Calabi–Yau Manifolds and Mirror Symmetry

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MINI- PROJECTS: see at the end (page 333) for details and a list.

The lecture course will cover the following topics:

*Complex Manifolds

*Kahler Manifolds

*Calabi-Yau Manifolds

*Examples, constructions

*Moduli Spaces: Special Kahler geometry, conifold transitions

*Introduction to Mirror Symmetry: the conjectures, Mirror Maps, Genus-zero Gromov-Witten invariants

*Time permitting: T-duality and the Strominger-Yau-Zaslow conjecture

Chapter 1 Complex Manifolds

Definition: An n–dimensional *complex manifold* M is a topological space together with a holomorphic (*ie* analytic) atlas.

That is: a collection of charts $(\mathcal{U}_j, z_j)$, where the z_j are 1–1 maps of the corresponding $\mathcal{U}_j$ to $\mathbb{C}^n$ such that for every non-empty intersection $\mathcal{U}_j \cap \mathcal{U}_k$, the maps $z_j z_k^{-1}$ are holomorphic.

We say that M has a *complex structure*.

Clearly, from the definition, every n–dim complex manifold is a 2n–dim real manifold.

A crucial difference between the definition of a complex manifold and the definition of a real manifold is that the transition functions f_{jk} which relate the coordinates in overlapping coordinate patches $\mathcal{U}_j$ and $\mathcal{U}_k$:

$$z_j^\mu = f_{jk}^\mu(z_k) \quad ,$$

are now required to be **holomorphic** rather than C^∞.

Obviously, $\mathbb{C}^n$ is complex.

An Important Example

The n–dimensional *complex projective space*, $\mathbb{P}^n$, is the space of complex lines through the origin.

Consider $\quad \mathbb{C}^{n+1} \setminus \{0\}$

and identify

$$(z^1, z^2, \ldots, z^{n+1}) \sim \lambda(z^1, z^2, \ldots, z^{n+1})$$

for any $\lambda \in \mathbb{C}^*$.

We can take sets $\mathcal{U}_j = \{z^j \neq 0\}$, $j = 1, \ldots, n+1$, as coordinate neighborhoods and choose coordinates

$$\xi_j^m = \frac{z^m}{z^j}$$

on each $\mathcal{U}_j$.

On the overlap $\mathcal{U}_j \cap \mathcal{U}_k$ we have

$$\xi_j^m = \frac{z^m}{z^j} = \frac{z^m/z^k}{z^j/z^k} = \frac{\xi_k^m}{\xi_k^j} \ .$$

ξ_k^j is not vanishing on the overlap and so ξ_j^m is a holomorphic function of ξ_k^m.

So, $\mathbb{P}^n$ is an n-dimensional complex manifold.

$\mathbb{P}^n$ is compact. Proof later in this lecture

One can think of $\mathbb{P}^n$ is a compactified form of $\mathbb{C}^n$ to which a hyperplane has been added at infinity.

Exercise: $\mathbb{P}^1$ is the Riemann sphere S^2.

Hint:

We want to show that one can cover $\mathbb{P}^1$ by two coordinate patches $\mathcal{U}_1$ and $\mathcal{U}_2$. On each we have coordinates z_1 and z_2.

On $\mathcal{U}_1 \cap \mathcal{U}_2$: $\qquad z_1 = z_2^{-1}.$

To see this: Project stereographically from the North and South poles and obtain two coordinate patches $\mathcal{U}_1 = S^2 \setminus$ North Pole, and $\mathcal{U}_2 = S^2 \setminus$ South Pole, with coordinates (x_1, y_1) and (x_2, y_2) respectively.

Now let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ and prove that $z_2 = z_1^{-1}$ on the overlap.

Exercise: Any orientable 2-dim Riemannian manifold is a complex manifold.

We will be interested in *compact* complex manifolds.

Compact submanifolds of $\mathbb{R}^n$ (such as S^{n-1}) provide many examples of compact **real** manifolds.

The following theorem shows that the situation is very different with respect to $\mathbb{C}^n$.

Theorem: A connected compact analytic submanifold of $\mathbb{C}^n$ is a point.

(By an analytic submanifold we mean a submanifold defined by analytic equations $Z^j = Z^j(x)$, where the $Z^i, i = 1, \ldots, n$ are coordinates on $\mathbb{C}^n$ and the $x^m, m = 1, \ldots, k$ are coordinates on the submanifold.)

Sketch of proof:

This rests on the maximum modulus principle.
Recall: from the theory of one complex varible, if f is a holomorphic function in some open set $\mathcal{U}$, then $|f|$ cannot have a maximum (or a minimum) at an interior point $p \in \mathcal{U}$ unless f is constant throughout $\mathcal{U}$. This result extends to the case of several complex variables by applying the one variable result to *lines* through p (*line* = 1-dim complex manifold = $\mathbb{C}$).

Suppose now that M is a complex compact submanifold embedded in $\mathbb{C}^n$. The coordinates Z^j of the embedding space are holomorphic functions on M. Since M is compact, each $|Z^j|$ must achieve a maximum somewhere on M hence each Z^j must be a constant. So, M is a point. $\square$

There are however compact complex manifolds that are submanifolds of $\mathbb{P}^n$.

$\mathbb{P}^n$ is compact and all its submanifolds are compact.

By a theorem of Chow (no proof here) any submanifold on $\mathbb{P}^n$ can be realized as the zero locus of a finite number of homogeneous polynomial equations
Example: Fermat quintic in $\mathbb{P}^4$ which defined by

$$z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0$$

Other examples of compact complex manifolds are quotient manifolds of $\mathbb{C}^n$.

Example: Complex torus
Let G be the group generated by translation by $2n$ complex vectors that are linearly independent over the reals

$$\mathbf{a}_j = (a_j^1, \ldots, a_j^n) \ , \qquad \mathbf{b}_j = (b_j^1, \ldots, b_j^n) \ .$$

Then $\mathbb{C}^n/G$ is a complex manifold.

Another Example: Exercise

Show that $S^{2r+1} \times S^{2s+1}$ is a complex manifold.

Hint:

Consider the Hopf fibration of S^{2r+1} (S^{2r+1} is a fiber bundle over $\mathbb{P}^r$, each fiber is a circle).

Let

$S^{2r+1} = \{(z^1, \ldots, z^{r+1}) \in \mathbb{C}^{r+1} | \sum_{i=1}^{r+1} |z^i|^2 = 1\}$.

But also we can think of $(z^1, \ldots, z^{r+1})$ as homogeneous coordinates of a point in $\mathbb{P}^r$.

Show that there is a map $\pi : S^{2r+1} \to \mathbb{P}^r$ which projects a S^1 to a point in $\mathbb{P}^r$.

Perform a Hopf construction for $S^{2r+1} \times S^{2s+1}$:

$$\pi : S^{2r+1} \times S^{2s+1} \longrightarrow \mathbb{P}^r \times \mathbb{P}^s$$

Each fiber is now $S^1 \times S^1 = T^2$.

Both the base $\mathbb{P}^r \times \mathbb{P}^s$, and the fiber are complex.

Write down an atlas and check that the transition functions are holomorphic.

Remarks:

**The construction of the Hopf fibration demonstrates that $\mathbb{P}^n$ is compact.
This is because $\mathbb{P}^n$ is the image of a compact space under a continuous map.

**Apart from possibly S^6, S^{2n} are NOT complex for $n > 1$.

Two manifolds can be **different** if regarded as complex manifolds and yet be diffeomorphic (*ie* equivalent as real manifolds).

Example: consider two 2-dim tori

T_1: $(x, y) \sim (x + 1, y) \sim (x, y + 1)$

T_2: $(\xi, \eta) \sim (\xi + 1, \eta) \sim (\xi, \eta + 2)$

T_1 and T_2 are diffeomorphic as real manifolds:

$$(\xi, \eta) = (x, 2y)$$

defines a C^∞ map between them.

If we set $z = x + iy$ and $\zeta = \xi + i\eta$, then

$$\zeta = \frac{3}{2}z - \frac{1}{2}\bar{z}$$

and it is not possible to eliminate $\bar{z}$.

Next: another way to characterize complex manifolds.

A word on notation and indices:

Definition: If a real manifold M admits a smooth mixed tensor $J_m{}^n$ satisfying

$$J_m{}^n J_n{}^k = -\delta_m{}^k ,$$

then M is an *almost complex manifold* and J is called an *almost complex structure* on M.

Questions:

When is an almost complex structure a complex structure?

Why are the two definitions equivalent?

Lecture 2:

Continue discussion on Complex Manifolds

Two definitions:

Definition: An n–dimensional *complex manifold* M is a topological space together with a holomorphic (*ie* analytic) atlas. We say that M has a *complex structure*.

Definition: If a real manifold M admits a smooth mixed tensor $J_m{}^n$ satisfying

$$J_m{}^n J_n{}^k = -\delta_m{}^k ,$$

then M is an *almost complex manifold* and J is called an *almost complex structure* on M.

Definition: Let X and Y be any smooth vector fields on M. We define

$$N_J(X,Y) = [X,Y] + J([JX,Y] + [X,JY]) - [JX,JY]$$

which is a new vector field.

** $[X,Y]$ is the Lie bracket of vector fields:

$$\mathcal{L}_X Y = [X,Y] = Z \ .$$

In indices: $Z^n = [X,Y]^n = X^m \partial_m Y^n - Y^m \partial_m X^n$.

**By JX we mean the new vector

$$(JX)^m = J_n{}^m X^n \ .$$

** N_J is a tensor (in fact $N_J(X,Y)$ is pointwise bilinear in X and Y) which is called the *Nijenhuis tensor* of J. In indices

$$N_{mn}{}^k = J_{[m}{}^k{}_{,n]} - J_{[m}{}^p J_{n]}{}^q J_p{}^k{}_{,q}$$

Theorem: (Newlander-Nirenberg Theorem)
An almost complex structure J is a complex structure if and only if $N_J = 0$.

That is, $N_J = 0$ is a necessary and sufficient condition for there to exist a holomorphic chart in the neighborhood of each point on M. Thus M has an atlas of holomorphic charts which makes M a complex manifold.

Sketch of Proof only.

If M is complex n-dimensional, and $\{z^\mu\}$ are local coordinates on a coordinate neighborhood U, we can define the tensor

$$J = idz^\mu \otimes \frac{\partial}{\partial z^\mu} - idz^{\bar{\mu}} \otimes \frac{\partial}{\partial z^{\bar{\mu}}} \; .$$

- (i) J is in fact a tensor: it's definition is independent of the choice of coordinates
- (ii) J is real (obviously)
- (iii) $J_m{}^n J_n{}^k = -\delta_m{}^n$
- (iv) $N_J = 0$

Proof of (i): Let w be coordinates on $\mathcal{U}_1$ and z be coordinates on $\mathcal{U}_2$. On $\mathcal{U}_1 \cap \mathcal{U}_2$, the w are analytic functions of the z. By the chain rule

$$dw^\mu \otimes \frac{\partial}{\partial w^\mu} = dz^\mu \otimes \frac{\partial}{\partial z^\mu}$$

and so

$$J = i dw^\mu \otimes \frac{\partial}{\partial w^\mu} - i dw^{\bar{\mu}} \otimes \frac{\partial}{\partial w^{\bar{\mu}}}$$

Proof of (iii): Easy.

In a complex basis $J_\mu{}^\nu = i\delta_\mu{}^\nu$ and $J_\mu{}^{\bar{\nu}} = 0$, or

$$J = \begin{pmatrix} i\mathbf{1} & 0 \\ 0 & -i\mathbf{1} \end{pmatrix}$$

In a real basis, with $z^\mu = x^\mu + iy^\mu$

$$J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$$

Proof of (iv): computational

Proving the converse of the Newlander–Niremberg Theorem is not so easy.

Given an almost complex structure J on M, we can define *two* projection operators:

$$P = \frac{1}{2}\left(\mathbf{1} - iJ\right) \qquad Q = \frac{1}{2}\left(\mathbf{1} + iJ\right)$$

Clearly

$$P^2 = P \qquad Q^2 = Q \qquad PQ = 0 \qquad P + Q = \mathbf{1}$$

These operators permit a refinement of the exterior calculus.

Given, say, a one-form

$$U = u_m dx^m = u_m \delta_n{}^m dx^n = u_m (P+Q)_n{}^m dx^n$$

we can *define* projected components

$$U^{(1,0)} = u_m P_n{}^m dx^n \qquad U^{(0,1)} = u_m Q_n{}^m dx^n$$

Let us seek a set of complex coordinates $z^\mu = z^\mu(x)$. Then the holomorphic coordinate differentials are

$$dz^\mu = \frac{\partial z^\mu}{\partial x^m} dx^m \ .$$

Inserting $P + Q$ in the RHS as above we have

$$dz^\mu = \frac{\partial z^\mu}{\partial x^m} P_n{}^m dx^n + \frac{\partial z^\mu}{\partial x^m} Q_n{}^m dx^n$$

If J is in fact a complex structure, P and Q are used to project out the 'holomorphic' and 'anti-holomorphic' components of tensors. In this case we can show (easy computation)

$$dz^\mu = \frac{\partial z^\mu}{\partial x^m} P_n{}^m dx^n \qquad \frac{\partial z^\mu}{\partial x^m} Q_n{}^m dx^n = 0 \qquad (1)$$

So, in fact the $(1,0)$ part of the RHS is precisely dz^μ.

To prove the non-trivial part of the Newlander–Niremberg Theorem we then ask the following: if J is an almost complex structure on M

(i) Do there exist locally coordinates z^μ such that these equations (above) are true?

(ii) Given a covering of M by local neighborhoods such that (i) is true, are the transition functions holomorphic?

We regard

$$\frac{\partial z^\mu}{\partial x^m} Q_n{}^m dx^n = 0$$

as a differential equation for the complex coordinates z^μ. A necessary and sufficient condition for this equation to be integrable is that the equation

$$\frac{\partial z^\mu}{\partial x^m} Q_n{}^m = 0 \ ,$$

should be integrable. By acting on this equation with $Q_k^l \partial/\partial x^l$ and after a bit of algebra we get

$$P_j{}^m Q_{[k}{}^l Q_{n]}{}^j{}_{,l} = 0 \ .$$

More algebra shows that both the real and imaginary parts of the equation are equivalent to $N_J = 0$.

We still need to show that in the ovelap of two coordinate patches $\mathcal{U}$ and $\mathcal{W}$, the respective coordinates are holomorphic functions of each other. Recall from (1) that

$$dz^\mu = \frac{\partial z^\mu}{\partial x^m} P_n{}^m dx^n \qquad \text{is a (1,0)- form.}$$

On $\mathcal{U} \cap \mathcal{W}$, we have in general that $z^\mu = z^\mu(w, \bar{w})$, where z and w are the complex coordinates on $\mathcal{U}$ and $\mathcal{W}$ respectively. So using the chain rule

$$dz^\mu = \frac{\partial z^\mu}{\partial w^\nu} dw^\nu + \frac{\partial z^\mu}{\partial w^{\bar{\nu}}} dw^{\bar{\nu}} \ .$$

But the LHS is a $(1,0)$ form on $\mathcal{U} \cap \mathcal{W}$. Therefore

$$\frac{\partial z^\mu}{\partial w^{\bar{\nu}}} = 0 \ ,$$

that is, the z^μ are holomorphic functions of w on the overlap $\square$

Decomposition of tensors

As suggested before, the significance of P and Q is that they can be used to project out the "holomorphic" and "anti–holomorphic" components of tensors.

For a one-form we had

$$U = u_m dx^m = u_m \delta_n{}^m dx^n = u_m (P+Q)_n{}^m dx^n$$

and we *defined* projected components

$$U^{(1,0)} = u_m P_n{}^m dx^n \qquad U^{(0,1)} = u_m Q_n{}^m dx^n$$

If J is a (almost) complex structure, a k-form ω can be decomposed in an analogous way

$$\omega = \sum_{p+q=k} \omega^{(p,q)} \ ,$$

where

$$\omega^{(p,q)}_{m_1\cdots m_k} = \omega_{n_1\cdots n_p r_1\cdots r_q} P_{m_1}{}^{n_1} \cdots P_{m_p}{}^{n_p} Q_{m_{p+1}}{}^{r_1} \cdots Q_{m_k}{}^{r_k}$$

Exercises:

If J is an almost complex structure on M

$$d\omega^{(p,q)} = (d\omega)^{(p-1,q+2)} + (d\omega)^{(p,q+1)} + (d\omega)^{(p+1,q)} + (d\omega)^{(p+2,q-1)}$$

If J is a complex structure

$$(d\omega)^{(p-1,q+2)} = (d\omega)^{(p+2,q-1)} = 0$$

Define operators ∂ and $\bar\partial$ on a manifold M with an almost complex structure J by

$$\partial\omega^{(p,q)} = (d\omega)^{(p+1,q)} \qquad \bar\partial\omega^{(p,q)} = (d\omega)^{(p,q+1)}$$

So, if M is complex the we can think of ∂ and $\bar\partial$ as the $(1,0)$ and $(0,1)$ parts of $d = \partial + \bar\partial$.
Show that the condition for ∂ to square to zero is that $N_J = 0$.

Chapter 2 Kähler Manifolds

Let M be a complex manifold with complex structure J.

Definition: A complex manifold M is *Hermitian* if it is endowed with a metric of the form

$$g(X,Y) = g(JX, JY)$$

for all vector fields X and Y.

Equivalently: $g_{mn} = J_m{}^k J_n{}^l g_{kl}$ or

$$ds^2 = g_{\mu\bar{\nu}} dz^\mu dz^{\bar{\nu}} \qquad (g_{\mu\nu} = g_{\bar{\mu}\bar{\nu}} = 0)$$

Definition: Let M be a Hermitian manifold with metric g and complex structure J.

The *Hermitian form* on M is a two-form ω defined by

$$\omega(X,Y) = g(JX, Y)$$

(Equivalently: $\omega_{mn} = J_m{}^k g_{kn}$)

It is in fact a two-form:

$$\begin{aligned} \omega(Y,X) = g(JY,X) &= g(X,JY) = g(JX, J^2 Y) \\ &= g(JX, -Y) = -g(JX,Y) = -\omega(X,Y) \end{aligned}$$

Moreover, ω is a $(1,1)$ form:

$$\omega_{\mu\nu} = J_\mu{}^{\bar{\rho}} g_{\bar{\rho}\nu} = i g_{\mu\nu} = 0$$

Covariant Derivaties and Curvature Tensors

Recall: in Riemannian geometry, the Christoffel connection is uniquely determined by requiring that the metric satisfies $\nabla g = 0$ and that the connection Γ be symmetric $\Gamma_{mn}{}^p = \Gamma_{nm}{}^p$ (*ie* ∇ is torsion free).

Theorem: On a Hermitian manifold, there exists a unique connection, called the *Chern connection*, with the properties
(i) $\nabla g = 0$
(ii) $\nabla J = 0$
(iii) The torsion $\Gamma_{[mn]}{}^r$ is pure in its indices.

Proof:
Recall: $2P = \mathbf{1} - iJ$ and $2Q = \mathbf{1} + iJ$.
(ii) implies that $\nabla P = 0$ and $\nabla Q = 0$ (the converse is also true).

Consider $\nabla P = 0$. Recall also that in complex coordinates

$$J = \begin{pmatrix} i\mathbf{1} & 0 \\ 0 & -i\mathbf{1} \end{pmatrix}$$

so

$$P = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$

$$0 = \nabla_k P_m{}^n = -\Gamma_{km}{}^l P_l{}^n + \Gamma_{kl}{}^n P_m{}^l$$

- $m \to \mu \,, \quad n \to \nu$: no content

$$0 = -\Gamma_{k\mu}{}^l P_l{}^\nu + \Gamma_{kl}{}^\nu P_\mu{}^l = -\Gamma_{k\mu}{}^\nu + \Gamma_{k\mu}{}^\nu$$

- $m \to \mu \,, \quad n \to \bar{\nu}$:

$$0 = -\Gamma_{k\mu}{}^l P_l{}^{\bar{\nu}} + \Gamma_{kl}{}^{\bar{\nu}} P_\mu{}^l = \Gamma_{k\mu}{}^{\bar{\nu}}$$

Therefore: $\Gamma_{\rho\mu}{}^{\bar{\nu}} = 0$ and $\Gamma_{\bar{\rho}\mu}{}^{\bar{\nu}} = 0$
Similarly: $\Gamma_{\bar{\rho}\bar{\mu}}{}^{\nu} = 0$ and $\Gamma_{\rho\bar{\mu}}{}^{\nu} = 0$

(Remaining components of the connection:

$$\Gamma_{\mu\bar{\rho}}{}^{\bar{\nu}} \,, \quad \Gamma_{\bar{\mu}\bar{\rho}}{}^{\bar{\nu}} \,, \quad \Gamma_{\bar{\mu}\rho}{}^{\nu} \,, \quad \Gamma_{\mu\rho}{}^{\nu} \quad)$$

Condition (iii) gives further information:

$$\Gamma_{[\lambda\bar{\mu}]}{}^{\nu} = 0 = \frac{1}{2}\left(\Gamma_{\lambda\bar{\mu}}{}^{\nu} - \Gamma_{\bar{\mu}\lambda}{}^{\nu}\right)$$

Then $\qquad \Gamma_{\bar{\mu}\lambda}{}^{\nu} = 0 \quad$ and also $\qquad \Gamma_{\mu\bar{\lambda}}{}^{\bar{\nu}} = 0$

So, the Chern connection is *pure*.

