perp \cap S_\sigma])\}_{\tau\prec\sigma}.$$

For each $\sigma \in \Delta$ with $\tau \prec \sigma$ we have a surjection of rings $\mathbb{C}[S_\sigma] \to \mathbb{C}[\tau^\perp \cap S_\sigma]$ given by mapping any χ^u with u outside of $\tau^\perp$ to zero. This uses that $\tau \prec \sigma$.

These affine closed immersions glue to embed $V(\tau)$ as a closed subscheme in X_Δ. Then $\overline{O_\tau} = V(\tau)$.

ORBIT DECOMPOSITION OF $V(\tau)$

Treating $V(\tau)$ as a toric variety and thinking about orbits as we did for X_Δ, we see that

$$V(\tau) = \coprod_{\bar{\sigma} \in Star(\tau)} O_{\bar{\sigma}} = \coprod_{\sigma \succ \tau} O_\tau \, .$$

MORE ON $V(\tau)$

The ideal in $\mathbb{C}[S_\sigma]$ defining $V(\tau) \cap U_\sigma$ is the kernel of $\mathbb{C}[S_\sigma] \to \mathbb{C}[\tau^\perp \cap S_\sigma]$. Clearly this is

$$span_{\mathbb{C}}\{\chi^u : u \in S_\sigma \backslash \tau^\perp\}.$$

Any T_N-stable closed subscheme of U_σ is defined by an ideal of the form above for some face τ[1,§5.3].

AN APPLICATION

Removing a maximal cone from a fan (but retaining its faces) corresponds to removing the corresponding closed orbit from X_Δ.

This often gives an easier way to compute X_Δ if $\Delta \subset \Delta'$ for an already known $X_{\Delta'}$.

For example, see the exercises in §1.4, page 22. These are easier computed now as $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{C}^2$ minus some orbits.

TOPOLOGY

AIMS

Compute the fundamental group of a toric variety.

Outline how one would explore the cohomology of a toric variety.

State some applications of knowing the cohomology (Euler characteristic and interpretation of second homology group).

ALGEBRAIC TORUS

The algebraic torus $T_N \equiv (\mathbb{C}^*)^n$ as an analytic space deformation retracts onto the n-torus $S^1 \times \cdots \times S^1$.

As such it has the topology of an n-torus, i.e.

$$\pi_1(T_N) \cong N,$$
$$H^i(T_N; \mathbb{Z}) = \Lambda^i M.$$

We go from T_N to X_Δ by adding some smaller orbits, which intuitively may 'close' holes in T_N.

ONE-PARAMETER SUBGROUPS

If T_N is the torus corresponding to a lattice N with dual M, then the 1-parameter subgroups of T_N, i.e. algebraic morphisms $\mathbb{C}^* \to T_N$ form a group isomorphic to N.

If $v \in N$, then remember $T_N = Hom(M, \mathbb{C}^*)$. Then

$$\lambda_v : \mathbb{C}^* \to T_N, \qquad z \mapsto \lambda_v(z),$$

Is defined by $\lambda_v(z)(u) = z^{<u,v>}$.

FACT

Let X be any normal variety and let $U \subset X$ be an open subvariety. Then there is a surjection

$$\pi_1(U) \to \pi_1(X),$$

induced by the inclusion of U into X.[2]

FULL DIMENSION FANS

We will argue that if Δ contains a full dimensional cone σ then it is simply connected.

First, by the previous fact, it suffices to show that any loop in T_N is contractible in U_σ and hence also X_Δ.

We have $N \cong \pi_1(T_N)$, with $v \in N$ corresponding to the loop

$$S^1 \subset \mathbb{C}^* \to T_N, \qquad z \mapsto \lambda_v(z),$$

where λ_v is the 1-parameter subgroup corresponding to v.

FULL DIMENSION FANS

We only need to show such a loop λ_v is contractible. Now if $v \in \sigma \cap N$ then the limit of $\lambda_v(z)$ as $z \to 0$ exists in U_σ.

Indeed if $x \in Hom_{sg}(S_\sigma, \mathbb{C})$, then for any $u \in S_\sigma$ we have

$$\lambda_v(z)(u) = z^{<u,v>}.$$

The limit as $z \to 0$ exists. It is 1 if $u \in \sigma^\perp$ and 0 otherwise.

We can therefore expand λ_v from $\mathbb{C}^*$ to U_σ, and

$$\lambda_{v,t}(z) = \lambda_v(tz), \qquad z \in S^1, t \in [0,1],$$

is a contraction of the loop λ_z to a point.

LOWER DIMENSION CONES

If σ is a k-dimensional cone in an n-dimensional lattice, then from earlier, the orbit O_σ of U_σ is identified with T_{N/N_σ}.

We know that if σ' is the cone σ considered in N_σ then

$$U_\sigma = U_{\sigma'} \times T_{N/N_\sigma}.$$

Now σ' has full dimension in N_σ by definition so

$$\pi_1(U_\sigma) \cong N/N_\sigma$$

FUNDAMENTAL GROUP

We now have $\pi_1(U_\sigma) \cong N/N_\sigma$ for any affine toric variety.

One can use van-Kampen's theorem on the open cover $\{U_\sigma\}_{\sigma\in\Delta}$ to proof that

$$\pi_1(X_\Delta) \cong N/\Sigma_{\sigma\in\Delta}N_\sigma = N/N'$$

Where N' is the sublattice generated by all $\sigma \cap N$.

DEFORMATION RETRACTION

We can make our previous statements on the fundamental groups even stronger.

In fact an affine toric variety U_σ deformation retracts onto its unique closed orbit O_σ.

A homotopy is straightforward to construct.

COHOMOLOGY, AFFINE CASE

An affine toric variety U_σ deformation retracts onto its torus $O_\sigma \equiv T_{N/N_\sigma}$.

The dual of N/N_σ is $\sigma^\perp \cap M$, therefore the cohomology of U_σ is given by

$$H^i(U_\sigma; \mathbb{Z}) \cong \Lambda^i(\sigma^\perp \cap M).$$

COHOMOLOGY, GENERAL CASE

One can explore the cohomology of a general toric variety X_Δ using spectral sequences. There are two interesting facts this gives.

The first is that the topological Euler characteristic of X_Δ is equal to the number of full-dimensional cones in Δ.

SECOND COHOMOLOGY GROUP

If all maximal cones have full-dimension, such as when X_Δ is compact, then spectral sequences give the second cohomology group $H^2(X_\Delta)$ as equal to the kernel of

$$\bigoplus_{i<j} (\sigma_i^\perp \cap \sigma_j^\perp \cap M) \rightarrow \bigoplus_{i<j<k} (\sigma_i^\perp \cap \sigma_j^\perp \cap \sigma_k^\perp \cap M).$$

The importance of this cohomology group is that it is isomorphic to the Picard group $Pic(X_\Delta)$.

OTHER REFERENCES

[1] T. Oda, *Lectures on Torus Embeddings and Applications*

[2] W. Fulton and R. Lazarsfeld, *Connectivity and its applications in algebraic geometry*