Toric Varieties Reading Group - Week 7
Toric Chow Groups

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Today's Talk

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Remark

In this talk many of the results are stated only for non-singular toric varieties for ease of exposition and to simplify some proofs. However many of today's results have analogues for simplicial toric varieties (i.e. orbifold toric varieties) which hold for Chow groups/homology groups with rational coefficients.

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Suppose $W \subset X$ is a subvariety of dimension $k+1$ and $r \in \mathbb{C}(W)^*$ is a rational function. We set $\mathrm{div}(r) = \sum_V \mathrm{ord}_V(r) \cdot [V]$, where the sum is taken over all k-dimensional varieties $V \subset W$ and where for $a, b \in \mathcal{O}_{V,W}$ we set

$$\mathrm{ord}_V(a/b) = \ell_{\mathcal{O}_{V,W}}(\mathcal{O}_{V,W}/(a)) - \ell_{\mathcal{O}_{V,W}}(\mathcal{O}_{V,W}/(b)).$$

The subgroup of $Z_k(X)$ generated by the divisors $\mathrm{div}(r)$ is denoted $R_k(X)$.

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The kth *Chow group* of X is $A_k(X) = Z_k(X)/R_k(X)$, the group of k-cycles modulo rational equivalence. If $n = \dim X$, we write $A^k(X) = A_{n-k}(X)$.

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Example

If X is a normal variety then $A_{n-1}(X) = \mathrm{Cl}(X)$ is the group of Weil divisors modulo linear equivalence.

Some Properties of Chow Groups

Proposition

For all $k \geq 0$, $A_k(X) = A_k(X_{\mathrm{red}})$.

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Proposition

$A_k(\mathbb{A}^n) = 0$ for all $0 \leq k \leq n-1$, and $A_n(\mathbb{A}^n) \cong \mathbb{Z}$ with generator $[\mathbb{A}^n]$.

Chow Groups of Toric Varieties

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Let $X_i = \bigcup_{\dim \sigma \geq n-i} V(\sigma)$, and give X_i the reduced subscheme structure. Then we have a filtration by closed subschemes $X = X_n \supset X_{n-1} \supset \cdots \supset X_{-1} = \emptyset$ and

$$X_i \setminus X_{i-1} = \bigsqcup_{\dim \sigma = n-i} \mathcal{O}_\sigma$$

using the orbit/orbit closure relations from Week 4.

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using the orbit/orbit closure relations from Week 4. We argue by induction on i. Consider the exact sequence

$$A_k(X_{i-1}) \longrightarrow A_k(X_i) \longrightarrow \bigoplus_{\dim \sigma = n-i} A_k(\mathcal{O}_\sigma) \longrightarrow 0.$$

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We have that each orbit $\mathcal{O}_\sigma$ is an open subscheme of $\mathbb{A}^i$, so $A_i(\mathcal{O}_\sigma) = \mathbb{Z}[\mathcal{O}_\sigma]$ and $A_k(\mathcal{O}_\sigma) = 0$ for $k \neq i$. Moreover the map $A_k(X_i) \to A_k(\mathcal{O}_\sigma)$ sends $[V(\tau)]$ to $[\mathcal{O}_\sigma]$ if $\tau = \sigma$ and to 0 if $\tau \neq \sigma$.

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Suppose D meets the irreducible subvariety V properly. We may then form the *intersection cycle*

$$D \cdot V := [D|_V] \in \mathrm{WDiv}(V).$$

Intersection Cycles on Toric Varieties

Now assume $X = X(\Delta)$ is toric, $D = \sum a_i D_i$ is T-Cartier and $V = V(\sigma)$. Then $D|_V$ is also T-Cartier, so

$$D \cdot V(\sigma) = \sum b_\gamma V(\gamma),$$

where the sum ranges over all cones γ containing σ with $\dim(\gamma) = \dim(\sigma) + 1$.

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where the sum ranges over all cones γ containing σ with $\dim(\gamma) = \dim(\sigma) + 1$.

We can compute the b_γ as follows. Suppose γ is spanned by σ and a finite set of minimal edge vectors v_i for $i \in I_\gamma$. The lattice N_γ / N_σ is one-dimensional; let e be the generator of this lattice such that the image of each v_i is a positive integer multiple s_i of e.

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Claim

$b_\gamma = a_i / s_i$ for all $i \in I_\gamma$, where a_i is the coefficient in D of $v_i \leftrightarrow D_i$.