We now prove the existence and uniqueness.

Solve $\nabla_m g_{nr} = 0$ in terms of the derivatives is g:

$$0 = \partial_m g_{nr} - \Gamma_{mn}{}^s g_{sr} - \Gamma_{mr}{}^s g_{ns}$$

- $(m, n, r) \longrightarrow (\mu, \nu, \bar{\rho})$

$$0 = \partial_\mu g_{\nu\bar{\rho}} - \Gamma_{\mu\nu}{}^\lambda g_{\lambda\bar{\rho}} - \Gamma_{\mu\bar{\rho}}{}^\lambda g_{\nu\lambda} \Longrightarrow \partial_\mu g_{\nu\bar{\rho}} = \Gamma_{\mu\nu}{}^\lambda g_{\lambda\bar{\rho}}$$

Contracting with $g^{\bar{\rho}\kappa}$:

$$\Gamma_{\mu\nu}{}^\kappa = g^{\bar{\rho}\kappa}\partial_\mu g_{\nu\bar{\rho}}$$

$\square$

This equation for Γ leads to a great simplification in the structure of the Riemann tensor.

Riemann tensor:

$$R_{mn}{}^k{}_l = 2\partial_{[m}\Gamma_{n]l}{}^k + \Gamma_{mr}{}^k\Gamma_{nl}{}^r - \Gamma_{nr}{}^k\Gamma_{ml}{}^r$$

- The only non-zero components are

$$R_{\mu\bar{\nu}}{}^\rho{}_\sigma, \quad R_{\mu\bar{\nu}}{}^{\bar{\rho}}{}_{\bar{\sigma}}, \quad R_{\bar{\mu}\nu}{}^\rho{}_\sigma, \quad R_{\bar{\mu}\nu}{}^{\bar{\rho}}{}_{\bar{\sigma}}$$

- Also

$$R_{\mu\bar{\nu}}{}^\rho{}_\sigma = -\partial_{\bar{\nu}}\Gamma_{\mu\sigma}{}^\rho$$

Ricci-form

$$\mathcal{R} = \frac{1}{4}R_{mn}{}^k{}_l J_k{}^l dx^m \wedge dx^n$$

Exercise: Show that $\quad \mathcal{R} = i\partial\bar{\partial} \ \log \ g^{1/2}$

Lecture 3

Continue with Hermitian and Kähler manifolds.

Definition: A complex manifold M with complex structure J is *Hermitian* if it is endowed with a metric of the form

$$g(X, Y) = g(JX, JY)$$

for all vector fields X and Y. In this case, we can define a *Hermitian form* on M is a two-form ω defined by

$$\omega(X, Y) = g(JX, Y)$$

<u>Ricci-form</u>

$$\mathcal{R} = \frac{1}{4} R_{mn}{}^{k}{}_{l} J_{k}{}^{l} dx^{m} \wedge dx^{n} = i\partial\bar{\partial}\ \log\ g^{1/2}$$

<u>More on the Ricci-form</u>

The Ricci-form is d-closed: $d\mathcal{R} = 0$

This is true because

$$\partial\bar{\partial} = -\frac{1}{2} d(\partial - \bar{\partial})$$

Note that we have not shown that $\mathcal{R}$ is *exact*. In fact, $g^{1/2}$ *is not* a coordinate scalar. However $\mathcal{R}$ is globally well defined.

Definition: The Ricci-form defines a cohomology class

$$c_1 = \left[\frac{1}{2\pi}\mathcal{R}\right]$$

which is called the *First Chern Class* of M.

The First Chern Class is an *analytic invariant*, that is, it is invariant under smooth changes of the complex structure on M.

Under a change in the metric $g_{mn} \longrightarrow g_{mn} + \delta g_{mn}$:

$$g^{1/2} \longrightarrow g^{1/2} + \delta g^{1/2}\,, \qquad \delta g^{1/2} = \frac{1}{2} g^{1/2} g^{mn} \delta g_{mn}\,.$$

So

$$\delta\mathcal{R} = i\partial\bar{\partial}(g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}) = -\frac{i}{2} d\left[(\partial - \bar{\partial}) g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}\right]$$

But $g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}$ is a coordinate scalar so $\delta\mathcal{R}$ is exact even though $\mathcal{R}$ may not be.

In other words: a smooth variation of the metric changes $\mathcal{R}$ but not c_1.

Definition: A Hermitian manifold M with Hermitian form ω is *Kähler* if ω is closed

$$d\omega = 0$$

In this case ω is called the *Kähler form.* It defines a class $[\omega] \in H^2(M, \mathbb{R})$ which is called the *Kähler class* of M.

Example: all 1-dim complex manifolds are Kähler since $d\omega = 0$ trivially ($d\omega$ is a 3-form).

<u>Consequences of $d\omega = 0$</u>

$$d\omega = \partial\omega + \bar{\partial}\omega = 0$$

Therefore

$$\partial\omega = 0 \qquad \text{and} \qquad \bar{\partial}\omega = 0$$

Using the fact that in complex coordinates

$$\omega_{\mu\bar{\nu}} = i g_{\mu\bar{\nu}}$$

we have

$$\partial\omega = 0 \qquad \Longrightarrow \qquad \partial_\rho g_{\mu\bar{\nu}} = \partial_\mu g_{\rho\bar{\nu}}$$

and

$$\bar{\partial}\omega = 0 \qquad \Longrightarrow \qquad \partial_{\bar{\rho}} g_{\mu\bar{\nu}} = \partial_{\bar{\nu}} g_{\mu\bar{\rho}}$$

These equations imply that on each coordinate neighborhood $\mathcal{U}_j$, there is a real scalar φ_j, such that

$$g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} \varphi_j$$

on $\mathcal{U}_j$, and hence

$$\omega = i\partial\bar{\partial}\varphi_j \ .$$

Another way to say this is that these equations are the integrability conditions for the existence of a local real $\varphi_j(z, \bar{z})$.

φ is called the *Kähler potential.*

Remark:

The φ_j *do not* fit together on the overlaps $\mathcal{U}_j \cap \mathcal{U}_k$ to give a globally defined function on M (if M is compact). The metric must be however globally defined so it must be that on the overlap $\mathcal{U}_j \cap \mathcal{U}_k$ of two coordinate patches, the two Kähler potentials φ_j and φ_k are related by

$$\varphi_j = \varphi_k + f_{jk}(z) + \overline{f_{jk}(z)}$$

That is,

$$e^{\varphi_j} = |e^f|^2 e^{\varphi_k}$$

so e^{φ} is a section of a non-trival line bundle over M.

To see why the φ_j do not fit together, note that

$$\omega = i\partial\bar{\partial}\varphi_j = -\frac{i}{2}d\left((\partial - \bar{\partial})\varphi_j\right) \ .$$

This ensures that $d\omega = 0$ (of course).

If $(\partial-\bar{\partial})\varphi_j$ were globally defined, then ω would also be exact. But this is impossible. Consider

$$\omega^n = \omega \wedge \omega \cdots \wedge \omega \propto \text{volume form of} \ \ M$$

So

$$\int_M \omega^n = n! \ vol(M) > 0 \ \ \text{and so} \ \ [\omega] \neq 0 \ .$$

If $\omega = dA$, that is, if ω is exact, then replacing ω by dA and invoking Stoke's theorem we would show that $vol(M) = 0$.

Another remark:

$\nabla J = 0$ and $\nabla g = 0$ imply that $\nabla\omega = 0$

(Because $\omega_{mn} = J_m{}^k \, g_{kn}$)

In particular, ω is divergence-free:

$$d^*\omega = -g^{mn}\nabla_n\omega_{mk} \, dx^k = 0 \ .$$

Recall that in even real dimensions, if α is a p-form

$$d^*\alpha = - * d * \alpha = -\frac{1}{p-1}g^{nm}\nabla_n\alpha_{mn_2\cdots n_p}dx^{n_2} \wedge \cdots \wedge dx^{n_p}$$

where d^* is the adjoint of the exterior derivative with respect to the inner product

$$(\alpha, \beta) = \int_M \alpha \wedge *\beta \ ,$$

α and β any two p–forms. The operator d^* maps k–forms into $k-1$–forms.

Thus ω is harmonic:

$$\triangle\omega = (dd^* + d^*d)\omega = 0$$

Example: $\mathbb{P}^n$ is Kähler.

Recall: If $(z^1, \ldots, z^{n+1}) \in \mathbb{C}^{n+1}$, we choose coordinate neighborhoods $\mathcal{U}_j$ such that $z_j \neq 0$ with coordinates

$$\xi_j^m = z^m / z^j \ .$$

There is a natural Kähler metric on $\mathbb{P}^n$, the Fubini-Study metric. This metric is given by the Kähler form

$$\begin{aligned} \omega &= i\partial\bar{\partial} \log \left(1 + |\xi|^2\right) \\ &= i \left(\frac{d\xi^m \wedge d\overline{\xi^m}}{1 + |\xi|^2} - \frac{\overline{\xi^m} d\xi^m \wedge \xi^n d\overline{\xi^n}}{(1 + |\xi|^2)^2} \right) \end{aligned}$$

where $1 + |\xi|^2 = \sum_{m=1}^{n+1} |\xi^m|^2$.

One can check that the Kähler potential is

$$\varphi = \log \left(1 + |\xi|^2\right)$$

and that on the overlap $\mathcal{U}_j \cap \mathcal{U}_k$ satisfies

$$\varphi_j = \varphi_k - \log(z^j / z^k) - \log \left(\overline{z^j} / \overline{z^k}\right)$$

Example: $M_{r,s} = S^{2r+1} \times S^{2s+1}$ are not Kähler.

A Kähler manifold has $b_2 \geq 1$ because of the existence of at least the non-trivial class $[\omega] \in H^2(M, \mathbb{R})$. However we can show that

$$b_2(M_{r,s}) = 0.$$

This follows from the fact that a harmonic 2-form A on M would admit a decomposition

$$A = \alpha_2 + \alpha_1 \wedge \beta_1 + \beta_2$$

where α_2 (β_2) are harmonic 2-forms on $S^{2r+1}(S^{2s+1})$, and α_1 (β_1) are harmonic 1-forms on $S^{2r+1}(S^{2s+1})$. But these do not exist on spheres, so $b_2 = 0$.
(For any S^r, $b_p = dim H_p = 0,\ 1 \leq p \leq r-1$ and $b_0 = b_r = 1$.)

Curvature and connection on a Kähler manifold

Recall that on a Hermitian manifold M with metric g, complex structure J, and Hermitian form ω we have

- $\Gamma_{\mu\nu}{}^{\kappa} = g^{\bar{\rho}\kappa}\partial_\mu g_{\nu\bar{\rho}}\ ,\quad \Gamma_{\bar{\mu}\bar{\nu}}{}^{\bar{\kappa}} = g^{\bar{\kappa}\rho}\partial_{\bar{\mu}} g_{\bar{\nu}\rho}$

(These are unique and are the only non-zero components of Γ: we call this the Chern connection)

- $R_{\mu\bar{\nu}}{}^{\rho}{}_{\sigma} = -\partial_{\bar{\nu}}\Gamma_{\mu\sigma}{}^{\rho}$

The only non-zero components are

$$R_{\mu\bar{\nu}}{}^{\rho}{}_{\sigma},\quad R_{\mu\bar{\nu}}{}^{\bar{\rho}}{}_{\bar{\sigma}},\quad R_{\bar{\mu}\nu}{}^{\rho}{}_{\sigma},\quad R_{\bar{\mu}\nu}{}^{\bar{\rho}}{}_{\bar{\sigma}}$$

If M is also Kähler, then

• The connection is now symmetric

$$\Gamma_{\mu\nu}{}^{\kappa} = \Gamma_{\nu\mu}{}^{\kappa} \ , \qquad \Gamma_{\bar{\mu}\bar{\nu}}{}^{\bar{\kappa}} = \Gamma_{\bar{\nu}\bar{\mu}}{}^{\bar{\kappa}},$$

that is, the torsion vanishes. In fact,

$$\Gamma_{\mu\nu}{}^{\kappa} = g^{\bar{\rho}\kappa}\partial_{\mu}g_{\nu\bar{\rho}} = g^{\bar{\rho}\kappa}\partial_{\nu}g_{\mu\bar{\rho}} = \Gamma_{\nu\mu}{}^{\kappa}$$

(Here we have used the result $\partial_{\rho}g_{\mu\bar{\nu}} = \partial_{\mu}g_{\rho\bar{\nu}}$)

As a consequence the Chern connection coincides with the Christoffel connection. In other words, ∇ is the same as the Levi-Civita connection.

• Additional simplifications for the Riemann tensor follow from the fact that the torsion vanishes.

For example, we have the familiar symmetries or the Riemann tensor with Christoffel connection:

$$R_{mnrs} = R_{rsmn}$$
$$R_{m[nrs]} = 0$$

The Ricci-tensor R_{mn}, which is defined as

$$R_{mn} = R_{pm}{}^{p}{}_{n}$$

only has mixed indices. For example,

$$R_{\mu\nu} = R_{\rho\mu}{}^{\rho}{}_{\nu} + R_{\bar{\rho}\mu}{}^{\bar{\rho}}{}_{\nu} = 0 \ .$$

Recall the Ricci-form

$$\begin{aligned}\mathcal{R} &= \frac{1}{4} R_{mn}{}^{k}{}_{l} J_{k}{}^{l} dx^{m} \wedge dx^{n} \\ &= i R_{\mu\bar{\nu}} dz^{\mu} \wedge dz^{\bar{\nu}} = i\partial\bar{\partial} \log g^{1/2}\end{aligned}$$

So the components of $\mathcal{R}$ are those of the Ricci tensor (hence the name).

As we have discussed $\mathcal{R}$ defines a cohomology class

$$c_1(M) = \left[\frac{1}{2\pi}\mathcal{R}\right]$$

Question: Given a Kähler manifold M, under what circumstances does M admit a Ricci-flat metric?

This is a very interesting question in both Mathematics and Physics.

In Physics: in 4-dim Ricci-flat metrics correspond to solutions of Einstein's equations in vacuum

In String Theory: Ricci-flat metrics correspond to certain supersymmetric solutions

Suppose g is any metric on M and suppose there is a another metric $\tilde{g}$ on M which is Ricci-flat. The corresponding Ricci-forms $\mathcal{R}(g)$ and $\mathcal{R}(\tilde{g})$ both belong to $c_1(M)$

$$\mathcal{R}(g) = \mathcal{R}(\tilde{g}) + dA$$

But $\mathcal{R}(\tilde{g}) = 0$. Therefore $\mathcal{R}(g)$ must be exact and c_1 must be a trivial cohomology class : $c_1 = 0$. This is a necessary condition for M to admit a Ricci–flat metric.

Exercise:

Show that the Ricci–form for $\mathbb{P}^n$ with the Fubini-Study metric satisfies

$$\mathcal{R} = -(n+1)\omega$$

where ω is the Kähler form. (This is an example of an Einstein–Kähler metric.)

Since ω is harmonic, then so is $\mathcal{R}$, so certainly it cannot be an exact form. Thus c_1 is non-trivial and $\mathbb{P}^n$ does not admit a Ricci-flat metric. $\square$

The necessity of $c_1 = 0$ for a Kähler manifold M to admit a Ricci-flat metric was first noticed by Calabi.

In the 50's Calabi conjectured that c_1 is <u>the only</u> topological obstruction.

Theorem: (Yau)

Given a complex manifold with $c_1 = 0$, and any Kähler metric g with Kähler form ω, there is a unique Ricci–flat metric $\tilde{g}$ whose Kähler form $\tilde{\omega}$ is in the same cohomology class as ω

Proof: see

Yau,

or Joyce (Riemannian Holonomy Groups and Calibrated Geometry, Chapter 6),

or A Moroianu (Lectures on Kähler Geometry, Chapter 18, for an overview.)

$\square$

Chapter 3 Calabi–Yau Manifolds

Definition: A Calabi-Yau manifold is a compact Kähler manifold of vanishing First Chern Class.

We will discuss later non-compact Calabi–Yau manifolds motivated by the study, for example, of singular cases, like the conifold and its desingularizations.

Example: T^2 is a Calabi–Yau manifold

We will discuss several different equivalent characterizations of Calabi–Yau manifolds. Let M be a n-dim Calabi–Yau manifold:

- (A) M has a Ricci–flat metric (✓)
- (B) The Holonomy Group Hol of M is a subgroup of SU(n) On a Kähler manifold, $Hol \subseteq U(n) = U(1) \times SU(n)$; the $U(1)$ is generated by $\mathcal{R}$.
- (C) The canonical class is trivial
  The bundle of $(n, 0)$–forms, $\Lambda^n T^{*(1,0)} M$, is trivial. This bundle is often called the "canonical bundle".
- (D) There is a unique (up to a constant) holomorphic and nowhere vanishing $(n, 0)$–form Ω.
- (E) M admits a pair of globally defined covariantly constant spinors.

Cohomology and The Hodge Diamond

Theorem: A Calabi–Yau manifold M with non-zero Euler number has $b_1 = 0$.

Proof:

It is sufficient to establish the result for the Ricci–flat metric because b_1 is a topological invariant.

Let $U = u_m dx^m$ be a 1-form.

The Laplacian on 1-forms is

$$\triangle U = \left(-g^{jk}\nabla_j\nabla_k u_m + R_{mp}u_n g^{pn}\right) dx^m$$

where $\triangle = dd^* + d^*d$.

If U is a harmonic 1-form $\triangle U = 0$.

But R_{mn} vanishes. Then

$$
\begin{aligned}
& 0 = g^{jk}\nabla_j\nabla_k u_m \\
\implies \quad & 0 = \int_M u^m g^{jk}\nabla_j\nabla_k u_m g^{1/2} d^6x \\
\implies \quad & 0 = \int_M g^{jk}(\nabla_j u^m)(\nabla_k u_m) g^{1/2} d^6x \\
\implies \quad & 0 = \nabla_m u_n
\end{aligned}
$$

That is, u_m must be covariantly constant.

A vector field on a manifold with Euler number χ has at least $|\chi|$ zeros. Thus u_m must have a zero and because $\nabla_m u_n = 0$, then the u_m must vanish identically. $\square$

As discussed earlier, on an (almost) complex manifold we can use the projection operators,

$$P = \frac{1}{2}(\mathbf{1} - iJ), \quad \text{and} \quad Q = \frac{1}{2}(\mathbf{1} + iJ) \ ,$$

to decompose of k–forms into a sum of (p, q) forms

$$\omega = \sum_{p+q=k} \omega^{(p,q)} \ .$$

On a *complex* manifold we defined operators ∂ and $\bar{\partial}$ such that

$$d = \partial + \bar{\partial}$$

and $\partial^2 = 0$ and $\bar{\partial}^2 = 0$. If ω is a (p, q)–form, then $\partial\omega$ is the $(p + 1, q)$ component of $d\omega$ and $\bar{\partial}\omega$ is the $(p, q + 1)$ component of $d\omega$.

We define the Dolbeault cohomology groups $H_{\bar{\partial}}^{(p,q)}(M)$. Its elements are those (p, q)–forms which are $\bar{\partial}$–closed

$$\bar{\partial}\alpha = 0$$

modulo those which are $\bar{\partial}$–exact (the $\bar{\partial}$ of a $(p, q-1)$–form).

We define the *Hodge numbers* $h_{p,q}$ as

$$h_{p,q} = \dim H_{\bar{\partial}}^{(p,q)}(M)$$

Lecture 4

Continue introduction to Calabi–Yau manifolds.

We are discussing the Cohomology of Calabi–Yau manifolds.

So far:

- $b_1(M) = 0$ when $\chi \neq 0$
- Dolbeaut cohomology groups $H^{(p,q)}_{\bar{\partial}}(M)$.

We define the *Hodge numbers* $h_{p,q}$ as

$$h_{p,q} = \dim H^{(p,q)}_{\bar{\partial}}(M)$$

On a complex manifold with a Hermitian metric, we can define adjoint operators ∂^* and $\bar{\partial}^*$ for ∂ and $\bar{\partial}$, respectively, with respect to the inner product.

The natural inner product between two (p, q)–forms is given by

$$(\varphi, \psi) = \int_M \varphi \wedge *\overline{\psi} \ .$$

We define $\bar{\partial}^*$ as

$$(\alpha, \bar{\partial}\beta) = (\bar{\partial}^*\alpha, \beta) \ .$$

and we find $\bar{\partial}^*\alpha = - * \bar{\partial} * \alpha$.

The operator $\bar{\partial}^*$ maps (p, q)–forms into $(p, q-1)$–forms.

We define Laplacians:

$$\triangle_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial \ , \qquad \text{and} \qquad \triangle_{\partial} = \partial\partial^* + \partial^*\partial \ .$$

And we can represent cohomology classes in $H^{(p,q)}_{\bar\partial}(M)$ by $\bar\partial$–harmonic forms.

Exercise: If M is Kähler:

$$\triangle_d = 2\triangle_\partial = 2\triangle_{\bar\partial}$$

$\square$

Therefore

$$\mathcal{H}^k = \oplus_{p+q=k}\mathcal{H}^{(p,q)}(M) \ ,$$

where $\mathcal{H}^k$ is the set of harmonic k–forms. This result shows that deRham cohomology and $\bar\partial$-cohomology are equivalent.

The k-th Betti number, b_k, then decomposes

$$b_k = \sum_{p+q=k} h_{p,q} \ .$$

Not all Hodge numbers are independent:

• Hodge-star duality (valid for smooth compact spaces)

$$h_{p,q} = h_{n-q,n-p}$$

• Complex conjugation: on a Kähler manifold

$$h_{p,q} = h_{q,p}$$

• For Calabi–Yau manifolds: $h_{n,0} = h_{0,n} = 1$

This is guaranteed by the existence of Ω (property (D)).

Exercise: Show that, because of property (D),

$$h_{0,q} = h_{0,n-q}$$

For example, for $n = 3$, this means that $h_{01} = h_{02}$.

It is customary to display the Hodge numbers of a complex manifold in an array called the **Hodge diamond**. For $n = 3$:

$$\begin{array}{ccccccc} & & & h_{00} & & & \\ & & h_{10} & & h_{01} & & \\ & h_{20} & & h_{11} & & h_{02} & \\ h_{30} & & h_{21} & & h_{12} & & h_{03} \\ & h_{31} & & h_{22} & & h_{13} & \\ & & h_{23} & & h_{32} & & \\ & & & h_{33} & & & \end{array}$$

Due to the properties above, the diamond has a number of symmetries.

Complex conjugation gives $h_{pq} = h_{qp}$ so the diamond is symmetric under reflection in the vertical axis.

$$\begin{array}{ccccccc} & & & h_{00} & & & \\ & & h_{10} & & h_{10} & & \\ & h_{20} & & h_{11} & & h_{20} & \\ h_{30} & & h_{21} & & h_{21} & & h_{30} \\ & h_{31} & & h_{22} & & h_{31} & \\ & & h_{23} & & h_{23} & & \\ & & & h_{33} & & & \end{array}$$

Hodge–star duality, $h_{pq} = h_{n-q,n-p}$, makes the diamond symmetric also under reflection in a horizontal axis.

$$\begin{array}{ccccccc}
 & & & h_{00} & & & \\
 & & h_{10} & & h_{10} & & \\
 & h_{20} & & h_{11} & & h_{20} & \\
h_{30} & & h_{21} & & h_{21} & & h_{30} \\
 & h_{20} & & h_{11} & & h_{20} & \\
 & & h_{10} & & h_{10} & & \\
 & & & h_{00} & & &
\end{array}$$

Finally, as mentioned above, the existence of a holomorphic n-form implies $h_{p0} = h_{n-p,0}$.

$$\begin{array}{ccccccc}
 & & & h_{00} & & & \\
 & & h_{10} & & h_{10} & & \\
 & h_{10} & & h_{11} & & h_{10} & \\
h_{30} & & h_{21} & & h_{21} & & h_{30} \\
 & h_{10} & & h_{11} & & h_{10} & \\
 & & h_{10} & & h_{10} & & \\
 & & & h_{00} & & &
\end{array}$$

We also have:

- $h_{0,0} = b_0 = 1$ for a single connected piece
- $b_{2n} = 1 = h_{n,n}$ (unique) volume form
- $b_1 = 0$ so $h_{1,0} = h_{0,1} = 0$ if $\chi \neq 0$
- $h_{n,0} = h_{0,n} = 1$

T^2

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$$

$K3$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & h_{11} & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \qquad \text{with} \qquad h_{11} = 20$$

$n = 3$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h_{11} & & 0 & \\
1 & & h_{21} & & h_{21} & & 1 \\
 & 0 & & h_{11} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

<u>Different characterizations of Calabi–Yau manifolds.</u>

Let M be a n-dim Calabi–Yau manifold:

- (A) M has a Ricci–flat metric (✓)
- (B) The Holonomy Group Hol of M is a subgroup of SU(n) On a Kähler manifold, $Hol \subseteq U(n) = U(1) \times SU(n)$; the $U(1)$ is generated by $\mathcal{R}$.
- (C) The <u>canonical class</u> is trivial
  The bundle of $(n, 0)$–forms, $\Lambda^n T^{*(1,0)} M$, is trivial. This bundle is often called the "canonical bundle".
- (D) There is a unique (up to a constant) <u>holomorphic</u> and <u>nowhere vanishing</u> $(n, 0)$–form Ω.
- (E) M admits a pair of globally defined covariantly constant spinors.

(C) The <u>canonical class</u> is trivial

Assume M is simply connected.

Let K_M be the *canonical line bundle*

$$K_M = \Lambda^n T^{*(1,0)} M \ ,$$

the bundle of $(n, 0)$–forms.

(Recall that the projection operators P and Q can be used to project T^*M

$$T^*_{\mathbb{C}} M = T^{*(1,0)} M \oplus T^{*(0,1)} M \ ,$$

where $T^{*(1,0)}M$ is the holomorphic sub-bundle spanned by $\{dz^\mu\}$.)

The Kähler metric g on M induces a metric and a connection on K_M. One can show that the curvature of this connection is the Ricci–form of M. (This is because $[\nabla^K, \nabla^K]\alpha = -\mathcal{R}\alpha$, where α is a $(n, 0)$–form.)

Therefore: $\quad [\mathcal{R}] = 2\pi c_1(K_M) = 2\pi c_1(M) = 0.$

(B) The Holonomy Group Hol of M is a subgroup of SU(n) On a Kähler manifold, $Hol \subseteq U(n) = U(1) \times SU(n)$; the $U(1)$ is generated by $\mathcal{R}$.

A connection ∇ is a rule for parallel transport of vectors in M.

To each closed loop C in M, we associate a linear transformation $S(C)$ measuring the rotation of a vector which results when a vector is parallel transported around C.

Thus, $S(C) \in O(d)$, where $d =$ (real) dim M.

Hol is the set of all such $S(C)$ for all possible curves C. It gives information about the curvature of M.
So $Hol \subseteq O(d)$.
In our case $d = 2n$: $Hol \subseteq O(2n)$.
If M is orientable: $Hol \subseteq SO(2n)$

Simple Examples:

$\mathbb{R}^d$	$Hol = \mathbf{1}$
S^d	$Hol = SO(d)$

If M is Kähler: $Hol \subseteq U(n)$
This is a consequence of the connection being pure in its indices.
A vector V in $T^{(1,0)}M$ (which has only holomorphic components) is carried into another such vector after parallel transport around a closed loop C.

Let
$$e_\alpha = e_\alpha{}^\mu \frac{\partial}{\partial z^\mu}$$
be a basis of vectors at a point p such that
$$g(e_\alpha, e_{\bar{\beta}}) = \delta_{\alpha\bar{\beta}} \ .$$

Then

$$e_\alpha \qquad \longrightarrow \qquad S_\alpha{}^\beta \; e_\beta$$

under parallel transport.

The fact that $g(e_\alpha, e_{\bar{\beta}}) = \delta_{\alpha\bar{\beta}}$ implies that $S_\alpha{}^\beta$ is a unitary matrix.

If M is moreover a Calabi–Yau manifold,

$$Hol \subseteq SU(n)$$

Let $V \in TM$, and consider the change induced in V by parallel transport around an infinitesimal rectangle of area δa^{mn} with edges that are parallel to $\frac{\partial}{\partial x^m}$ and $\frac{\partial}{\partial x^n}$.

It is a standard result that

$$V'^k = V^k + \delta a^{mn} R_{mn}{}^k{}_\ell V^\ell.$$

The matrices

$$\delta^k{}_\ell + \delta a^{mn} R_{mn}{}^k{}_\ell$$

are the elements of Hol that are infinitesimally close to the identity.

As discussed, $\delta a^{mn} R_{mn}{}^k{}_\ell$ must be in the Lie algebra of $U(n)$. The $U(1)$ in

$$U(n) = U(1) \times SU(n)$$

is generated by the trace

$$\delta a^{mn} R_m{}^k{}_{nk} = -4\delta a^{\mu\bar{\nu}} R_{\mu\bar{\nu}}$$

Thus the $U(1)$ part of the holonomy is generated by the Ricci tensor.

So if the manifold is both Kähler and Ricci-flat then the holonomy group is contained in $SU(n)$.

This statement is true for the **local holonomy group**, which is the subgroup of the holonomy group associated with paths C that may be continuously shrunk to a point.

However if the manifold is multiply connected then there are paths that cannot be continuously shrunk to a point. The statement that the holonomy group of a Ricci-flat Kähler manifold is contained in $SU(n)$ even when the manifold is multiply connected is nevertheless true though the proof is more involved.

(D) There is a unique (up to a constant) holomorphic and nowhere vanishing $(n, 0)$–form Ω.

Theorem:

A compact Kähler manifold M has $c_1(M) = 0$ iff M admits a unique (up to a constant) nowhere vanishing holomorphic $(n, 0)$–form which is globally well defined

$$\Omega = \frac{1}{n!} \Omega_{\mu_1 \cdots \mu_n}(z) dz^{\mu_1} \wedge dz^{\mu_2} \wedge \cdots \wedge dz^{\mu_n} \ .$$

Proof:

Suppose first that there is a nowhere vanishing holomorphic $(n, 0)$–form

$$\Omega = \frac{1}{n!} \Omega_{\mu_1 \cdots \mu_n}(z) dz^{\mu_1} \wedge dz^{\mu_2} \wedge \cdots \wedge dz^{\mu_n} \ .$$

It must be the case that

$$\Omega_{\mu_1 \cdots \mu_n}(z) = f(z) \epsilon_{\mu_1 \cdots \mu_n}$$

where $f(z)$ is holomorphic, nowhere vanishing and $\epsilon_{\mu_1 \cdots \mu_n}$ is the permutation symbol.

Let

$$\begin{aligned} ||\Omega||^2 &= \frac{1}{n!} |f(z)|^2 \epsilon_{\mu_1 \cdots \mu_n} \epsilon_{\bar{\nu}_1 \cdots \bar{\nu}_n} g^{\mu_1 \bar{\nu}_1} \cdots g^{\mu_n \bar{\nu}_n} \\ &= |f(z)|^2 g^{-1/2} \end{aligned}$$

Then $\qquad g^{1/2} = |f(z)|^2 / ||\Omega||^2.$

Recall

$$\mathcal{R} = i\partial\bar{\partial}\log g^{1/2}$$

Then

$$\begin{aligned}\mathcal{R} &= i\partial\bar{\partial}\left(-\log||\Omega||^2 + \log f + \log \bar{f}\right)\\ &= -i\partial\bar{\partial}\left(\log||\Omega||^2\right)\end{aligned}$$

But $\log||\Omega||^2$ is a coordinate scalar which by hypothesis is globally well defined.

Therefore $\mathcal{R}$ is exact and $c_1(M) = 0$.

<ins>Conversely</ins>: now we want to show that if $c_1(M) = 0$, then there is a unique $(n, 0)$–form Ω which is holomorphic and nowhere vanishing.

A holomorphic $(n, 0)$–form Ω is a section of the canonical line bundle

$$K_M = \Lambda^n T^{*(1,0)}M$$

But K_M is trivial ($c_1(K_M) = 0$), then Ω is nowhere vanishing.

This is because the triviality of K_M implies that the total space of K_M is $M\times\mathbb{C}$. The unit section, that is, the constant function 1, is a globally defined nowhere vanishing holomorphic $(n, 0)$–form, Ω.

Lecture 5

Finish introduction to Calabi–Yau manifolds.

(And start discussing examples.)

Let M be a n-dim Calabi–Yau manifold:

- (A) M has a Ricci–flat metric (✓)
- (B) The Holonomy Group Hol of M is a subgroup of SU(n) (✓)
- (C) The canonical class is trivial (✓)
- (D) There is a unique (up to a constant) holomorphic and nowhere vanishing $(n, 0)$–form Ω.
- (E) M admits a pair of globally defined covariantly constant spinors.

(D) There is a unique (up to a constant) holomorphic and nowhere vanishing $(n, 0)$–form Ω.

Theorem:

A compact Kähler manifold M has $c_1(M) = 0$ iff M admits a unique (up to a constant) nowhere vanishing holomorphic $(n, 0)$–form which is globally well defined

$$\Omega = \frac{1}{n!}\Omega_{\mu_1\cdots\mu_n}(z)dz^{\mu_1} \wedge dz^{\mu_2} \wedge \cdots \wedge dz^{\mu_n} \ .$$

Continue with Proof:

We showed that if there is a nowhere vanishing holomorphic $(n,0)$–form

$$\Omega = \frac{1}{n!}\Omega_{\mu_1\cdots\mu_n}(z)dz^{\mu_1}\wedge dz^{\mu_2}\wedge\cdots\wedge dz^{\mu_n}\ ,$$

then

$$\mathcal{R} = -i\partial\bar{\partial}\left(\log||\Omega||^2\right)$$

is exact and $c_1(M)=0$.

<ins>Conversely</ins>: now we want to show that if $c_1(M)=0$, then there is a unique $(n,0)$–form Ω which is holomorphic and nowhere vanishing.

A holomorphic $(n,0)$–form Ω is a section of the canonical line bundle

$$K_M = \Lambda^n T^{*(1,0)}M$$

But K_M is trivial ($c_1(K_M)=0$), then Ω is nowhere vanishing.

This is because the triviality of K_M implies that the total space of K_M is $M\times\mathbb{C}$. The unit section, that is, the constant function 1, is a globally defined nowhere vanishing holomorphic $(n,0)$–form, Ω.

Another way to see this is to use Yau's theorem and Čech cohomology.

If $c_1(M) = 0$, Yau's theorem implies that there is a Ricci–flat metric g

$$\partial_\mu \partial_{\bar{\nu}} \log g^{1/2} = 0 \ .$$

This means that there must exist functions $f_j(z)$ on each coordinate patch $\mathcal{U}_j$ such that

$$g_j^{1/2} = |f_j|^2$$

and moreover, each f_j must be non–vanishing on $\mathcal{U}_j$, otherwise g would be singular.

The idea of the proof is to use Čech cohomology to show that it is possible to choose a set of phases θ_j such that

$$e^{-i\theta_j} f_j(z) dz_j^1 \wedge \ldots \wedge dz_j^n$$

is independent of j, and which is in fact Ω.

First, on $\mathcal{U}_i \cap \mathcal{U}_j$, we study the transformation properties of the f_j. Under a coordinate transformation

$$g_i^{1/2} \left| \frac{\partial(z_i)}{\partial(z_j)} \right|^2 = g_j^{1/2}.$$

Taking this together with $g_j^{1/2} = |f_j|^2$, we make a separation of variables argument

$$\frac{f_i \frac{\partial(z_i)}{\partial(z_j)}}{f_j} = \overline{\left(\frac{f_j}{f_i \frac{\partial(z_i)}{\partial(z_j)}} \right)} = e^{i\theta_{ij}}.$$

The first quantity is a function of the z^μ while the second is a function of the $z^{\bar{\mu}}$: these can be equal only if they are both equal to a constant which we have written as $e^{i\theta_{ij}}$. It is immediate that the θ_{ij} are real.

Considering the inverse transfomation we obtain

$$\theta_{ij} = -\theta_{ji}$$

So the θ_{ij} are a set of *constants* associated with the nonempty overlaps $\mathcal{U}_i \cap \mathcal{U}_j$.

We consider also nonempty triple overlaps $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$. By considering succesive transformations we find that

$$\theta_{ij} + \theta_{jk} + \theta_{ki} = 0$$

Now in the language of Čech cohomology θ_{ij} is a one-cochain and is the analogue of a one-form (the counting is that a zero-cochain is a set of constants θ_j defined on the patch $\mathcal{U}_j$ and is the counterpart of a zero-form *i.e.* a function).

The relation

$$\theta_{ij} + \theta_{jk} + \theta_{ki} = 0$$

is the cocycle condition analogous to the statement that the corresponding form is closed.

Čech cohomology is equivalent to deRham cohomology so if $b_1 = 0$ the cocycle condition can be satisfied only if θ_{ij} is in fact a coboundary, *i.e.* the analogue of an exact 1–form,

$$\theta_{ij} = \theta_i - \theta_j$$

for some choice of constants θ_i associated with each $\mathcal{U}_i$. We have the transformation law for $g^{1/2}$ above

$$e^{-i\theta_i} f_i \frac{\partial(z_i)}{\partial(z_j)} = e^{-i\theta_j} f_j.$$

So we set

$$\Omega = e^{-i\theta_j} f_j(z) dz_j^1 \wedge \cdots \wedge dz_j^n$$

which is independent of the coordinates used in its definition. $\square$

Remarks

• Ω is unique up to a constant: let $\tilde{\Omega} = h(z)\Omega$, for some holomorphic function $h(z)$. By the maximum modulus principle, a globally defined holomorphic function is a constant, so $\tilde{\Omega}$ is a constant multiple of Ω.

• $d\Omega = 0$

In fact $d\Omega = \partial\Omega + \bar{\partial}\Omega = 0$ because $\partial\Omega$ is a $(n+1, 0)$–form and therefore vanishes, and $\bar{\partial}\Omega = 0$ because Ω is holomorphic.

• These imply that $h_{n,0} = 1$.

• Ω is covariantly constant in the Ricci-flat metric.

Let $n = 3$.

Note that

$$\nabla_{\bar{\kappa}} \Omega_{\mu\nu\rho}(z) = 0$$

since Ω is holomorphic.

Consider therefore

$$\nabla_\kappa \Omega_{\mu\nu\rho} = \partial_\kappa \Omega_{\mu\nu\rho} - 3\Gamma_{\kappa[\mu}{}^\sigma \Omega_{\nu\rho]\sigma}.$$

Now $\Gamma_{\kappa[\mu}{}^\sigma \Omega_{\nu\rho]\sigma}$, being skew in (μ, ν, ρ), must be proportional to $\Omega_{\mu\nu\rho}$.

In fact

$$
\begin{aligned}
\Gamma_{\kappa[\mu}{}^{\sigma}\Omega_{\nu\rho]\sigma} &= \frac{\bar{\Omega}^{\xi\eta\zeta}}{3!\|\Omega\|^2}\Gamma_{\kappa\xi}{}^{\sigma}\Omega_{\eta\zeta\sigma}\Omega_{\mu\nu\rho} \\
&= \frac{1}{3}\Gamma_{\kappa\sigma}{}^{\sigma}\Omega_{\mu\nu\rho} \\
&= \frac{1}{3}\partial_\kappa(\log g^{\frac{1}{2}})\Omega_{\mu\nu\rho}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\nabla_\kappa\Omega_{\mu\nu\rho} &= \Big(\partial_\kappa f - f\partial_\kappa \log(|f|^2)\Big)\,\epsilon_{\mu\nu\rho} \\
&= 0.
\end{aligned}
$$

$\square$

So far we have:

Let M be a n-dim Calabi–Yau manifold:

- (A) M has a Ricci–flat metric (✓)
- (B) The Holonomy Group Hol of M is a subgroup of SU(n) (✓)
- (C) The <u>canonical class</u> is trivial (✓)
- (D) There is a unique (up to a constant) <u>holomorphic</u> and <u>nowhere vanishing</u> $(n, 0)$–form Ω. (✓)
- (E) M admits a pair of globally defined covariantly constant spinors.

(E) M admits a pair of globally defined covariantly constant spinors.

Let $n = 3$ and let $Hol = SU(3)$.

Theorem: A Calabi-Yau manifold admits a globally defined pair of spinors $\zeta, \bar{\zeta}$ (where $\bar{\zeta}$ is the complex conjugate of ζ), which are covariantly constant

$$\nabla^S \zeta = 0 \ , \qquad \nabla^S \bar{\zeta} = 0 \ ,$$

where ∇^S is the spin connection on the spin–bundle S on M which is induced from the Levi-Civita connection on the $SU(3) \subset SO(6)$ bundle on M.

Proof:

Wang, *Parallel Spinors and Parallel Forms*, Annals of Global Analysis and Geometry 7 (1989), 59-68.

Let M be an oriented, d-dimensional Riemannian *spin* manifold with metric g.