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Fix a cone γ containing σ as a facet and let $u(\gamma) \in M/M(\gamma)$ be the linear function on γ corresponding to D:

$$\Gamma(U_\gamma, \mathcal{O}(D)) = \mathbb{C}[S_\gamma] \cdot \chi^{u(\gamma)} \iff D|_{U_\gamma} = \operatorname{div}(\chi^{-u(\gamma)}).$$

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The condition that $V(\sigma) \not\subset \mathrm{supp}(D)$ translates as $u(\gamma) \in M(\sigma)/M(\gamma)$. By passing to $\mathrm{Star}(\sigma)$ and using the formula $\mathrm{ord}_{V(\tau)}(\mathrm{div}(\chi^u)) = \langle u, v_\tau \rangle$, we have

$$b_\gamma = -\langle u(\gamma), e \rangle.$$

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$$a_i = -\langle u(\gamma), v_i \rangle.$$

As $u(\gamma) \in M(\sigma)/M(\gamma)$ and as the image of v_i in N_γ/N_σ is $s_i e_i$, it follows that

$$a_i = -\langle u(\gamma), s_i e \rangle = s_i b_\gamma$$

as claimed.

Intersection Cycles on Toric Varieties

In the case where X is non-singular, there is only one element $i = i(\gamma) \in I_\gamma$, and $s_i = 1$. Hence $b_\gamma = a_{i(\gamma)}$. In other words,

$$D_k \cdot V(\sigma) = \begin{cases} V(\gamma) & \text{if } \sigma \text{ and } v_k \text{ span a cone } \gamma \in \Delta, \\ 0 & \text{otherwise.} \end{cases}$$

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Fact

In the former case, D_k and $V(\sigma)$ meet transversally. In the latter case, D_k and $V(\sigma)$ are disjoint.

Intersection Cycles Revisited

Let D be a Cartier divisor on a variety X. Suppose this time V is an irreducible subvariety contained in $\mathrm{supp}(D)$. We can still make sense of $D \cdot V$ as an element of $A_{\dim V-1}(V)$, by first finding a Cartier divisor E on V such that $\mathcal{O}_V(E) \cong \mathcal{O}_X(D)|_V$, then setting $D \cdot V$ to be the rational equivalence class of the cycle corresponding to E.

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If $f \in \mathbb{C}(X)$ is such that V is not contained in the support of $D' = D + \mathrm{div}(f)$, then $D \cdot V$ is represented by the cycle $D' \cdot V$ defined previously, since rationally equivalent divisors on X determine rationally equivalent cycles on V.

Intersection Cycles Revisited

If $X = X(\Delta)$ is toric, but this time $V(\sigma)$ is contained in the support of the T-Cartier divisor D, one can check that $D' = D + \mathrm{div}(\chi^u)$ works, where $u \in M$ is any element mapping to $u(\sigma) \in M/M(\sigma)$, where $u(\sigma)$ is the linear function on σ corresponding to D.

Intersection Products

Let X be a non-singular quasi-projective variety over $\mathbb{C}$.

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Theorem

There exists a unique associative, commutative, graded ring structure with identity on $A^\bullet(X)$, called the intersection pairing, satisfying the axioms A1-A7 of Hartshorne Appendix A.

Intersection Products

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- If Y and Z are subvarieties of X which *intersect properly*, meaning that every irreducible component of $Y \cap Z$ has codimension equal to $\mathrm{codim}(Y) + \mathrm{codim}(Z)$, then $Y \cdot Z = \sum i(Y, Z; W_j) W_j$, where the sum runs over the irreducible components W_j of $Y \cap Z$ and where the integer $i(Y, Z; W_j)$ depends only on a neighbourhood of the generic point of W_j on X.

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- If Y is a subvariety of X and if Z is an effective Cartier divisor meeting Y properly, then $Y \cdot Z$ is the cycle associated to the Cartier divisor $Y \cap Z$ on Y.

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- If Y is a subvariety of X and if Z is an effective Cartier divisor meeting Y properly, then $Y \cdot Z$ is the cycle associated to the Cartier divisor $Y \cap Z$ on Y.

In particular, if Y and Z are non-singular subvarieties intersecting transversally (i.e. $T_pY + T_pZ = T_pX$ for all $p \in Y \cap Z$), then each $i(Y, Z; W_j) = 1$.

Intersection Products on Toric Varieties

If $X = X(\Delta)$ is any toric variety, and if $\sigma, \tau \in \Delta$ are cones, then *as schemes* $V(\sigma) \cap V(\tau) = V(\gamma)$ if σ and τ span the cone γ, and $V(\sigma) \cap V(\tau) = \emptyset$ if σ and τ do not span a cone; to see this, recall that $V(\sigma)$ has the affine open cover $\{U_\rho(\sigma)\}$, where ρ varies over all cones in Δ containing σ as a face.