A *spinor* ζ, is a section of the *spin bundle* S on M

$$\zeta \in C^\infty(S)$$

and they fall into complex representations, Δ^d, of $Spin(d)$ (the double cover of $SO(d)$).

If $d = 2n$, the dimension of Δ^{2n} is 2^n, and it splits into irreducible representations

$$\Delta^{2n} = \Delta^n_+ \oplus \Delta^n_-$$

with dimension of $\Delta^n_\pm$ being 2^{n-1}. Correspondingly $S = S_+ \oplus S_-$, with $S_\pm$ vector sub–bundles on M with fibre $\Delta_\pm$.

In six dimensions, a spinor has eight components which transform in the $\mathbf{4} \oplus \bar{\mathbf{4}}$ of $SO(6) \approx SU(4)$.

With respect to the $SU(3) \times U(1)$ subgroup of $SU(4)$, the representation $\mathbf{4}$ of $SU(4)$ decomposes as

$$\mathbf{4} = \mathbf{3} \oplus \mathbf{1}.$$

Thus there is an $SU(3)$ singlet, that is, it is invariant under the action of Hol. There is a $1-1$ correspondence between covariantly constant spinors and the singlets in $\Delta^3_\pm$ of the Hol.

If M is simply connected, the Hol group of the spin connection ∇^S follows from the Riemannian Hol groups

$$Hol(\nabla^S) = SU(3)$$

Remarks:

- One can prove directly that if M is a Riemannian manifold which admits a covariantly constant spinor then it should be Ricci-flat.

  This follows from

  $$[\nabla^S_m, \nabla^S_n]\zeta = -\frac{1}{4} R_{mnpq}\gamma^{pq}\zeta = 0$$

  Contract with γ^n and use the identity

  $$\begin{aligned} \gamma^n\gamma^{pq} =& \frac{1}{2}\big(\{\gamma^{pq}, \gamma^n\} - [\gamma^{pq}, \gamma^n]\big) \\ =& \frac{1}{2}\big(2\gamma^{npq} + 4g^{n[p}\gamma^{q]}\big) \end{aligned}$$

  The quantity $R_{mnpq}\gamma^{npq}$ vanishes ($R_{m[npq]} = 0$). We are left with $\quad R_{mn} = 0.$

• One can write Ω and J in terms of ζ and $\bar{\zeta}$

$$\Omega_{mnr} = \zeta^T \gamma_{mnr} \zeta$$
$$J_m{}^n = -i\bar{\zeta}^T \gamma_m{}^n \zeta$$

It is not too hard to prove that these have the required properties ($J^2 = -\mathbf{1}$, $N_J = 0$, etc)

Chapter 4
Examples of Calabi–Yau manifolds

• 1 complex dimension

$\mathbb{P}_1$ is not a Calabi–Yau manifold

Σ_g, a Riemann surface of genus g is a Calabi–Yau manifold only for $g = 1$.

T^2 is the only topological type

$Hol = \mathbf{1}$

• 2 complex dimensions

* There is only one topological type of a Calabi–Yau manifold with $Hol = SU(2)$.

These are the $K3$ surfaces.

* With $Hol \subset SU(2)$: T^4

• 3 complex dimensions

* There are more than 10^5 3-folds with $Hol = SU(3)$. The easiest to construct are those which are hypersurfaces and complete intersections in toric varieties.
* We also have those with $Hol \subset SU(3)$, as for example T^6 and $T^2 \times K3$.

Chern Classes

There are cohomology classes which are analytic invariants of the manifold and defined by polynomials of the curvature 2-form, Θ.

Let E be a holomorphic vector bundle with Hermitian metric. Let A be a connection matrix, that is, a Lie algebra valued matrix of 1-forms.

The *curvature 2-form* is then

$$\Theta = dA + A \wedge A$$

which is a Lie algebra valued $n \times n$ matrix of 2-forms.

Θ satisfies a Bianchi identity

$$D\Theta = d\Theta + A \wedge \Theta - \Theta \wedge A = 0$$

where D is the covariant derivative with respect to the connection A.

Definition: an *invariant polynomial* in Θ, $P(\Theta)$, is any polynomial in Θ which is invariant under all unitary frame transformations

$$e^{\alpha} \to \phi^{\alpha}{}_{\beta} e^{\beta} \ .$$

Here $\{e^{\alpha}, e^{\bar{\alpha}}\}$ is a basis for $T^{*(1,0)}M \oplus T^{*(0,1)}M$. (So, for example we have $e^{\alpha} = e^{\alpha}{}_{\mu} dz^{\mu}$.)

Examples of invariant polynomials:

$$Tr\Theta\,, \quad Tr(\Theta\wedge\Theta)\,, \quad Tr(\Theta\wedge\Theta\wedge\Theta)+Tr\Theta\wedge Tr\Theta\wedge Tr\Theta$$

indices fully contracted

Proposition: Each invariant polynomial, $P(\Theta)$, defines a cohomology class which is an analytic invariant.

To prove:

(i) Show that each invariant $P(\Theta)$ is closed.

(ii) Show that under variations of A, the change in the invariant polynomial $P(\Theta)$ is exact.

(Polynomial invariants have topologically invariant integrals)

Proof of (i):

Follows from the BI: $D\Theta = d\Theta + A\wedge\Theta - \Theta\wedge A = 0$

Examples first:

- $$\begin{aligned} DTr\Theta &= D\Theta^{\alpha}{}_{\alpha} \\ &= dTr\Theta + Tr(A\wedge\Theta - \Theta\wedge A) = 0 \end{aligned}$$

  $\Longrightarrow \quad dTr\Theta = 0$

- $$\begin{aligned} dTr(\Theta\wedge\Theta) &= Tr(d\Theta\wedge\Theta + \Theta\wedge d\Theta) \\ &= Tr(D\Theta\wedge\Theta + \Theta\wedge D\Theta) = 0 \end{aligned}$$

Note that, because of the Tr, terms like

$$Tr\left[(A\wedge\Theta - \Theta\wedge A)\wedge\Theta\right]$$

vanish.

Clearly, $dP(\Theta)$ gives a sum of terms each of which contains a $D\Theta$ and so it vanishes: $dP(\Theta) = 0$.

Proof of (ii):

Consider variations of A: $\quad A \to A + \delta A$.

Recall: $\quad \Theta = dA + A\wedge A$.

Then

$$\begin{aligned} \delta\Theta &= \Theta(A+\delta A) - \Theta(A) \\ &= d\delta A + A\wedge\delta A + \delta A\wedge A \\ &= D\delta A \end{aligned}$$

Consider, for example, the polynomial $Tr(\Theta\wedge\Theta)$:

$$\begin{aligned} \delta Tr(\Theta\wedge\Theta) &= Tr(D\delta A\wedge\Theta) + Tr(\Theta\wedge D\delta A) \\ &= d(Tr(\delta A\wedge\Theta + \Theta\wedge\delta A)) \end{aligned}$$

In a similar way any invariant polynomial $P(\Theta)$ varies by an amount that is exact

$$\delta P(\Theta) = d\delta Q \ .$$

To establish the result for finite differences let $\tilde{A}$ and A be two connections. Suppose that

$$A_t = (1-t)A + t\tilde{A} \ , \qquad 0 \leq t \leq 1$$

so that $A_0 = A$ and $A_1 = \tilde{A}$.

We have just shown above that

$$\frac{d}{dt} P(\Theta_t) = dQ_t$$

for some Q_t. Integrating

$$P(\tilde{\Theta}) - P(\Theta) = d \int_0^1 Q_t \, dt \ .$$

So $P(\tilde{\Theta})$ and $P(\Theta)$ differ by an exact form, and thus, their integrals over manifolds without boundary give the same results. □

Definition: Let E be a Hermitian vector bundle over a complex smooth manifold M with connection A and curvature 2-form Θ.

The *Chern Polynomials*, or *Chern Classes*, $c_j(\Theta)$, are defined by the formal expansion of the *Total Chern Form*, $c(\Theta)$, of the vector bundle E

$$\begin{aligned} c(\Theta) &= \det\left(\mathbf{1} + \frac{i}{2\pi}\Theta\right) \\ &= c_0(\Theta) + c_1(\Theta) + c_2(\Theta) + \ldots + c_n(\Theta) \end{aligned}$$

where the $c_j(\Theta)$ are invariant polynomials of degree j in Θ

$$c_j(\Theta) \in H^{2j}(M)$$

They are independent of the connection (we proved that $P(\Theta)$ and $P(\tilde{\Theta})$ differ by an *exact* form).

We have for example:

$$c_0(\Theta) = 1$$
$$c_1(\Theta) = \frac{i}{2\pi} Tr\Theta$$
$$c_2(\Theta) = -\frac{1}{8\pi^2}\left[Tr(\Theta \wedge \Theta) - Tr\Theta \wedge Tr\Theta\right]$$
$$c_3(\Theta) = -\frac{1}{48\pi^3}[-2Tr(\Theta \wedge \Theta \wedge \Theta)$$
$$+ 3Tr(\Theta \wedge \Theta) \wedge Tr\Theta - Tr\Theta \wedge Tr\Theta \wedge Tr\Theta]$$

Note that $c_j = 0$ for $2j > n = \dim M$, so $c(\Theta)$ is a finite sum.

Note: if E is the Tangent bundle

$$c_1 = \frac{i}{2\pi} Tr\Theta = \frac{i}{2\pi} R_{\mu\bar{\nu}k}{}^{k} dz^{\mu} \wedge dz^{\bar{\nu}} = \frac{1}{2\pi}\mathcal{R}$$

Lecture 6

Examples of Calabi–Yau manifolds.

Definition: Let E be a Hermitian vector bundle over a complex smooth manifold M with connection A and curvature 2-form Θ. The *Chern Polynomials*, or *Chern Classes*, $c_j(\Theta)$, are defined by the formal expansion of the *Total Chern Form*, $c(\Theta)$, of the vector bundle E

$$c(\Theta) = \det\left(\mathbf{1} + \frac{i}{2\pi}\Theta\right) = c_0(\Theta) + c_1(\Theta) + \ldots + c_n(\Theta)$$

where the $c_j(\Theta)$ are invariant polynomials of degree j in Θ

$$c_j(\Theta) \in H^{2j}(M, \mathbb{Z})$$

They are independent of the connection (we proved that $P(\Theta)$ and $P(\tilde{\Theta})$ differ by an *exact* form).

We have for example:

$$c_0(\Theta) = 1$$
$$c_1(\Theta) = \frac{i}{2\pi} Tr\Theta$$
$$c_2(\Theta) = -\frac{1}{8\pi^2}\left[Tr(\Theta \wedge \Theta) - Tr\Theta \wedge Tr\Theta\right]$$
$$\begin{aligned} c_3(\Theta) = -\frac{1}{48\pi^3}[&-2Tr(\Theta \wedge \Theta \wedge \Theta) \\ &+ 3Tr(\Theta \wedge \Theta) \wedge Tr\Theta - Tr\Theta \wedge Tr\Theta \wedge Tr\Theta] \end{aligned}$$

Note: if E is the Tangent bundle

$$c_1 = \frac{i}{2\pi} Tr\Theta = \frac{i}{2\pi} R_{\mu\bar{\nu}k}{}^{k} dz^{\mu} \wedge dz^{\bar{\nu}} = \frac{1}{2\pi}\mathcal{R}$$

Let

$$x = \frac{i}{2\pi}\Theta \ .$$

Associated with the Chern polynomials is the *Chern Character*, $ch(E)$, and the *symmetric polynomials* $S_k(E)$ defined by

$$S_k(E) = Tr(x^k)$$
$$ch(E) = Tr(e^x) = \sum \frac{1}{k!} S_k(E)$$

The utility of the Chern character being that it behaves well under addition and multiplication of bundles

$$ch(E \oplus F) = ch(E) + ch(F)$$
$$ch(E \otimes F) = ch(E) \wedge ch(F) \ .$$

Also the S_k are useful to compute the Chern classes $c_j(\Theta)$ from the Chern form $c(\Theta)$.

Let λ_m, $m = 1, \ldots, n$, be the eigenvalues of x. Then

$$\begin{aligned} c(x) &= \det(\mathbf{1} + x) = \prod_m (1 + \lambda_m) \\ &= \mathbf{1} + \sum_m \lambda_m + \sum_{m>n} \lambda_m \lambda_n + \sum_{m>n>r} \lambda_m \lambda_n \lambda_r + \cdots \\ &= \mathbf{1} + c_1 + c_2 + \ldots + c_n \end{aligned}$$

Now, since

$$S_k = Tr(x^k) = \sum_m \lambda_m^k$$

there are relations between the Chern polynomials and the S_k which are convenient to write as follows.

$$\begin{aligned} c_1(\Theta) &= \sum_m \lambda_m = S_1 = \frac{i}{2\pi} Tr\Theta \\ c_2(\Theta) &= \sum_{m>n} \lambda_m \lambda_n = \frac{1}{2} \left(\sum_{m,n} \lambda_m \lambda_n - \sum_m \lambda_m^2 \right) \\ &= \frac{1}{2} \left(c_1^2 - S_2 \right) = \left(\frac{i}{2\pi} \right)^2 \frac{1}{2} \left[(Tr\Theta)^2 - Tr\Theta^2 \right] \\ c_3(\Theta) &= \frac{1}{3} \left[-S_3 + c_1^3 + 3c_1 \wedge c_2 \right] \\ c_4(\Theta) &= \frac{1}{4} \left[-S_4 + c_1^4 - 4c_1^2 \wedge c_2 + 4c_1 \wedge c_3 + 2c_2^2 \right] \\ & etc \end{aligned}$$

An alternative way of generating these equations is from the Newton formulæ

$$S_k - c_1 \wedge S_{k-1} + \cdots + (-1)^k k c_k = 0 \ .$$

These polynomials give rise to the *Chern numbers*, which, of course, are analytic invariants. The Chern classes actually belong to integer classes and

$$\int_{\alpha_j} c_j(\Theta) \in \mathbb{Z} \ , \qquad \alpha_j \in H_{2j}(M, \mathbb{Z}) \ .$$

The Chern numbers are numbers obtained from integrating the classes over the manifold M.

If $\dim M = 2$, there are only two numbers

$$\mathcal{C}_2 = \int_M c_2(\Theta)$$
$$\mathcal{C}_1 = \int_M c_1(\Theta) \wedge c_1(\Theta)$$

By the Gauss-Bonnet theorem, $\mathcal{C}_2 = \chi$, the Euler number, when E is the Tangent Bundle of M.

If $\dim M = 3$, there are only three numbers

$$\mathcal{C}_1 = deg(M) = \int_M c_1(\Theta)^3$$
$$\mathcal{C}_2 = \int_M c_1(\Theta) \wedge c_2(\Theta)$$
$$\mathcal{C}_3 = \int_M c_3(\Theta)$$

By the Gauss-Bonnet theorem, $\mathcal{C}_3 = \chi$, the Euler number, when E is the Tangent Bundle of M. If the First Chern class vanishes, then only the Euler number is non-trivial.

Splitting Principle (Illustrate only)

Consider a matrix α and the identity

$$\det \alpha = e^{(Tr \log \alpha)} \ .$$

If α is a diagonalizable matrix with eigenvalues λ_j then

$$\det \alpha = \prod_i \lambda_i = e^{\sum_i \log \lambda_i} = e^{Tr \log \alpha}$$

This identity holds for any matrix

Suppose Θ, the $n \times n$ matrix of curvature 2-forms, is diagonalizable into n 2-forms Θ_j. Then

$$\begin{aligned} c(E) &= \det \left(\mathbf{1} + \frac{i}{2\pi} \Theta \right) \\ &= \prod_i \left(\mathbf{1} + \frac{i}{2\pi} \Theta_i \right) = \prod_i (\mathbf{1} + x_i) \end{aligned}$$

where $x_j = \frac{i}{2\pi} \Theta_j$.

We can interpret each $(\mathbf{1} + x_i)$ as

$$\begin{aligned} c(L_j) &= \text{Chern form of a 1-dim line bundle } L_j \\ &= \mathbf{1} + c_1(L_j) = \mathbf{1} + \frac{i}{2\pi} \Theta_j = \mathbf{1} + x_j \end{aligned}$$

If E is a k-dim vector bundle such that

$$E = L_1 \oplus L_2 \oplus \cdots \oplus L_k$$

then

$$c(E) = \prod_j c(L_j) = \prod_j (1 + x_j)$$

with $x_j = c_1(L_j)$.

The main point is that we can deform smoothly a holomorphic vector bundle E of rank k into $\oplus_{i=1}^k L_i$, where the L_i are line bundles. The Chern classes are invariant under such deformations.

Example: Chern Classes of $\mathbb{P}^n$

We obtained previously

$$c_1 = \frac{1}{2\pi}\mathcal{R} = -\frac{1}{2\pi}(n+1)\omega$$

where ω is the Fubini-Study Kähler form.

Moreover, using the splitting principle

$$c(\mathbb{P}^n) = (\mathbf{1} + x)^{n+1}$$

where

$$x = \frac{i}{2\pi}\Theta = -\frac{i}{2\pi}(n+1)\omega = c_1(L)$$

is the fundamental generator of $H^2(\mathbb{P}^n, \mathbb{Z})$ and L is the natural line bundle over $\mathbb{P}^n$ with the natural curvature Θ.

Euler Characteristic: $\chi(\mathbb{P}^n) = \sum_k (-1)^k b_k = n+1$

A question:

Recall the Chow's theorem: any submanifold of $\mathbb{P}^n$ can be realized as the zero locus of a finite number of homogeneous polynomial equations.

Consider first the case of one polynomial.

Question: Consider a compact complex Kähler manifold M which is a hypersurface in $\mathbb{P}^n$ defined by the zero locus of a degree d homogeneous polynomial P:

$$P(x_1, x_2, \ldots, x_{n+1}) = 0$$

Notation: $\qquad \mathbb{P}^n[d]$

Can we choose d such that $c_1(M) = 0$?

We have

$$T\,\mathbb{P}^n\big|_M = TM \oplus \mathcal{N}$$

and so

$$c(T\,\mathbb{P}^n\big|_M) = c(TM) \wedge c(\mathcal{N})$$

Equivalently, we can write

$$c(M) = c(T\,\mathbb{P}^n\big|_M)/c(\mathcal{N})$$

where the right hand side is understood as a formal (Taylor) series expansion.

Remark: Let E, E' and E'' be vector bundles over M. If

$$0 \to E' \to E \to E'' \to 0$$

is a short exact sequence then

$$c(E) = c(E') \wedge c(E'')$$

This includes the case $E = E' \oplus E''$. One can prove this using the splitting principle.

Compute:

Let $\mathcal{O}(1)$ be the hyperplane bundle over M (the sections of this line bundle are linear polynomials over M). Then $\mathcal{N} = \mathcal{O}_{\mathbb{P}^n}(d)$. We have

$$\begin{aligned} c(\mathcal{N}) &= 1 + d\,\omega \\ c(M) &= (1+\omega)^{n+1}/(1+d\,\omega) \\ &= 1 + (n+1-d)\omega + c_2(M) + \ldots \end{aligned}$$

Thus, $c_1(M) = 0$ and M is an $n-1$ dimensional Calabi–Yau manifold if $d = n+1$:

$$M \in \mathbb{P}^n[n+1]$$

- $n-1=1$ $\mathbb{P}^2[3]$ elliptic curve
- $n-1=2$ $\mathbb{P}^3[4]$ K3 surface
- $n-1=3$ $\mathbb{P}^4[5]$ quintic 3-fold

Useful Theorem:

Let M be a complex submanifold of X with $\dim M = n$ and $\dim X = n+r$ and with normal bundle $\mathcal{N}$ with $\mathrm{rank}(\mathcal{N}) = r$.

If α is any closed (n,n) form on X then

$$\int_M \alpha = \int_X \mu \wedge \alpha$$

where μ is a closed (r,r)-form on X whose restriction to M is the top Chern class of $\mathcal{N}$.

The closed form μ “lifts” the integration on M to X.

$K3$ surfaces

Theorem: any two $K3$ surfaces are diffeomorphic to each other.

Compute for $\mathbb{P}^3[4]$:

$$\begin{aligned} c(M) &= (1+\omega)^4/(1+4\,\omega) \\ &= (1+4\omega+6\omega^2)(1-4\omega+16\omega^2) \\ &= 1+(16-16+6)\omega^2 = 1+6\omega^2 \end{aligned}$$

So $c_2(M) = 6\omega^2$.