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Fact

Assume X is non-singular, and $\sigma \in \Delta$ has minimal generators $v_{i_1}, \ldots, v_{i_k}$. Then $V(\sigma)$ is the transversal intersection of the divisors $D_{i_1}, \ldots, D_{i_k}$.

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Fact

Assume X is non-singular, and $\sigma \in \Delta$ has minimal generators $v_{i_1}, \ldots, v_{i_k}$. Then $V(\sigma)$ is the transversal intersection of the divisors $D_{i_1}, \ldots, D_{i_k}$.

As a consequence, if $V(\sigma)$ and $V(\tau)$ have non-empty and proper intersection, then $V(\sigma) \cdot V(\tau) = V(\gamma)$ in $A^\bullet(X)$. As the classes $V(\sigma)$ generate the Chow ring, this completely determines the intersection pairing on X!

Example: the Chow Ring of $\mathbb{P}^2$

Consider $\mathbb{P}^2$ with its usual fan. The orbit closure of the unique 0-dimensional cone is the whole of $\mathbb{P}^2$, so $A^0(\mathbb{P}^2) = \mathbb{Z}[\mathbb{P}^2] \cong \mathbb{Z}$.

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Example: the Chow Ring of $\mathbb{P}^2$

It follows that the only non-trivial intersection product is $[H] \cdot [H]$. But we can compute this by choosing two of the one-dimensional cones σ and τ and using the formula

$$[H]^2 = [V(\sigma)] \cdot [V(\tau)] = [V(\gamma)],$$

where γ is the unique 2-dimensional cone which has σ and τ as faces. In other words we have $[H]^2 = [P]$, so

$$A^\bullet(\mathbb{P}^2) = \frac{\mathbb{Z}[H]}{[H]^3}, \quad \deg([H]) = 1.$$

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This generalises: $A^\bullet(\mathbb{P}^n) = \mathbb{Z}[H]/[H]^{n+1}$, with $[H] \in A^1(\mathbb{P}^n)$ the class of a hyperplane.

Singular Homology of Toric Varieties

Let $X = X(\Delta)$ be a complete non-singular toric variety of dimension n. We say that Δ is *good* if there exists an ordering $\sigma_1, \ldots, \sigma_m$ of the top-dimensional cones of X such that, if $\tau_i \subset \sigma_i$ is the cone formed by intersecting σ_i with all σ_j such that $j > i$ and σ_j meets σ_i in a cone of dimension $n-1$, then

$$\tau_i \subset \sigma_j \Rightarrow i \leq j.$$

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Call such an ordering a *good ordering*.

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Any non-singular projective fan is good.

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Suppose $X(\Delta)$ is projective, so admits a very ample divisor $D = \sum a_i D_i$ corresponding to the strictly Fulton convex/Ibáñez-Núñez concave function ψ. The $u(\sigma)$, as σ ranges over the n-dimensional cones in Δ, are the vertices of the polytope

$$P_D = \{u \in M_{\mathbb{R}} : \langle u, v_i \rangle \geq -a_i = \psi(v_i)\}.$$

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We may choose some $v \in N$ such that the $h(\sigma) = \langle u(\sigma), v \rangle$ are all distinct. We then order the cones σ by their heights:

$$h(\sigma_1) < \cdots < h(\sigma_m).$$

Singular Homology of Toric Varieties

Proposition

Any non-singular projective fan is good.

On the other hand, the fan corresponding to the polytope P_D is exactly Δ (exercise). In particular, there is an inclusion-reversing correspondence between cones in Δ and faces of P_D, and one can check that τ_i is the cone corresponding to the smallest face F_i of P_D containing $u(\sigma_i)$ and all edges connecting $u(\sigma_i)$ to $u(\sigma_j)$ with $j > i$. As this face contains no $u(\sigma_j)$ with $j < i$ then

$$\tau_i \subset \sigma_j \Rightarrow i \leq j.$$

Singular Homology of Toric Varieties

Theorem

Suppose Δ is good, complete and non-singular. Then the classes $[V(\tau_i)]$ given by a choice of a good ordering of Δ form a basis for $A_\bullet(X) \cong H_{2\bullet}(X_{\mathrm{an}}, \mathbb{Z})$.

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2. If γ is a face of γ' then $i(\gamma) \leq i(\gamma')$.

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For $1 \leq i \leq m$ set

$$Y_i = \bigcup_{\tau_i \subset \gamma \subset \sigma_i} \mathcal{O}_\gamma = \bigcup_{i(\gamma)=i} \mathcal{O}_\gamma = V(\tau_i) \cap U_{\sigma_i}$$

and

$$Z_i = \bigcup_{j \geq i} Y_j.$$

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From the first consequence of the good ordering property, X is the disjoint union of the Y_i. Each Z_i is closed because of the second consequence and because

$$\overline{\mathcal{O}_\gamma} = \bigcup_{\gamma \subset \gamma'} \mathcal{O}_{\gamma'}.$$

The first statement follows.