Calculate the Euler number:

$$\begin{aligned} \chi &= \int_M c_2(M) = \int_{\mathbb{P}^3} c_1(\mathcal{N}) \wedge c_2(M) \\ &= \int_{\mathbb{P}^3} 4\omega \wedge 6\omega^2 = 24 \int_{\mathbb{P}^3} \omega^3 = 24 \end{aligned}$$

Using the fact that $\chi = \sum_i b_i(-1)^i$, we can calculate b_2 and h_{11}:

$$\begin{aligned} 24 = 1+0+b_2+0+1 \quad &\Longrightarrow \quad b_2 = 22 \\ b_2 = 1+h_{11}+1 = 22 \quad &\Longrightarrow \quad h_{11} = 20 \end{aligned}$$

Hodge Diamond:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & h_{11} & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \qquad \text{with} \qquad h_{11} = 20$$

$\underline{M \in \mathbb{P}^4[5]}$

Compute:

$$
\begin{aligned}
c(M) &= (1+\omega)^5/(1+5\,\omega) \\
&= (1+5\omega+10\omega^2+10\omega^3)(1-5\omega+25\omega^2-125\omega^3) \\
&= 1+(5-5)\omega+(25-25+10)\omega^2 \\
&\quad +(-125+125-50+10)\omega^3 \\
&= 1+10\omega^2-40\omega^3
\end{aligned}
$$

So: $\quad c_2(M) = 10\omega^2 \quad$ and $\quad c_3 = -40\omega^3 \quad .$

Recall the Hodge Diamond for Calabi–Yau manifolds in 3 dimensions:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h_{11} & & 0 & \\
1 & & h_{21} & & h_{21} & & 1 \\
 & 0 & & h_{11} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

We have: $b_3 = 2(1+h_{21})$ and $b_2 = h_{11}$.

Euler number: $\chi = \sum_i b_i(-1)^i = 2(h_{11}-h_{21})$

For the quintic 3-fold: $\quad b_2 = h_{11} = 1.$

This is an exercise: prove it using the Lefschetz-Hyperplane Theorem.

Want to compute h_{21}.

Calculate the Euler number:

$$\begin{aligned}\chi &= \int_M c_3(M) = \int_{\mathbb{P}^4} c_1(\mathcal{N}) \wedge c_3(M) \\ &= \int_{\mathbb{P}^4} 5\omega \wedge (-40)\omega^3 = -200 \int_{\mathbb{P}^4} \omega^4 = -200\end{aligned}$$

We have: $b_3 = 2(1 + h_{21})$ and $b_2 = h_{11}$.

Euler number: $\chi = \sum_i b_i(-1)^i = 2(h_{11} - h_{21})$

Thus

$-200 = 2(1 - h_{12})$

$\quad h_{21} = 101 \quad \text{and} \quad b_3 = 2(101 + 1) = 2(102) = 204$

Exercise: Prove that the most general quintic polynomial in $\mathbb{P}^4$ has 101 arbitrary parameters (but there are 126 possible terms). This is not a coincidence....

Smooth example:

Consider the one-parameter family of quintic 3-folds, M_ψ, defined by the quintic polynomials

$$P = \sum_{i=1}^{5} x_i^5 - 5\psi\ x_1 x_2 x_3 x_4 x_5$$

This family is smooth for every value of ψ except when $\psi^5 = 1$.

<u>$\mathbb{P}^{3+n}$ with n polynomials P^α, $\alpha = 1, \ldots, n$.</u>

Notation: $\quad M \in \mathbb{P}^{3+n}[d_1, d_2, \ldots, d_n]$

where d_α is the degree of P^α.

Normal bundle:

$$\mathcal{N} = \oplus \mathcal{N}_\alpha$$

Then

$$c(\mathcal{N}) = \wedge_\alpha\, c(\mathcal{N}_\alpha)$$

The Chern Form is

$$\begin{aligned} c(M) &= (1+\omega)^{n+4} \Big/ \prod_\alpha (1 + d_\alpha\, \omega) \\ &= 1 + \left(n + 4 - \sum_{\alpha=1}^{n} d_\alpha \right) \omega + \ldots \end{aligned}$$

Therefore $M \in \mathbb{P}^{3+n}[d_1, d_2, \ldots, d_n]$ is a Calabi–Yau manifold if

$$\sum_{\alpha=1}^{n} d_\alpha = n + 4$$

Each $d_\alpha \geq 2$ (to be non-trivial). We have:

$$\begin{array}{ll} n = 1: & \mathbb{P}^4[5]_{-200} \\ n = 2: & \mathbb{P}^5[4,2]_{-176}\ , \quad \mathbb{P}^5[3,3]_{-144} \\ n = 3: & \mathbb{P}^6[3,2,2]_{-144} \\ n = 4: & \mathbb{P}^7[2,2,2,2]_{-128} \end{array}$$

All these have $h_{11} = 1$. As an exercise you can try to compute χ and h_{12} for at least one of these.

Let $X = \prod_{i=1}^{f} \mathbb{P}^{n_f}$ and consider n polynomials P^α each with multi-degrees $deg(\alpha)_j$, $\alpha = 1, \ldots, n$ and $j = 1, \ldots, f$.

($deg(\alpha)_j$ = degree of P^α with respect to $\mathbb{P}^{n_j}$.)

Notation: put the degrees in a matrix. For example

$$\begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \begin{bmatrix} 3 \\ 3 \end{bmatrix} , \qquad \begin{array}{c} \mathbb{P}^3 \\ \mathbb{P}^3 \end{array} \begin{bmatrix} 1\,3\,0 \\ 1\,0\,3 \end{bmatrix} , \qquad \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \begin{bmatrix} 1\,1 \\ 3\,0 \\ 0\,3 \end{bmatrix} .$$

Exercise: Show that the Calabi–Yau condition is

$$\sum_\alpha deg(\alpha)_i = n_i + 1, \quad i = 1, \ldots, f$$

Lecture 7

The Eguchi-Hanson Geometry

The Eguchi-Hanson geometry, EH_n is a non-compact Ricci-flat geometry which is useful for repairing the singularities of orbifolds. It is of additional interest because it can be constructed explicitly.

(Our discussion follows that of A.Strominger and E.Witten, Commun. Math. Phys. **101**, (1985) 341.)

Let us seek a Ricci-flat Kähler metric of the form

$$ds^2 = [A(\sigma)\delta_{\mu\bar{\nu}} + B(\sigma)z_\mu z_{\bar{\nu}}]\, dz^\mu dz^{\bar{\nu}}$$

where we adopt the convention that

$$z_\mu = z^{\bar{\mu}} \quad , \quad z_{\bar{\mu}} = z^\mu$$

and

$$\sigma = z_\mu z^\mu = z_{\bar{\mu}} z^{\bar{\mu}}.$$

First we impose the Kähler condition. We have

$$\partial_\rho g_{\mu\bar{\nu}} = A' z_\rho \delta_{\mu\bar{\nu}} + B z_\mu \delta_{\rho\bar{\nu}} + B' z_\rho z_\mu z_{\bar{\nu}}.$$

For the metric to be Kähler this must be symmetric in ρ and μ. This requires

$$B = A'.$$

The condition that the metric be Ricci-flat is that

$$\partial_\mu \partial_{\bar{\nu}} \log g^{\frac{1}{2}} = 0.$$

Note that the determinant of the metric is a function of σ only and for an arbitrary such function we have

$$\partial_\mu \partial_{\bar{\nu}} F(\sigma) = F''(\sigma) z_\mu z_{\bar{\nu}} + F'(\sigma) \delta_{\mu\bar{\nu}}.$$

The only solution to the equation

$$\partial_\mu \partial_{\bar{\nu}} F(\sigma) = 0$$

is therefore

$$F = const.$$

Since we may rescale the metric by a constant the Ricci-flatness condition is equivalent to the condition

$$\det(g_{\mu\bar{\nu}}) = 1.$$

Now

$$\begin{aligned}\det(g_{\mu\bar{\nu}}) = \frac{1}{n!}\, \epsilon^{\mu_1 \cdots \mu_n}\, \epsilon^{\bar{\nu}_1 \cdots \bar{\nu}_n}\, & (A\delta_{\mu_1\bar{\nu}_1} + A' z_{\mu_1} z_{\bar{\nu}_1}) \\ & \cdots (A\delta_{\mu_n\bar{\nu}_n} + A' z_{\mu_n} z_{\bar{\nu}_n}) \\ = & A^{n-1}(A + \sigma A') = 1\end{aligned}$$

Multiplying by $n\sigma^{n-1}$ we have

$$\frac{d}{d\sigma}\left((\sigma A)^n\right) = n\sigma^{n-1}.$$

Thus

$$A(\sigma) = \sigma^{-1}(c+\sigma^n)^{\frac{1}{n}} \ , \ \ B(\sigma) = -c\sigma^{-2}(c+\sigma^n)^{\frac{1}{n}-1}$$

and the metric is

$$g_{\mu\bar{\nu}} = \sigma^{-1}(c+\sigma^n)^{\frac{1}{n}}\left\{\delta_{\mu\bar{\nu}} - \frac{cz_\mu z_{\bar{\nu}}}{\sigma(c+\sigma^n)}\right\}.$$

We shall take the constant c to be positive since otherwise the manifold has a singularity where $\sigma^n = -c$. As $\sigma \to \infty$ we find $g_{\mu\bar{\nu}} \to \delta_{\mu\bar{\nu}}$ so the metric is asymptotically flat. We can set $c = 1$ by scaling the coordinates $z^\mu \to c^{1/2n} z^\mu$.

The metric becomes

$$g_{\mu\bar{\nu}} = \sigma^{-1}(1+\sigma^n)^{\frac{1}{n}}\left\{\delta_{\mu\bar{\nu}} - \frac{z_\mu z_{\bar{\nu}}}{\sigma(1+\sigma^n)}\right\}.$$

We need to examine the singularity of the metric at $\sigma = 0$. We will show that it is merely a coordinate singularity which may be removed by a proper choice of coordinates.

First note that $\sigma = 0$ is the limit in which all the z^μ are zero. It is preferable to work with coordinates such that $\sigma = 0$ corresponds to the vanishing of just one of the coordinates.

Set

$$y = z^n \text{ and } y^i = \frac{z^i}{z^n}\,, \quad i = 1, \ldots, n-1.$$

and to save writing define: $\rho = 1 + y^i y_i.$

The limit we shall take is $y \to 0$ with y^i constant. Substitution into the metric and expansion of the metric components to leading order in powers of $|y|$ yields

$$ds^2 \sim \frac{1}{\rho}\left(dy_i dy^i - \frac{1}{\rho} y_i dy^i y_{\bar{\jmath}} dy^{\bar{\jmath}}\right) + \rho^n |y|^{2n-2} dy d\bar{y}.$$

In this form the metric is still singular at $y = 0$ since the coefficients of all the terms containing dy and $d\bar{y}$ vanish there, so at $y = 0$ the metric has a vanishing row and column and hence has no inverse.

To cure this we make a further change of variables. Set

$$w = \frac{1}{n} y^n \text{ so that } y^{n-1} dy = dw.$$

In terms of (w, y^i) coordinates the metric becomes

$$ds^2 \sim \frac{1}{\rho}\left(dy_i dy^i - \frac{1}{\rho} y_i dy^i y_{\bar{\jmath}} dy^{\bar{\jmath}}\right) + \rho^n dw d\bar{w}$$

In these coordinates the metric is regular at $w = 0$.

Note however two important facts.

Firstly $w = 0$ corresponds to an $(n-1)$-dimensional submanifold and the metric of this submanifold is given by setting $dw = 0, d\bar{w} = 0$. We recognize the resulting metric as the Fubini-Study metric.
Thus $\sigma = 0$ corresponds in fact to a $\mathbb{P}^{n-1}$.

Secondly we see from the relations

$$w = \frac{1}{n}(z^n)^n \quad , \quad y^i = \frac{z^i}{z^n}$$

that the n points $\alpha^k z^\mu, \ k = 0, \ldots, n-1$, where $\alpha = e^{2\pi i/n}$ is an n'th root of unity, determine the same values of (w, y^i).

Since it is in the (w, y^i) coordinates that the manifold is nonsingular and these must be single valued we must identify the points

$$z^\mu \approx \alpha^k z^\mu \quad , \quad k = 1, \ldots, n-1.$$

Because of this $\mathbb{Z}_n$ identification EH_n is not asymptotically like $\mathbb{R}_{2n}$. The statement that the metric is asymptotically flat is only true locally. The surface corresponding to $\sigma = R^2$, with R a large constant, is not an S^{2n-1} but rather $S^{2n-1}/\mathbb{Z}_n$.

Orbifolds

An n-dimensional complex *orbifold* is a singular complex manifold with singularities locally isomorphic to quotient singularities $\mathbb{C}^n/G$, where G is a finite subgroup of $GL(n, \mathbb{C})$.

The condition that G is finite is because we would consider *crepant resolutions* of orbifolds, that is, resolutions that are Calabi–Yau manifolds.

Example: Weighted Projective Spaces

The n–dimensional *weighted projective space*, denoted by $\mathbb{P}^n_{w_1,w_2,\ldots,w_{n+1}}$, is the space

$$\left(\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}\right) \big/ \mathbb{C}^*$$

where $\mathbb{C}^*$ acts on $\left(\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}\right)$ by the identifications

$$(z^1, z^2, \ldots, z^{n+1}) \sim (\lambda^{w_1} z^1, \lambda^{w_2} z^2, \ldots, \lambda^{w_{n+1}} z^{n+1})$$

for $\lambda \in \mathbb{C}^*$, and $w_1, w_2, \ldots, w_{n+1}$, are positive integers.

Clearly,

$$\mathbb{P}^n = \mathbb{P}^n_{1,1,\ldots,1} \ .$$

Consider for example the point $p = (0, 0, \ldots, 1)$.

If $\lambda \in \mathbb{C}^*$, then

$$(0, 0, \ldots, 1) \sim (0, 0, \ldots, \lambda^{w_{n+1}}) \ .$$

Then, if $w_{n+1} > 1$, p is left fixed by the finite subgroup G of $\mathbb{C}^*$

$$G = \{\lambda \in \mathbb{C}^* : \lambda^{w_{n+1}} = 1\} \ .$$

G is isomorphic to $\mathbb{Z}_{w_{n+1}}$.

It can be shown that the open set

$$\mathcal{U}_{n+1} = \{(z^1, z^2, \ldots, z^{n+1}) \in \mathbb{P}^n_{w_1,w_2,\ldots,w_{n+1}} : z^{n+1} \neq 0\}$$

is isomorphic to $\mathbb{C}^n/G$, where G acts on $\mathbb{C}^n$ by

$$(z^1, z^2, \ldots, z^n) \to (\lambda^{w_1} z^1, \lambda^{w_2} z^2, \ldots, \lambda^{w_n} z^n) \ , \lambda \in G$$

$\square$

A complex orbifold M is Kähler with Kähler metric g, if g is Kähler every where in $M \setminus \{\text{orbifold points}\}$. Moreover, in an open set of M which is locally $\mathbb{C}^n/G$ the metric g can be identified with a G–invariant Kähler metric near $\mathbf{0} \in \mathbb{C}^n$.

For example, $M = \mathbb{P}^n_{w_1,w_2,\ldots,w_{n+1}}$ is Kähler.
A Kähler metric on M would be a generalization of the Fubini-Study metric on $\mathbb{P}^n$.

The Calabi conjecture holds: If M is a compact Kähler orbifold with $c_1(M) = 0$ and Kähler class $[\omega]$, then there is a unique Ricci–flat Kähler metric with Kähler class in $[\omega]$.

The First Chern Class is well defined.

A Calabi–Yau orbifold M is a complex Kähler orbifold with $Hol \subseteq SU(n)$. The Holonomy group is defined on $M \setminus \{\text{orbifold points}\}$.

Similar characterizations for Calabi–Yau orbifolds as in Chapter 3.

Example:

The $\mathbb{Z}_3$ Orbifold and its Crepant resolution.

Consider the complex torus $\mathbb{T}$ obtained by making in $\mathbb{C}^3$ the identifications (*ie* defining a lattice Λ in $\mathbb{C}^3$)

$$z^k \sim z^k + 1 \sim z^k + e^{\pi i/3} , \quad k = 1, 2, 3 \ .$$

That is $\mathbb{T} = \mathbb{C}^3/\Lambda$.

On $\mathbb{T}$ we impose the additional $\mathbb{Z}_3$ identification

$$z^k \sim e^{2\pi i/3} z^k.$$

There are 27 fixed points.

In each z^k-plane there are three fixed points

$$\zeta_r = \frac{r}{3}\left(\frac{3}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{r}{\sqrt{3}} e^{i\pi/6} \quad , \quad r = 0, 1, 2$$

corresponding to points for which $e^{2i\pi/3}\zeta_r = \zeta_r + r$.

It is a singular manifold with δ-function curvature concentrated at the fixed points.

To see this consider the curve C of radius ϵ of the figure. It is a closed curve in view of the identifications and has length $2\pi\epsilon/3$. This shows that there is curvature concentrated at the fixed point. We see also from the figure that after parallel transport around C the vector v becomes v' which is rotated by $2\pi/3$ with respect to v. The holonomy group of the orbifold is $\mathbb{Z}_3$.

The orbifold can be turned into a smooth manifold by the following process.

Excise a small ball around each of the 27 fixed points. The boundary of each ball is $S^5/\mathbb{Z}_3$ which is the same as the hypersurface $\sigma = R^2$ in EH_3. The interior of this hypersurface in EH_3 may be glued in place of the excised balls in the orbifold. Since, the Eguchi-Hanson metric has a size parameter that measures the region in which the curvature is concentrated, the join can be made arbitrarily smooth and it can be shown that a Calabi-Yau manifold results.

Exercise:

The way in which this manifold was constructed permits us to calculate its Euler number in a simple manner.

It may be shown that the Euler number of EH_n is n.

The Euler number of a torus is zero and the Euler number of a ball is unity.

In order to construct the manifold we took a torus, excised 27 balls, took a quotient by $\mathbb{Z}_3$ and glued in 27 copies of EH_3. Hence the Euler number is

$$\chi = \frac{(0-27)}{3} + 27 \times 3 = 72.$$

<u>The Hodge numbers:</u>

The group $H^{(1,0)}$ is trivial even though it is not for the torus $\mathbb{T}$.

For $\mathbb{T}$ the group is spanned by the basis

$$dz^k \ , \ k = 1, 2, 3.$$

However these forms are not invariant under the group action $z^k \to \alpha z^k$ with $\alpha = e^{2\pi i/3}$.

$H^{(1,1)}$:

As a basis we take the 27 forms

$$\omega_A \ , \quad A = 1, \ldots, 27,$$

that correspond to the $\mathbb{P}^2$'s located at the centers of each EH_3, and the nine forms

$$\nu_{i\bar{\jmath}} \sim dz^i \wedge dz^{\bar{\jmath}}.$$

By this notation is meant that $\nu_{i\bar{\jmath}}$ is asymptotically equal to $dz^i \wedge dz^{\bar{\jmath}}$ far from the regions where the curvature is concentrated.

The $\nu_{i\bar{\jmath}}$ is dual to the four-surface to which $dz^i \wedge dz^{\bar{\jmath}}$ is dual on $\mathbb{T}$. The dimension of $H^{(1,1)}$ is

$$h_{1,1} = 27 + 9 = 36.$$

$H^{(2,1)}$:

This group is trivial.

Quantities such as $dz^i \wedge dz^j \wedge dz^{\bar{k}}$ and $dz^i \wedge \omega_A$ are not invariant under the $\mathbb{Z}_3$ action. Thus $h_{2,1} = 0$.

As a check we have

$$\chi = 2(h_{1,1} - h_{2,1}) = 72$$

in agreement with our previous assertion.

$\square$

We have outlined the construction of the simplest orbifold and its Crepant resolution to construct a Calabi–Yau manifold. Many other constructions are possible and these have been the object of considerable study owing to the fact that strings can propagate consistently on orbifold backgrounds.

It is an interesting problem to what extent it is possible to blow up the singularities of orbifolds to obtain smooth Calabi-Yau manifolds.

Complete Intersection Calabi–Yau Manifolds

We have already discussed submanifolds X that are *complete intersections* of a number of N polynomials P^{α}, $\alpha = 1, \ldots, N$ in a product of projective spaces of total dimension $N + 3$.

The construction can be generalized to weighted projective spaces and toric varieties.

By a complete intersection is meant that the N–form

$$\Psi = dP^1 \wedge dP^2 \wedge \cdots \wedge dP^n$$

does not vanish on X.

This condition guards against the polynomials describing a surface with cusps or nodes. Ψ describes the N directions normal to X. If X is smooth then Ψ cannot vanish. If Ψ were to vanish at a point $p \in X$ then this would imply that X did not have well defined normal directions at p and so could not be smooth.

The assumption that Ψ does not vanish is quite restrictive.

Of course one expects that giving N equations in an $N+3$ dimensional space will describe a 3 dimensional manifold locally. The restrictive assumption is that they should in fact do so globally.

Consider the case of a hypersurface X in $M = \mathbb{P}^n_{w_1,w_2,\ldots,w_{n+1}}$, defined by one transverse polynomial P of degree d.

M is an orbifold.

The transversality condition on P implies that the only singularities in X are orbifold points of M, so X is an orbifold.

Exercise:

$$c_1(X) = 0 \qquad \Longleftrightarrow \qquad d = \sum_{j=1}^{n+1} w_j$$

□

When $n = 4$, there is a theorem (Roan) that guarantees that in this case there is a Crepant resolution of X which is a 3 dimensional Calabi–Yau manifold.

The first experimental evidence for mirror symmetry for Calabi–Yau manifolds, came from a plot by Candelas, Lynker and Schrimmrigk in the late 80's of all such 3 dimensional Calabi–Yau manifolds as transverse hypersufaces in $M = \mathbb{P}^n_{w_1,w_2,\ldots,w_{n+1}}$

They found 2339 distinct pairs of Hodge numbers (h_{11}, h_{12}).

The plot had a symmetry (almost): for every Calabi–Yau manifold X with Hodge numbers (h_{11}, h_{12}), there was another one Y with Hodge numbers (h_{12}, h_{11}).

Batyrev generalized this construction to hypersurfaces in toric varieties (one polynomial only). His construction is mirror symmetric and contains those found by Candelas et al.

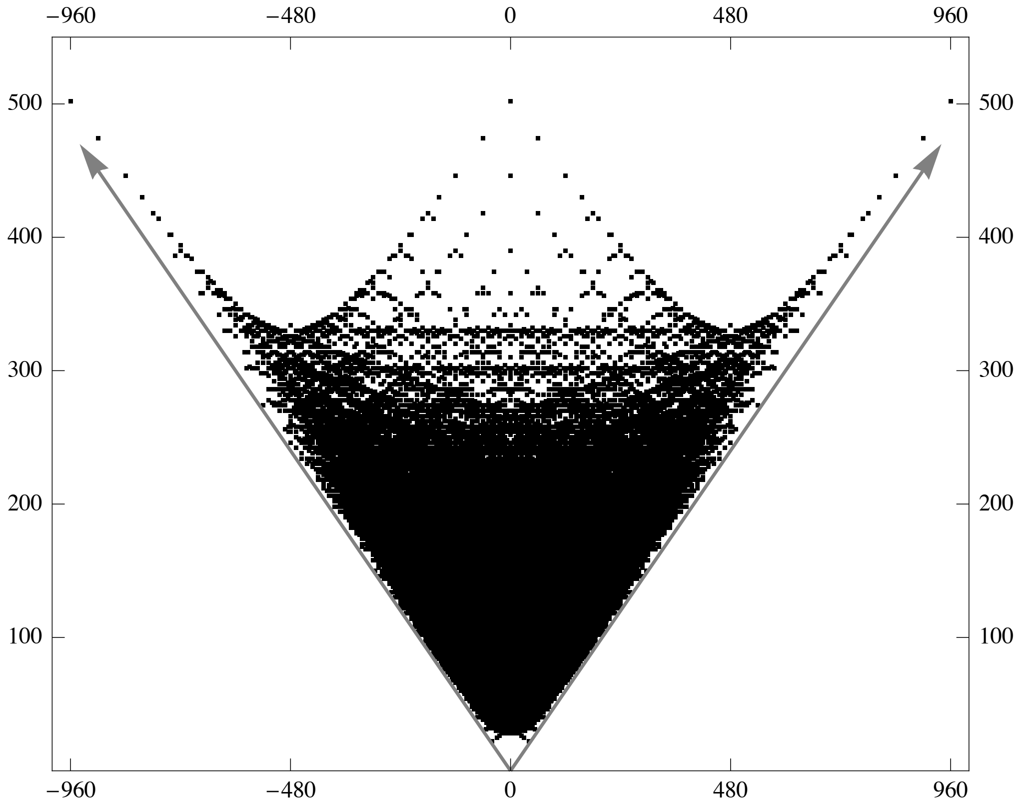


Figure 1: A plot of the Hodge numbers of the Kreuzer–Skarke list. $\chi = 2(h_{11} - h_{21})$ is plotted horizontally and $h_{11} + h_{21}$ is plotted vertically. The oblique axes bound the region $h_{11} \geq 0$, $h_{21} \geq 0$.

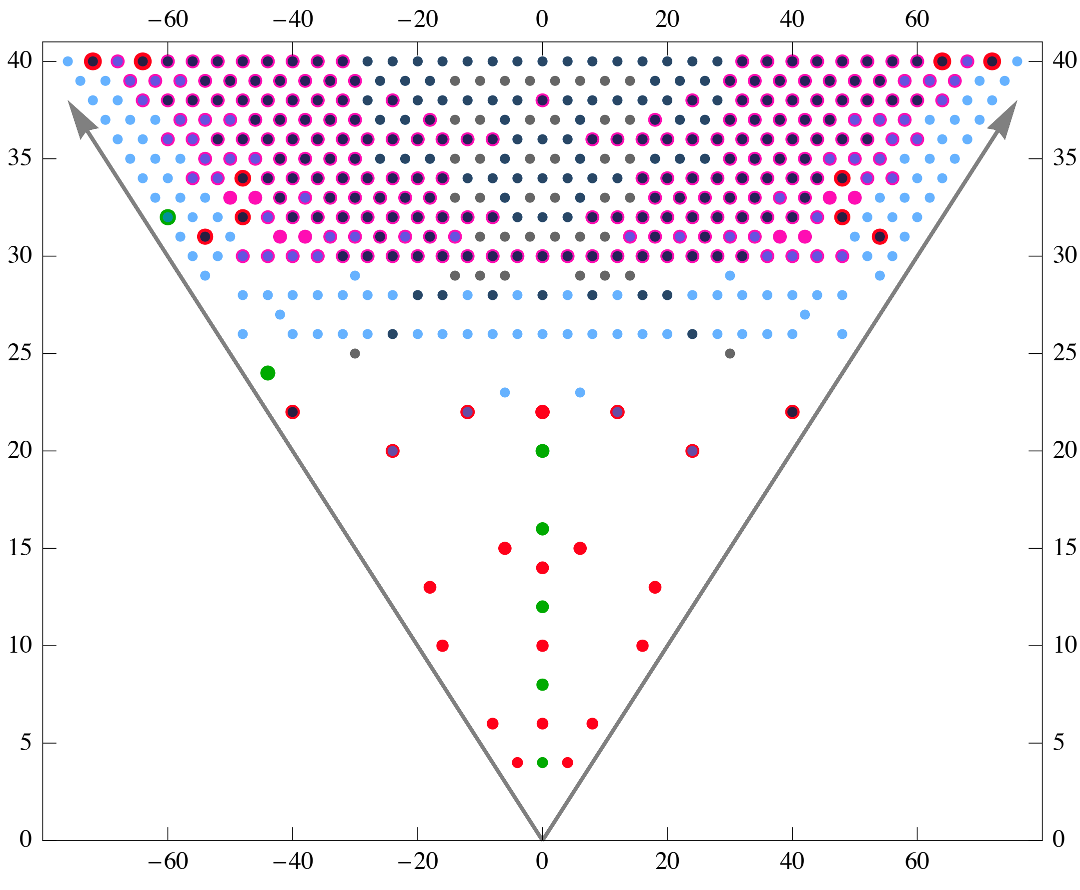


Figure 2: The tip of the landscape. $\chi = 2(h_{11} - h_{21})$ is plotted horizontally, $h_{11} + h_{21}$ is plotted vertically and the oblique axes bound the region $h_{11} \geq 0$, $h_{21} \geq 0$. The points are coloured according to provenance and have partial transparency in order to show overlays.

- The Kreuzer–Skarke list.
- The CICY's and their mirrors.
- The toric CICY's, and their mirrors.
- Quotients by freely acting groups and their mirrors.
- The Gross–Popescu, Borisov-Hua and Tonoli manifolds.

Lecture 8

Chapter 5 Moduli Spaces

Let X be a 3-dimensional Calabi–Yau manifold with $h_{10} = h_{20} = 0$.

Let J be the complex structure of X and ω its Kähler form. Both J and ω can be deformed.

Recall Yau's theorem:

Let X be a complex compact Kähler manifold with Kähler class $[\omega]$ and with $c_1(X) = 0$. There is a unique Ricci-flat metric g with Kähler form $\omega' \in [\omega]$.

We will consider the parameter space of Ricci–flat Kähler metrics.

Let g and $g + \delta g$ be two Ricci–flat metrics

$$R_{mn}(g) = 0 \ , \qquad R_{mn}(g + \delta g) = 0 \ .$$

Then δg_{mn} satisfies

$$g^{pq} \nabla_p \nabla_q \delta g_{mn} + 2 R_m{}^p{}_n{}^q \, \delta g_{pq} = 0$$

This equation is called the Lichnerowicz equation.

But X is Kähler:

- Christoffel symbols are "pure": $\Gamma_{\mu\nu}{}^{\kappa}$ and $\Gamma_{\bar{\mu}\bar{\nu}}{}^{\bar{\kappa}}$ are the only non-zero components of Γ
- The only non-zero components of the Riemann curvature are $\quad R_{\mu\bar{\nu}}{}^{\rho}{}_{\sigma}, \ R_{\mu\bar{\nu}}{}^{\bar{\rho}}{}_{\bar{\sigma}}, \ R_{\bar{\mu}\nu}{}^{\rho}{}_{\sigma}, \ R_{\bar{\mu}\nu}{}^{\bar{\rho}}{}_{\bar{\sigma}}$

Then $\delta g_{\mu\bar{\nu}}$ and $\delta g_{\mu\nu}$ satisfy the Lichnerowicz equation **separately**.

Variations of type $\delta g_{\mu\bar{\nu}}$:

We have $\omega \to \omega + \delta\omega$ with

$$\delta\omega = i \delta g_{\mu\bar{\nu}} dx^{\mu} \wedge dx^{\bar{\nu}}$$

The mixed variation $\delta g_{\mu\bar{\nu}}$ is harmonic if and only if it satisfies the Lichnerowicz equation.

That is: the "mixed" deformations of Ricci–flat metrics $\delta g_{\mu\bar{\nu}}$ are harmonic $(1,1)$ forms.

They are in 1-1 correspondence with variations of the Kähler class $[\omega]$, which in turn are also in 1-1 correspondence with the classes in $H^{(1,1)}(X)$.

Thus: the (real) dimension of the space of Ricci–flat metrics corresponding to deformations of the Kähler class is h_{11}.

Variations of type $\delta g_{\mu\nu}$ and $\delta g_{\bar{\mu}\bar{\nu}}$:

To each $\delta g_{\bar{\mu}\bar{\nu}}$, we can associate a complex $(2,1)$–form

$$\Omega_{\mu\nu\rho}\delta g_{\bar{\lambda}\bar{\sigma}} g^{\rho\bar{\lambda}} dx^{\mu} \wedge dx^{\nu} \wedge dx^{\bar{\sigma}}$$

This form is harmonic if and only if $\delta g_{\bar{\mu}\bar{\nu}}$ satisfies the Lichnerowicz equation.

In fact

$$H_{\bar{\partial}}^{(0,1)}(X,T) \simeq H^{(2,1)}(X)$$

because $\delta g_{\bar{\lambda}\bar{\sigma}} g^{\rho\bar{\lambda}} dx^{\bar{\sigma}}$ is in $H_{\bar{\partial}}^{(0,1)}(X,T)$ if $\delta g_{\bar{\mu}\bar{\nu}}$ satisfies the Lichnerowicz equation.

Thus: the space of Ricci-flat metrics corresponding to "pure" deformations of the metric are in 1-1 correspondence with classes in $H^{(2,1)}(X)$ and so there are h_{21} complex parameters.

Claim: The "pure" variations of the metric, which give rise to h_{21} complex parameters, correspond to variations of the complex structure.

Proof:

To see this consider

$$g_{mn} + \delta g_{mn}$$

to be a Kähler metric close to the original g_{mn}.

There must exist a coordinate system in which the pure parts of the metric vanish.

Under $x^m \to x^m + f^m(x)$ the metric undergoes the change

$$\delta g_{mn} \to \delta g_{mn} - (\partial_m f^r) g_{rn} - (\partial_n f^r) g_{mr} \; .$$

If f^μ is holomorphic then $\delta g_{\mu\nu}$ is invariant

$$\delta g_{\mu\nu} \to \delta g_{\mu\nu} - (\partial_\mu f^{\bar\rho}) g_{\bar\rho\nu} - (\partial_\nu f^{\bar\rho}) g_{\mu\bar\rho}$$

Thus, the pure part of the variation can be removed by a coordinate transformation BUT it cannot be removed by a holomorphic coordinate transformation.

Pure parts of the metric variation correspond precisely to changes in the complex structure.

$\square$

Therefore:

Pure parts of the metric variation are in correspondence with deformations of the complex structure and are in 1-1 correspondence with elements of

$$H^{(2,1)}(X) \oplus H^{(1,2)}(X)$$

Moreover, deformations of the complex structure are not obstructed (Tian&Todorov).

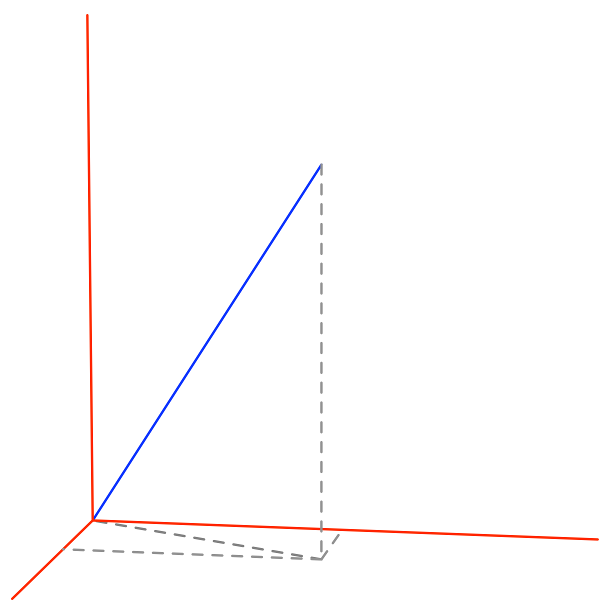
Section 5.1

The Complex Structure Moduli Space $\mathcal{M}_{\mathbb{C}S}$

Recall that an important property of a Calabi–Yau manifold is that there exists a unique holomorphic $(3,0)$ form

$$\Omega = \frac{1}{3!}\Omega_{\mu\nu\rho}dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho} \ .$$

We will study the variation of the Complex Structure by considering how Ω moves inside $H^3(X)$ following the work of Bryant&Griffiths.



Let (A^a, B_b), with $a, b = 0, 1, \ldots, h_{21}$, be a canonical homology symplectic basis of $H_3(X)$

$$A^a \cap A^b = B_a \cap B_b = 0 \qquad A^a \cap B_b = \delta^a_b$$

Let (α_a, β^b), $a, b = 0, 1, \ldots, h_{21}$, form a symplectic cohomology basis of $H^3(X)$ which is Poincaré dual to (A, B)

$$\int_{A^b} \alpha_a = \int_X \alpha_a \wedge \beta^b = \delta_a{}^b$$

$$\int_{B_a} \beta^a = \int_X \beta^b \wedge \alpha_a = -\delta_a{}^b$$

$$\int_X \alpha_a \wedge \alpha_b = \int_X \beta^a \wedge \beta^b = 0$$

We can write Ω in terms of the cohomology basis (α, β) as

$$\Omega = z^a \alpha_a - \mathcal{G}_b \, \beta^b \ .$$

The coefficients $(z^a, \mathcal{G}_b)$ are the *periods* of Ω

$$z^a = \int_{A^a} \Omega \qquad \mathcal{G}_a = \int_{B_a} \Omega \ .$$

Bryant& Griffiths: locally in moduli space, the complex structure of X is entirely determined by the set of periods $\{z^a\}$, and $\mathcal{G}_b = \mathcal{G}_b(z)$.

The $\{z^a\}$ define "special" projective coordinates on $\mathcal{M}_{\mathbb{C}S}$:

- $\{z^a\}$ cannot all vanish (we will see later that if they all vanish then $\int_X \Omega \wedge \bar{\Omega} = 0$)
- Under $\Omega \to \lambda\Omega$ for $\lambda \in \mathbb{C}^*$: $z^a \to \lambda z^a$

Therefore $(z^0, z^1, \ldots, z^{h_{12}}) \in \mathbb{P}_{h_{12}}$

Note that under $\Omega \to \lambda\Omega$, we also have that

$$\mathcal{G}_a \to \lambda \mathcal{G}_a$$

so $\mathcal{G}_a(z)$ is homogeneous of degree 1.

Kodaira:

Let t^α , $\alpha = 1, \ldots, h_{12}$, be affine coordinates. Then

$$\frac{\partial \Omega}{\partial t^\alpha} \in H^{(3,0)} \oplus H^{(2,1)}$$

and we can write

$$\frac{\partial \Omega}{\partial t^\alpha} = K_\alpha \Omega + \varphi_\alpha$$

Similarly:

$$\frac{\partial^2}{\partial t^\alpha \partial t^\beta}\Omega \in H^{(3,0)} \oplus H^{(2,1)} \oplus H^{(1,2)}$$

$$\frac{\partial^3}{\partial t^\alpha \partial t^\beta \partial t^\gamma}\Omega \in H^{(3,0)} \oplus H^{(2,1)} \oplus H^{(1,2)} \oplus H^{(0,3)}$$

Griffiths transversality condition:

It is clear that

$$\int_X \Omega \wedge \frac{\partial \Omega}{\partial z^a} = 0$$

This implies that $\mathcal{G}_a(z)$ is the gradient of a holomorphic *prepotential*, $\mathcal{G}(z)$, which is homogeneous of degree 2

$$\mathcal{G}_a = \mathcal{G}_a(z) = \frac{\partial}{\partial z^a}\mathcal{G}(z) \ .$$

Note:

There is no new information from

$$\int_X \Omega \wedge \frac{\partial^2 \Omega}{\partial z^a \partial z^b} = 0 \qquad \text{and} \qquad \int_X \Omega \wedge \frac{\partial^3 \Omega}{\partial z^a \partial z^b \partial z^c} = 0$$

The choice of a symplectic basis (α_a, β^b) of $H^3(X)$ is unique up to a symplectic transformation

$$\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(b_3) = Sp(2(2h_{21}+1))$$

Recall that $Sp(2n)$ is the group of $2n \times 2n$ matrices which preserve

$$\begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$$

So $Sp(b_3)$ preserves the symplectic inner product.

Under these transformations the periods transform as

$$\begin{pmatrix} \tilde{\partial}\tilde{\mathcal{G}} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \partial\mathcal{G} \\ z \end{pmatrix}$$

Note that the $\tilde{z}^a(z)$ are non-linear in the z's.

The period matrix

$$\mathbb{G}_{ab} = \frac{\partial^2}{\partial z^a \partial z^b}\,\mathcal{G}$$

transforms as

$$\mathbb{G} \to \tilde{\mathbb{G}} = (C\mathbb{G}+D)^{-1}(A\mathbb{G}+B)$$

(Think of the analogy with period matrices of Riemann surfaces!)

The transformation law for the $\{z^a\}$ is

$$z \to \tilde{z} = (C\mathbb{G} + D)z$$

that is, $\{z^a\}$ form a vector valued modular form of $Sp(b_3)$.

Metric on $\mathcal{M}_{\mathbb{C}S}$

Define:

$$e^{-K} = i \int_X \Omega \wedge \bar{\Omega}$$

Using $\quad \Omega = z^a \alpha_a - \mathcal{G}_a \beta^a \quad$ we can write e^{-K} in terms of the periods

$$e^{-K} = i(\mathcal{G}_a \bar{z}^a - z^a \bar{\mathcal{G}}_a)$$

(By the way, the reason why not all z's can vanish is that the volume form would vanish if they did.)

There is a natural Kähler metric on $\mathcal{M}_{\mathbb{C}S}$ with Kähler potential K

$$\begin{aligned} g_{\alpha\bar{\beta}} &= \partial_\alpha \partial_{\bar{\beta}} K \\ &= -ie^K \int_X \varphi_\alpha \wedge \bar{\varphi}_{\bar{\beta}} \end{aligned}$$

where $\varphi_\alpha \in H^{(2,1)}(X)$ and $\frac{\partial\Omega}{\partial t^\alpha} = K_\alpha \Omega + \varphi_\alpha$.

Tian: This is really the only metric one can write which is invariant under $\Omega \to f(z)\Omega$.

Ω takes values in a line bundle L: for each X_z, Ω is defined up to multiplication by a complex number $f(z)$.

Under $\Omega \to f(z)\Omega$ the Kähler potential changes as

$$e^{-K} \to |f|^2 e^{-K}$$

Then $K \to K - \log f - \log \bar{f}$, which leaves the metric invariant.

As we will see later

$$c_1(L) = \omega_{\mathcal{M}}$$

where $\omega_{\mathcal{M}}$ is the Kähler form on $\mathcal{M}_{\mathbb{C}S}$.

Another way to prove that $\mathcal{M}_{\mathbb{C}S}$ is Kähler is to construct the metric on the space of metrics.