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1. Each Z_i is closed in X, $Z_1 = X$ and $Y_i = Z_i \setminus Z_{i+1}$.
2. Each $Y_i \cong \mathbb{A}^{n-k_i}$, where $k_i = \dim(\tau_i)$.

For the second assertion, recall that any non-singular affine toric variety U_σ is a product of affine space $\mathbb{A}^{\dim\sigma}$ with a torus $\mathbb{G}_m^{\mathrm{codim}(\sigma)}$. We have that $Y_i = V(\tau_i) \cap U_{\sigma_i}$ is an affine open of $V(\tau_i)$ corresponding to a maximal $(n-k_i)$-dimensional cone in $N(\tau_i) = N/N_{\tau_i}$; consequently $Y_i \cong \mathbb{A}^{n-k_i}$.

Singular Homology of Toric Varieties

To prove the theorem, we will use descending induction on i to show that the canonical map $A_\bullet(Z_i) \to H_{2\bullet}((Z_i)_{\mathrm{an}}, \mathbb{Z})$ is an isomorphism, with these groups having as a basis the classes of the $V(\tau_j) = \overline{Y_j}$ for $j \geq i$.

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$$\begin{tikzcd}
 & A_k(Z_{i+1}) \arrow[r] \arrow[d] & A_k(Z_i) \arrow[r] \arrow[d] & A_k(Y_i) \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] H^{BM}_{2k+1}(Y_i, \mathbb{Z}) & H_{2k}(Z_{i+1}, \mathbb{Z}) \arrow[r] & H_{2k}(Z_i, \mathbb{Z}) \arrow[r] & H^{BM}_{2k}(Y_i, \mathbb{Z}) \arrow[r] & H_{2k-1}(Z_i, \mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd}$$

where the bottom row is the LES in Borel-Moore homology (singular homology with locally finite singular chains).

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Y_i is an affine space, so $A_\bullet(Y_i) \cong H^{BM}_{2\bullet}(Y_i, \mathbb{Z}) = \mathbb{Z}[Y_i]$ (in particular this covers the base case $Z_m = Y_m$).

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Betti Numbers of Toric Varieties

Corollary

Let $X(\Delta)$ be a non-singular projective toric variety. Let d_k denote the number of k-dimensional cones in Δ and let $\beta_k = \mathrm{rank}(A_k(X)) = \mathrm{rank}(H_{2k}(X_{\mathrm{an}}, \mathbb{Z}))$. Then

$$d_k = \sum_{j=0}^{k} \binom{n-j}{n-k} \beta_{n-j};$$

equivalently

$$\beta_k = \sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} d_{n-j}.$$

Betti Numbers of Toric Varieties

It suffices to prove the first equation. Use the polytope P_D from before. Then specifying a k-dimensional cone γ with $\tau_i \subset \gamma \subset \sigma_i$ is equivalent to specifying a $(n-k)$-dimensional face of P_D contained in the $(n-k_i)$-dimensional face F_i and containing the vertex $u(\sigma_i)$; i.e., specifying $(n-k)$ distinct vertices of F_i, all of which are distinct from $u(\sigma_i)$. The number of such choices is $\binom{n-k_i}{n-k}$.

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$$d_k = \sum_{j=0}^{m} \sum_{\substack{\tau_j \subset \gamma \subset \sigma_j \\ \dim \gamma = k}} 1 = \sum_{j=0}^{k} \binom{n-j}{n-k} \beta_{n-j},$$

as β_q equals the number of τ_i with $\dim(\tau_i) = q$ (of course, by Poincaré duality $\beta_q = \beta_{n-q}$).

Singular Cohomology of Toric Varieties

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Proposition

Let $X = X(\Delta)$ be a non-singular projective toric variety. Then as rings,

$$A^\bullet(X) \cong H^\bullet(X_{\mathrm{an}}, \mathbb{Z}) \cong \frac{\mathbb{Z}[D_1, \dots, D_d]}{I},$$

where the D_i are the irreducible T-divisors (with corresponding minimal generators v_i), and where I is the ideal generated by the following elements:

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1. $D_{i_1} \cdot \cdots \cdot D_{i_k}$ for $v_{i_1}, \ldots, v_{i_k}$ not in a cone of Δ;
2. $\sum_{i=1}^{d} \langle u, v_i \rangle D_i$ for each $u \in M$.