In the physics literature this was done by Bryce de-Witt (1967) in connection with the quantization of the gravitational field.

The natural line element on the space of metrics is

$$ds^2 = \frac{1}{4V} \int_X ||\delta g_{\mu\nu}||^2 + ||i\delta g_{\mu\bar{\nu}} + \delta B_{\mu\bar{\nu}}||^2$$

The first term corresponds to the variation of the complex structure and the second to the variations of $B + i\omega$ (what is B?).

One can show that the part for the complex structure coincides with the Weil–Petersson metric, which in turn can be shown to be Kähler.

Note: this metric is block–diagonal.

Lecture 9 Continue with $\mathcal{M}_{\mathbb{C}S}$

We are discussing the variations of the complex structure a Calabi–Yau manifold X in terms of Ω.

Study how Ω moves inside $H^3(X)$.

We have chosen a fixed symplectic real basis (α, β) of $H^3(X)$ and locally projective coordinates $\{z^a\}$ on $\mathcal{M}_{\mathbb{C}S}$

$$z^a = \int_{A^a} \Omega \in \mathbb{P}_{h_{21}} , \qquad \mathcal{G}_a = \int_{B_a} \Omega = \mathcal{G},_a(z)$$

which are the *periods* of Ω. These periods are homogeneous of degree 1 under $\Omega \to f(z)\Omega$.

$\mathcal{G}$, *the prepotential*, is homogenous of degree 2 in $\{z^a\}$.

In the symplectic basis (α, β) of $H^3(X)$, Ω can be written as

$$\Omega = z^a \alpha_a - \mathcal{G}(z),_b \beta^b$$

In summary: $\mathcal{M}_{\mathbb{C}S}$ is Kähler with holomorphic prepotential. We say that $\mathcal{M}_{\mathbb{C}S}$ is *Special Kähler*. (Note that this is not the most general definition of a *Special Kähler* space.)

The Kähler metric on $\mathcal{M}_{\mathbb{C}S}$ is given by

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K = -ie^K \int_X \varphi_\alpha \wedge \bar{\varphi}_{\bar{\beta}}$$

where

$$e^{-K} = i \int_X \Omega \wedge \bar{\Omega} = i(\mathcal{G}_a \bar{z}^a - z^a \bar{\mathcal{G}}_a) = -2i\Im(z^a \bar{\mathcal{G}}_a)$$

Recall:
$$\frac{\partial \Omega}{\partial t^\alpha} = K_\alpha \Omega + \varphi_\alpha$$

What is K_α? The answer is $K_\alpha = -\partial_\alpha K$

$$\begin{aligned}\int_X \frac{\partial \Omega}{\partial t^\alpha} \wedge \bar{\Omega} &= K_\alpha \int_X \Omega \wedge \bar{\Omega} + \int_X \varphi_\alpha \wedge \bar{\Omega} \\ &= K_\alpha \int_X \Omega \wedge \bar{\Omega}\end{aligned}$$

Using $\int_X \Omega \wedge \bar{\Omega} = -ie^{-K}$ we have
$$K_\alpha e^{-K} = \partial_\alpha e^{-K} = -e^{-K} \partial_\alpha K$$

Then
$$K_\alpha = -\partial_\alpha K$$

Connection on L

(Metric compatible hermitian connection)

Recall: Ω takes values in a line bundle L. For each X_z, Ω is defined up to multiplication by a complex number $f(z)$.

Write Kodaira's equation as
$$\mathcal{D}_\alpha \Omega = \left(\frac{\partial}{\partial t^\alpha} + K,_\alpha \right) \Omega = \varphi_\alpha \in H^{(2,1)}$$

We will regard $\mathcal{D}$ as a connection on L.

Under $\Omega \to f(z)\Omega$: $\quad \mathcal{D}_\alpha \Omega \to f(z) \mathcal{D}_\alpha \Omega$

So $\varphi_\alpha \to f(z)\varphi_\alpha$ and φ_α takes values in L.

Note that we also have

$$\mathcal{D}_{\bar{\beta}}\bar{\Omega} = \varphi_{\bar{\beta}} \in H^{(1,2)}$$

where $\varphi_{\bar{\beta}} = \overline{\varphi_{\beta}}$ and $\varphi_{\bar{\beta}}$ takes values in $\bar{L}$.

Want to find $\mathcal{D}_\alpha \mathcal{D}_\beta \Omega$ and $\mathcal{D}_\alpha \mathcal{D}_\beta \mathcal{D}_\gamma \Omega$.

Let Ψ take values in $L^a \otimes \bar{L}^b$, that is,

$$\text{under} \quad \Omega \to f(z)\Omega : \qquad\qquad \Psi \to f^a \bar{f}^b \Psi$$

(For example, e^{-K} takes values in $L \otimes \bar{L}$, and both $\varphi_{\bar{\beta}}$ and $\bar{\Omega}$ take values in $\bar{L}$)

We extended the definition of the covariant derivative $\mathcal{D}$ to Ψ so that the Leibnitz rule is respected

$$\mathcal{D}_\alpha \Psi = (\partial_\alpha + aK,_\alpha)\Psi$$
$$\mathcal{D}_{\bar{\beta}} \Psi = (\partial_{\bar{\beta}} + bK,_{\bar{\beta}})\Psi$$

Proposition:

$$
\begin{aligned}
[\mathcal{D}_\alpha, \mathcal{D}_{\bar\beta}]\, \Omega &= -g_{\alpha\bar\beta}\, \Omega \\
[\mathcal{D}_\alpha, \mathcal{D}_{\bar\beta}]\, \bar\Omega &= \quad g_{\alpha\bar\beta}\, \bar\Omega \\
[\mathcal{D}_\alpha, \mathcal{D}_{\beta}]\, \Omega &= [\mathcal{D}_{\bar\alpha}, \mathcal{D}_{\bar\beta}]\, \Omega = 0
\end{aligned}
$$

Proof:

$\square$

Thus $c_1(L) = \omega_{\mathcal{M}}$

Tian: $c_1(L)$ is the natural choice of Kähler form on $\mathcal{M}_{\mathbb{C}S}$.

Note that e^{-K} is covariantly constant:

In fact:

Find: $\mathcal{D}_\alpha \varphi_\beta$

A priori: $\mathcal{D}_\alpha \varphi_\beta \in H^{(2,1)} \oplus H^{(1,2)}$

Then

$$\mathcal{D}_\alpha \varphi_\beta = A_{\alpha\beta}{}^\gamma \varphi_\gamma + B_{\alpha\beta}{}^{\bar\gamma} \varphi_{\bar\gamma}$$

By considering

$$\int_X (\mathcal{D}_\alpha \varphi_\beta) \wedge \varphi_{\bar\delta}$$

we find that $A = 0$.

By considering

$$\int_X \varphi_\delta \wedge (\mathcal{D}_\alpha \varphi_\beta)$$

we find

$$B_{\alpha\beta}{}^{\bar\gamma} = -i\ e^K\ y_{\delta\alpha\beta}\ g^{\delta\bar\gamma}$$

where we define the *Yukawa couplings* as

$$y_{\alpha\beta\gamma} = -\int_X \Omega \wedge \mathcal{D}_\alpha \mathcal{D}_\beta \mathcal{D}_\gamma \Omega$$

Therefore

$$\mathcal{D}_\alpha \varphi_\beta = \mathcal{D}_\alpha \mathcal{D}_\beta \Omega = -i\ e^K\ y_{\alpha\beta\delta}\ g^{\delta\bar{\gamma}} \varphi_{\bar{\gamma}} \in H^{(1,2)}$$

Similarly

$$\mathcal{D}_\alpha \varphi_{\bar{\beta}} = g_{\alpha\bar{\beta}} \bar{\Omega} \qquad \text{and} \qquad \mathcal{D}_\alpha \bar{\Omega} = 0$$

The last equation is elementary.

Table of derivatives of the basis $(\Omega, \varphi_\alpha, \varphi_{\bar{\beta}}, \bar{\Omega})$

	Spans
Ω	$H^{(3,0)}$
$\mathcal{D}_\alpha \Omega = \varphi_\alpha$	$H^{(2,1)}$
$\mathcal{D}_\alpha \varphi_\beta = -i\ e^K\ y_{\alpha\beta\delta}\ g^{\delta\bar{\gamma}} \varphi_{\bar{\gamma}}$	$H^{(1,2)}$
$\mathcal{D}_\alpha \varphi_{\bar{\beta}} = g_{\alpha\bar{\beta}} \bar{\Omega}$	$H^{(0,3)}$
$\mathcal{D}_\alpha \bar{\Omega} = 0$	

Curvature

$$[\mathcal{D}_\alpha, \mathcal{D}_{\bar\beta}]\,\varphi_\gamma = -g_{\alpha\bar\beta}\varphi_\gamma + R_{\alpha\bar\beta\gamma}{}^{\delta}\,\varphi_\delta$$

where (using the table)

$$R_{\alpha\bar\beta\gamma\bar\delta} = -g_{\alpha\bar\beta}g_{\gamma\bar\delta} + g_{\alpha\bar\delta}g_{\gamma\bar\beta} - e^{2K}y_{\alpha\gamma\kappa}y_{\bar\beta\bar\delta\bar\epsilon}g^{\kappa\bar\epsilon}$$

Flat connection (Gauss-Manin)

There is a flat connection $\mathbb{D}$ acting on the basis $(\Omega, \varphi_\alpha, \varphi_{\bar\beta}, \bar\Omega)$. That is, there exists a 1-form $\mathbb{A}$ such that

$$\mathbb{D}\Lambda = \mathcal{D}\Lambda + \mathbb{A} \wedge \Lambda = 0$$

for every $\Lambda \in \{\Omega, \varphi_\alpha, \varphi_{\bar\beta}, \bar\Omega\}$

Exercise: Find such $\mathbb{A}$

Yukawa couplings

Largely determined by the prepotential (which also determines the metric)

$$y_{\alpha\beta\gamma} = -\int_X \Omega \wedge \mathcal{D}_\alpha \mathcal{D}_\beta \mathcal{D}_\gamma \Omega$$

- $\partial_{\bar{\delta}} y_{\alpha\beta\gamma} = 0$ that is, y is holomorphic.

This is because $\mathcal{D}_\alpha \mathcal{D}_\beta \mathcal{D}_\gamma \Omega$ is the $(0,3)$ part of $\partial^3\Omega/\partial t^\alpha \partial t^\beta \partial t^\gamma$

$$y_{\alpha\beta\gamma} = -\int_X \Omega \wedge \mathcal{D}_\alpha \mathcal{D}_\beta \mathcal{D}_\gamma \Omega = -\int_X \Omega \wedge \frac{\partial^3 \Omega}{\partial t^\alpha \partial t^\beta \partial t^\gamma}$$

Then $\partial_{\bar{\delta}} y_{\alpha\beta\gamma} = 0$

- $y_{\alpha\beta\gamma}$ is symmetric in its indices and takes values in $L^2 = L \otimes L$.

This is obvious.

- Let $\chi \in H_{\bar{\partial}}^{(0,1)}(X, T) \simeq H^{(2,1)}(X)$

Then

$$\chi_\alpha{}^\mu = \chi_{\alpha\bar{\nu}}{}^\mu dx^{\bar{\nu}} = \frac{1}{2||\Omega||^2} \bar{\Omega}^{\mu\rho\sigma} \varphi_{\alpha\rho\sigma\bar{\nu}} dx^{\bar{\nu}}$$

In terms of these

$$y_{\alpha\beta\gamma} = \int_X \Omega \wedge (\chi_\alpha{}^\mu \wedge \chi_\beta{}^\nu \wedge \chi_\gamma{}^\rho \, \Omega_{\mu\nu\rho})$$

Therefore

$$y: \quad H^{(2,1)} \otimes H^{(2,1)} \otimes H^{(2,1)} \to \mathbb{C}$$

This is a cubic form valued in $\mathbb{C}$ and a complicated holomorphic function of the periods.

In terms of the "special" projective coordinates, the Yukawa coupling can be expressed in terms of the prepotential

$$y_{abc} = \int_X \Omega \wedge \Omega_{,abc} = \mathcal{G}_{,abc}$$

Summary

The moduli space of the complex structure of a Calabi–Yau manifold X, $\mathcal{M}_{\mathbb{C}S}(X)$, has complex dimension h_{21} and has the following properties

• It is Special Kähler, with metric $g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K$, and with Kähler potential

$$e^{-K} = i \int_X \Omega \wedge \bar{\Omega} = -2i\Im(z^a \overline{\mathcal{G},_a})$$

where

$$\Omega = z^a \alpha_a - \mathcal{G},_b \beta^b$$

and the 3-forms (α, β) form a symplectic basis of H^3. It has a flat symplectic structure.

• There is cubic form valued in $\mathbb{C}$, the *Yukawa coupling*,

$$y_{abc} = \frac{\partial^3 \mathcal{G}}{\partial z^a \partial z^b \partial z^c}$$

Question: Can we determine the periods? Can we determine the prepotential?

Of course, given the periods, we know what the prepotential (and the Yukawa coupling) is:

$$\mathcal{G} = \frac{1}{2}\mathcal{G}_a z^a$$

If yes, we can compute the metric of $\mathcal{M}_{\mathbb{C}S}(X)$.

Calculability of the periods

The *periods* are generalized hypergeometric functions that satisfy linear differential equations with regular singularities.

I will not explain why the equations are Fuchsian however it is easy to see why they satisfy linear differential equations.

Consider the forms

$$\Omega \ , \ \ \Omega' \ , \ \ \Omega'' \ , \ \ \Omega''' \ , \ \ \ldots$$

where the prime represents the variation of Ω with respect to the complex structure parameters.

These are all closed three-forms and at most $b_3 = 2h_{21} + 2$ are linearly independent.
Then there must be a linear relation between the first $b_3 + 1$ of them. This is a linear differential equation of order b_3,

$$\mathcal{L}\,\Omega \ = \ \text{exact} \ .$$

where $\mathcal{L}$ is a linear differential operator of order b_3.

Since the period integrals are taken over a fixed basis in $H_3(X)$, it follows that the periods satisfy the same differential equation as Ω

$$\mathcal{L}\, z^a \ = \ 0 \qquad\qquad \mathcal{L}\, \mathcal{G}_b \ = \ 0 \ .$$

This differential equation is known as the **Picard–Fuchs equation** and there is a well defined prescription to calculate $\mathcal{L}$ (Gel'fand, Kapranov, Zelevinskii) and its solutions.
General Method:

- build a basis of differential 3-forms $\{\varphi_\alpha\}$
- differentiate this basis and write in terms of $\{\varphi_\alpha\}$

We will illustrate this for the mirror quintic and we will describe explicitly $\mathcal{M}_{\mathbb{C}S}(X)$.

Lecture 10

Summary: The moduli space of the complex structure of a Calabi–Yau manifold X, $\mathcal{M}_{\mathbb{C}S}(X)$, has complex dimension h_{21} and it is Special Kähler, with metric $g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K$, and with Kähler potential

$$e^{-K} = i \int_X \Omega \wedge \bar{\Omega} = -2i\Im(z^a \overline{\mathcal{G},_a})$$

where $\Omega = z^a \alpha_a - \mathcal{G},_b \beta^b$ and the 3-forms (α, β) form a symplectic basis of H^3.

It has a flat symplectic structure. There is cubic form valued in $\mathbb{C}$, the *Yukawa coupling*,

$$y_{abc} = \frac{\partial^3 \mathcal{G}}{\partial z^a \partial z^b \partial z^c} \, .$$

Both, the metric and the Yukawa coupling are determined in terms of the periods.

Section 5.2

The Moduli Space of the Kähler Class $\mathcal{M}_{KC}$

Recall:

The "mixed" deformations of Ricci–flat metrics $\delta g_{\mu\bar{\nu}}$ are harmonic $(1,1)$ forms.

They are in 1-1 correspondence with variations of the Kähler class $[\omega]$

$$\omega = i g_{\mu\bar{\nu}} dx^\mu \wedge dx^{\bar{\nu}} \longrightarrow \omega + \delta\omega$$

The (real) dimension of the space of Ricci–flat metrics corresponding to deformations of the Kähler class is h_{11}.

First expectation from mirror symmetry:

Let X and Y be two Calabi–Yau manifolds which are mirror pairs. Then

$$h_{11}(X) = h_{12}(Y)$$

A highly non–trivial prediction of the Mirror Symmetry conjecture is that

$$\mathcal{M}_{\mathbb{C}S}(Y) \simeq \mathcal{M}_{KC}(X)$$

We do not know what we mean by $\mathcal{M}_{KC}(X)$ yet, but we can start by requiring that at least the dimensions should match.

If by $\mathcal{M}_{KC}(X)$ we mean the dimension of the space of Ricci-flat metrics corresponding to deformations of the Kähler class $[\omega]$, then the dimensions do not match since $h_{11}(X) \neq 2h_{12}(Y)$.

This problem is solved naturally in String Theory by the fact that what is relevant instead is the space of deformations of

$$B + i\omega$$

where B is called the *B–field*. The B–field defines a class in $H^{(1,1)}(X, \mathbb{R})$, but it is defined only modulo $H^2(X, \mathbb{Z})$. We call the combination $B + i\omega$ the *Complexified Kähler Class.*

Let $\mathcal{M}^o_{KC}(X)$ be the moduli space of the complexified Kähler class.

It is now certainly the case that

$$\dim \mathcal{M}^o_{KC}(X) = 2h_{11}(X) = 2h_{12}(Y).$$

We begin with a discussion of the geometry of $\mathcal{M}^o_{KC}(X)$.

Let $\{e_i\}, i = 1, \ldots, h_{11}$ be a basis for $H^2(X, \mathbb{Z})$.
Then we can write

$$B + i\omega = t^i e_i$$

where the $\{t^i\}$ are affine complex coordinates in $\mathcal{M}^o_{KC}(X)$.

Consider the holomorphic section

$$\mathcal{F}^o = \frac{1}{3!}\, y^o_{ijk}\, \frac{w^i w^j w^k}{w^0}$$

This is homogeneous of degree two in the homogeneous coordinates $\{w^0, w^1, \ldots, w^{h_{11}}\}$.
We define the affine coordinates as $t^i = w^i / w^o$.

We define

$$y^o_{ijk} = \int_X e_i \wedge e_j \wedge e_k$$

which is an intersection number. Then

$$y^o_{ijk} = \partial_i \partial_j \partial_k \mathcal{F}^o$$

where $\mathcal{F}^o$ is given above.

We have

$$y^o \ : \ H^2 \otimes H^2 \otimes H^2 \longrightarrow \mathbb{Z}$$

which is a cubic form valued in $\mathbb{Z}$.

So far this seems trivial. The surprise however is that one can prove the following proposition.

Proposition

$\mathcal{M}^o_{KC}(X)$ is Kähler with holomorphic prepotential. Moreover,

$$g^o_{i\bar{j}} = \frac{1}{4V} \int_X e_i \wedge * e_j = \partial_i \partial_{\bar{j}} K^o$$

where

$$e^{-K^o} = \frac{4}{3} \int_X \omega \wedge \omega \wedge \omega = -2i \Im (w^I \overline{\mathcal{F}^o_I})$$

and $I, J = 0, 1, \ldots, h_{11}$.

Proof:

We follow a similar procedure from that followed for the complex structure.

Recall:

The natural line element on the space of metrics is

$$ds^2 = \frac{1}{4V}\int_X ||\delta g_{\mu\nu}||^2 + ||\delta g_{\mu\bar{\nu}} + \delta B_{\mu\bar{\nu}}||^2$$

The first term corresponds to the variation of the complex structure and the second to the variations of $B + i\omega$.

We consider

$$ds^2_{KC} = \frac{1}{2V}\int_X g^{\kappa\bar{\mu}} g^{\lambda\bar{\nu}} (\delta g_{\kappa\bar{\nu}} \delta g_{\lambda\bar{\mu}} + \delta B_{\kappa\bar{\nu}} \delta B_{\lambda\bar{\mu}}) g^{1/2} d^6x$$

The corresponding inner product on $H^{(1,1)}$ is

$$g(\rho,\sigma) = \frac{1}{2V}\int_X \rho_{\mu\bar{\nu}} \sigma_{\rho\bar{\sigma}} g^{\mu\bar{\sigma}} g^{\rho\bar{\nu}} g^{1/2} d^6x = \frac{1}{2V}\int_X \rho\wedge*\sigma$$

for real ρ and σ in $H^{(1,1)}(X,\mathbb{R})$.

Note that $g(,)$ is positive.

Let $\kappa(\rho,\sigma,\tau)$ be a cubic form defined as

$$\kappa(\rho,\sigma,\tau)=\int_X \rho\wedge\sigma\wedge\tau$$

In virtue of the identities

$$V=\frac{1}{3!}\int_X \omega^3=\frac{1}{3!}\,\kappa(\omega,\omega,\omega)$$

$$*\sigma=-\omega\wedge\sigma+\frac{3\,\kappa(\sigma,\omega,\omega)}{2\,\kappa(\omega,\omega,\omega)}\,\omega\wedge\omega$$

the inner product can be written entirely in terms of the cubic form κ.

$$\begin{aligned} g(\rho,\sigma) &= \frac{1}{2V}\int_X \rho\wedge *\sigma \\ &= -3\left\{\frac{\kappa(\rho,\sigma,\omega)}{\kappa(\omega,\omega,\omega)}-\frac{3}{2}\,\frac{\kappa(\rho,\omega,\omega)\kappa(\sigma,\omega,\omega)}{\kappa^2(\omega,\omega,\omega)}\right\}\end{aligned}$$

Consider now

$$\frac{1}{2}\,g(e_i,e_j)=-\frac{3}{2}\left\{\frac{\kappa(e_i,e_j,\omega)}{\kappa(\omega,\omega,\omega)}-\frac{3}{2}\,\frac{\kappa(e_i,\omega,\omega)\kappa(e_j,\omega,\omega)}{\kappa^2(\omega,\omega,\omega)}\right\}$$

Then we can prove that

$$-\partial_i \partial_{\bar{j}} \log \kappa(\omega, \omega, \omega) = \frac{1}{2}\, g(e_i, e_j)$$

Note: In deriving this relation one needs

$$\omega = \frac{1}{2i}(t^i - \overline{t^i})e_i$$

so

$$\partial_i \omega = \frac{1}{2i} e_i \quad \text{and} \quad \partial_{\bar{i}} \omega = -\frac{1}{2i} e_i$$

For convenience we will shift the Kähler potential by a constant factor and we have

$$g^o_{i,\bar{j}} = \frac{1}{2}\, g(e_i, e_j) = -\partial_i \partial_{\bar{j}} \log \frac{4}{3}\, \kappa(\omega, \omega, \omega)$$

We set

$$e^{-K^o} = \frac{4}{3}\, \kappa(\omega, \omega, \omega)$$

We still need to prove that

$$e^{-K^o} = \frac{4}{3} \int_X \omega \wedge \omega \wedge \omega = -2i \Im(w^I \overline{\mathcal{F}^o_I})$$

which is an **exercise** for you.

$\square$

Now, $\mathcal{M}^o_{KC}(X)$ is not "isomorphic" to $\mathcal{M}_{\mathbb{C}S}(Y)$ at all, even though they are both Special Kähler. We will see this very clearly in an example soon.

But we do not want an "isomorphism" between these spaces because this is not what is consistent with Mirror Symmetry in String Theory.

	Complexified Kähler Cone	Complex Structure
Number of projective coords.	$b_{11}+1$	$b_{21}+1$
Cohomology group	$H^1(\mathcal{M},\mathcal{T}^*)$	$H^1(\mathcal{M},\mathcal{T})$
Prepotentials	$\mathcal{F}(w)$	$\mathcal{G}(z)$
$\exp(-K)$	$\frac{4}{3}\int J^3{=}2\Im m\Big(w^j\,\frac{\partial\bar{\mathcal{F}}}{\partial w^{\bar{\jmath}}}\Big)$	$i\int\Omega\bar{\Omega}{=}2\Im m\Big(z^a\,\frac{\partial\bar{\mathcal{G}}}{\partial z^{\bar{a}}}\Big)$
Period matrices	$\Im m\Big(\frac{\partial^2\mathcal{F}(w)}{\partial w^i\partial w^j}\Big)$	$\Im m\Big(\frac{\partial^2\mathcal{G}(z)}{\partial z^a\partial z^b}\Big)$
Signature of the period matrix	$(b_{11},1)$	$(b_{21},1)$

Table 2.1: A table of closely analogous quantities describing the spaces of complex structures and Kähler–forms.

Section 5.3

A Mirror Symmetry Conjecture in String Theory

Use Mirror Symmetry to learn about $\mathcal{M}_{KC}(X)$

Let (X, Y) be a Mirror Pair.

In String Theory a Mirror Pair is one for which

$$\Gamma_X \text{``} = \text{''} \Gamma_Y$$

For example, the *massless spectrum* of both theories is the same:

$$\# \text{ of generations} = |h_{11} - h_{12}| = \frac{1}{2}|\chi|$$

Therefore X and Y have the same massless spectrum.

There is much more.

If $\Gamma_X \text{``} = \text{''} \Gamma_Y$ is true, then it must be the case that

$$\begin{aligned} \mathcal{M}_{\mathbb{C}S}(Y) &\simeq \mathcal{M}_{KC}(X) \\ &\simeq \text{quantum corrected } \mathcal{M}^o_{KC}(X) \end{aligned}$$

If we call ψ the parameters of the complex structure of Y and t the parameters of the Kähler class of X, then the map $\psi(t)$ is called the *mirror map.*

The reason for this is that $\Gamma_X \text{``} = \text{''} \Gamma_Y$ holds if all correlation functions are **equal**. For example

- 2–point correlation function (kinetic terms in the Lagrangian for Γ) corresponds to the metrics in the moduli space
- 3–point correlation function corresponds to the Yukawa coupling.

In terms of the prepotentials: if X_t and Y_ψ is a mirror pair, then

$$\mathcal{F}(t) = \mathcal{G}(\psi)$$

where t and $\psi(t)$ are related by the mirror map.

What happens is that

$$\mathcal{F}^o(t) = \mathcal{G}(\psi)$$

is not true (except in a limit as we will see).

What we should have is

$$\mathcal{G}(\psi) = \mathcal{F}^o(t) + \Delta\mathcal{F}(t)$$

The left hand side is calculable, and so is $\mathcal{F}^o(t)$. Then we will use $\mathcal{G}$ as a generating function for $\Delta\mathcal{F}$.

But, how about the mirror map?

More on the structure of $\mathcal{M}_{KC}$ later: it is clear that it will contain the data in $\mathcal{M}^o_{KC}$ (as for example the intersection numbers); but it will contain other information, the Gromov-Witten invariants.

Lecture 11

Continue: $\mathcal{M}_{\mathbb{C}S}(Y_\psi) \simeq \mathcal{M}_{KC}(X_t)$

Recall: we do not know yet what $\mathcal{M}_{KC}(X)$ is.

From String Theory:
We want $\mathcal{M}_{KC}(X_t)$ to be a *deformation* of the special Kähler $\mathcal{M}^o_{KC}(X_t)$ (which has cubic prepotential defined by the triple intersection numbers) which preserves the Special Kähler structure. Moreover, we want the prepotential $\mathcal{F}$ for $\mathcal{M}_{KC}(X)$ to satisfy

$$\mathcal{G}(\psi) = \mathcal{F}^o(t) + \Delta\mathcal{F}(t)$$

where $\psi(t)$ is the mirror map (which we have not defined).

Chapter 6
Mirror Symmetry and the Quintic 3-fold

Quintic 3-fold:

Let $X \in \mathbb{P}^4[5]$. We know that

$$\chi(X) = -200 \qquad h_{21} = 101 \qquad h_{11} = 1$$

For definiteness, we consider a one parameter family of quintics X_ψ:

$$P(x,\psi) = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5\psi x_1x_2x_3x_4x_5$$

This family has a large group of automorphisms

$$G = \mathbb{Z}_5^3$$

Section 6.1
The Mirror of the Quintic 3-fold

The mirror Y_ψ of the quintic 3-fold was first constructed by Greene and Plesser. Let

$$\tilde{Y}_\psi = X_\psi / \mathbb{Z}_5^3$$

where the generators of $\mathbb{Z}_5^3$ are given by

$$\begin{aligned} g_0 &: (1,0,0,0,4) \\ g_1 &: (0,1,0,0,4) \\ g_2 &: (0,0,1,0,4) \\ g_3 &: (0,0,0,1,4) \end{aligned}$$

where, for example

$$g_1 : (x_1, \cdots, x_5) \longrightarrow (x_1, \zeta x_2, x_3, x_4, \zeta^4 x_5)\,, \quad \zeta = e^{2\pi i/5}$$

Note that only 3 of these are independent because of the identifications in $\mathbb{P}^4$

$$(x_1, \cdots, x_5) \longrightarrow \lambda(x_1, \cdots, x_5) \qquad \lambda \in \mathbb{C}^*$$

Note also that we are considering the most general quintic polynomial which in invariant under $\mathbb{Z}_5^3$. This is

$$P(x,\psi) = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5\psi x_1 x_2 x_3 x_4 x_5$$

(It is not a coincidence that there is only one free parameter.)

The mirror Y of the quintic 3-fold is obtained by resolving the fixed point singularities.

After resolving the singularities, we obtain

$$h_{11}(Y) = 101 \qquad h_{21}(Y) = 1$$

The 100 extra Kähler forms arise from the blow ups required to resolve the singularies associated with the fixed points of G.

Resolution of the singularities

We need to cut out curves and points of X which are left invariant by symmetries and, after the identifications by $\mathbb{Z}_5^3$, these curves and points are replaced by their smooth equivalents.

The $\mathbb{Z}_5^3$ action has 10 fixed curves

$$\mathcal{C}_{ijk} : x_i^5 + x_j^5 + x_k^5 = 0 \ , \quad i, j, k \ \text{ distinct}$$

Each of these curves is in $\mathbb{P}^2[5]$ and it is invariant under a $\mathbb{Z}_5$ subgroup.

(For example, g_1 leaves the curve $x_2 = x_5 = 0$ invariant, and the other two generators identify it with itself.)

These curves intersect in 10 fixed points

$$p_{ij} : x_i^5 + x_j^5 = 0 \ , \quad i \neq j$$

each of these being left invariant by a $\mathbb{Z}_5 \times \mathbb{Z}_5$.

Each fixed curve contains 3 fixed points.

And: 3 fixed curves meet in each of the fixed points

Consider points and curves separately.

The Euler number of curves less the points (before identifications) is

$$\chi(\mathbb{P}^2[5]) - 3 \times 5 = -10 - 15 = -25$$

$$\begin{aligned}\chi &= \frac{1}{125}\left(-200 - 10 \times 5 - 10 \times (-25)\right) \\ &\quad + 10 \times \frac{1}{25}((-25) \times 5) + 10 \times \frac{1}{5}(5 \times 25) \\ &= +200\end{aligned}$$

$X_t \in \mathbb{P}^4[5]$

For Y_ψ, we work with a one parameter family of mirror quintic 3-folds:

$$Y_\psi : \quad P(x,\psi) = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5\psi x_1 x_2 x_3 x_4 x_5$$

together with the $\mathbb{Z}_5^3$ identifications with generators given by

$$\begin{aligned} g_0 &: (1,0,0,0,4) \\ g_1 &: (0,1,0,0,4) \\ g_2 &: (0,0,1,0,4) \\ g_3 &: (0,0,0,1,4) \end{aligned}$$

Section 6.2 Describing $\mathcal{M}_{\mathbb{C}S}(Y)$

- Y_ψ and $Y_{\alpha\psi}$, where $\alpha^5 = 1$, have the same complex structure:

$\psi \longrightarrow \alpha\psi$ can be undone by a coordinate transformation, say

$$x_1 \to \alpha^{-1} x_1 \qquad\qquad x_i \to x_i \ \ , i = 2, 3, 4, 5$$

Natural coordinate on $\mathcal{M}_{\mathbb{C}S}(Y)$ is ψ^5.

- $Y_{\psi=\infty}$

This is called the *large complex structure limit* (LCSL)

Y_∞ corresponds to the (very) singular quintic

$$x_1 x_2 x_3 x_4 x_5 = 0$$

Before identifications by $\mathbb{Z}_5^3$: This is 5 $\mathbb{P}^3$s meeting in 10 $\mathbb{P}^2$s meeting in 10 $\mathbb{P}^1$ meeting in 5 points.

- $Y_{\psi^5=1}$ is singular (conifold)

This space has a node, a double point singularity.

The polynomial fails to be transverse: there is a solution of $P = 0$ and $dP = 0$, however the matrix of second derivatives is non-degenerate ($\det \partial_i \partial_j P \neq 0$).

The polynomial has 125 singular points

$$x_i = \alpha^{n_i}, \qquad \sum n_i = 0 \; (mod\ 5)$$

which are all identified by $\mathbb{Z}_5^3$: there is one conifold singularity at $(1, 1, 1, 1, 1)$

A neighborhood of the node is locally a cone with base $S^2 \times S^3$

Let $x_i = 1 + y_i$, $i = 1, 2, 3, 4$, y_i small, and $x_5 = 1$. Into $P = 0$ and after a change of coordinates we obtain

$$Q = \sum_{A=1}^{4} (w_A)^2 = 0 \qquad \text{in } \mathbb{C}^4$$

which is a cone with apex at $\{w_A = 0\}$. The base of the cone, $\mathcal{N}$, is the intersection of the solutions of $Q = 0$ with a sphere of radius r in $\mathbb{C}^4 = \mathbb{R}^8$ defined by $\sum_{A=1}^{4} |w_A|^2 = r^2$.

To see that $\mathcal{N} = S^2 \times S^3$, we let $\mathbf{w} = (w_1, w_2, w_3, w_4)$ and

$$\mathbf{w} = \mathbf{x} + i\mathbf{y}$$

Then

$$\mathbf{x} \cdot \mathbf{x} = \frac{r^2}{2}\ , \quad \mathbf{y} \cdot \mathbf{y} = \frac{r^2}{2}\ , \quad \mathbf{x} \cdot \mathbf{y} = 0\ .$$

The first equation is a 3-sphere of radius $r/\sqrt{2}$. The second and third define an S^2 fiber bundle of S^3. All such bundles over S^3 are trivial thus $\mathcal{N}$ is topologically $S^2 \times S^3$.

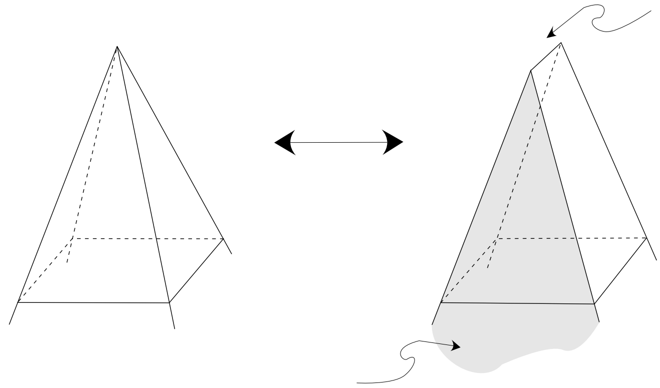


Figure 3: Conifold transition

In Summary:

We have special values for ψ:

- $\psi = 1$ conifold
- $\psi = \infty$ very singular, LCSL.
- $\psi = 0$

  While $Y_{\psi=0}$ is smooth, $\psi = 0$ is an orbifold point of the parameter space. This is because $Y_{\psi=0}$ is invariant under $\psi \to \alpha\psi$, $\alpha^5 = 1$.

More generaly

For example, in hypersufaces defined by one polynomial in a toric variety:

- $P = dP = 0$ gives parameter values (discriminat locus) for which the original family of hypersurfaces is singular.

  Conifold singularities are ubiquitous. But there are others.

- Very singular LCSL.

  We will define this in more generality in terms of the *monodromy* of the periods.

- Orbifold points of the parameter space.

Exercise: compute the discriminant locus of the following polynomial in $W\mathbb{P}_{[11222]}[8]$

$$P(x,\psi,\phi) = x_1^8+x_2^8+x_3^4+x_4^4+x_5^4-8\psi x_1x_2x_3x_4x_5-2\phi x_1^4x_2^4$$

Show that:

- The locus $\phi + 8\psi^4 = \pm 1$ corresponds to conifold singularites.
- The locus $\phi = \pm 1$ corresponds to more complicated singularites.
- What happens when $\psi = 0$?

See Candelas, XD, Font, Katz and Morrison: hepth 9308083

Lecture 12

We are working with the mirror pair $X_t \in \mathbb{P}^4[5]$ and Y_ψ.

For Y_ψ, we work with a one parameter family of mirror quintic 3-folds:

$$Y_\psi : \quad P(x,\psi) = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5\psi x_1 x_2 x_3 x_4 x_5$$

together with the $\mathbb{Z}_5^3$ identifications with generators given by

$$\begin{aligned} g_0 &: (1,0,0,0,4) \\ g_1 &: (0,1,0,0,4) \\ g_2 &: (0,0,1,0,4) \\ g_3 &: (0,0,0,1,4) \end{aligned}$$

In Summary:

We have special values for ψ in $\mathcal{M}_{\mathbb{C}S}$:

- $\psi = 1$ conifold
- $\psi = \infty$ very singular, LCSL.
- $\psi = 0$

  While $Y_{\psi=0}$ is smooth, $\psi = 0$ is an orbifold point of the parameter space. This is because $Y_{\psi=0}$ is invariant under $\psi \to \alpha\psi$, $\alpha^5 = 1$.

Symplectic homology basis

$$A^2 = S^3 \quad \text{that shrinks to 0 as} \quad \psi \to 1$$
$$B_2 \quad \text{is the cycle such that} \quad A^2 \cap B_2 = 1$$

(It is hard to prove that $A^2 \cap B_2 = 1$)

We take A^1 and B_1 to be remote from this neighborhood and do not intersect with A^2 and B_2

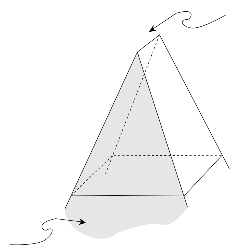


Figure 4: Deformation of the conifold

$$\begin{aligned} A^2 = \{\mathbf{x}|\ & x_5 = 1, x_i \in \mathbb{R}, i = 1,2,3; \\ & x_4 \ \text{branch of } P = 0 \text{ that is an } S^3 \text{ as } \ \psi \to 1\} \\ B_2 = \{\mathbf{x}|\ & x_5 = 1, |x_1| = |x_2| = |x_3| = \delta \\ & x_4 \ \text{soln of } P = 0 \text{ that tends to zero as } \psi \to \infty\} \end{aligned}$$

Notes on B_2:

* It is T^3 near the LCSL

* As $\psi \to \infty$ the other four branches for x_4 are $\mathcal{O}(\psi^{1/4})$.

Construct Ω

Consider the holomorphic 4-form in $\mathbb{P}^4$

$$\begin{aligned}\mu &= \frac{1}{5!}\,\epsilon_{\nu_1\nu_2\nu_3\nu_4\nu_5}\,x^{\nu_1}dx^{\nu_2}\wedge dx^{\nu_3}\wedge dx^{\nu_4}\wedge dx^{\nu_5}\\ &= \frac{1}{5}\,(x^5dx^1\wedge dx^2\wedge dx^3\wedge dx^4+\ldots)\end{aligned}$$

Now, $(x_1, x_2, x_3, x_4, x_5)$ and $\lambda(x_1, x_2, x_3, x_4, x_5)$, $\lambda \in \mathbb{C}^*$ are the same point in $\mathbb{P}^4$.

Then μ is not well defined in $\mathbb{P}^4$ since $\mu \longrightarrow \lambda^5\mu$ under $(x_1, x_2, x_3, x_4, x_5) \longrightarrow \lambda(x_1, x_2, x_3, x_4, x_5)$

Using *residues*, we can however construct a well defined 3-form on Y.

Let

$$\nu = \frac{\mu}{P}$$

- This has poles at $P = 0$
- It is well defined on $\mathbb{P}^4$.

This is the case because under

$$(x_1, x_2, x_3, x_4, x_5) \longrightarrow \lambda(x_1, x_2, x_3, x_4, x_5)$$

$P \to \lambda^5 P$ and then ν is invariant.

Note: that this is true only if P is of degree 5.

We will now consider a 1-dim contour C in $\mathbb{P}^4$ around $P = 0$. C is a small circle of radius ϵ that winds around $P = 0$ (a circle bundle inside the complex normal bundle).

Integrating ν over C we obtain a holomorphic (by construction) 3-form on the quintic 3-fold

$$(2\pi i)\Omega = \lim_{\epsilon \to 0} \int_C \frac{\mu}{P}$$

We can now show that, on a patch $x_5 \neq 0$

$$\Omega = -5\psi \left(\frac{1}{2\pi i}\right)^3 \frac{x_5 dx_1 \wedge dx_2 \wedge dx_3}{\partial_4 P(x, \psi)}$$

where $C = C_4$ is the circle with $|x_4| = \epsilon$.

Note that:

- there are no poles since $dP = 0$ and $P = 0$ cannot be true simultaneously.
- the value of x_4 for which $P = 0$ lies inside the circle C_4 for sufficiently large ψ and the residue of $1/P$ is $1/\partial_4 P$.

Periods

Consider the period

$$\varpi_0(\psi) = \int_{B_2} \Omega$$

Recall from last lecture:

$$\Omega = -5\psi \left(\frac{1}{2\pi i}\right)^3 \frac{x_5 dx_1 \wedge dx_2 \wedge dx_3}{\partial_4 P(x,\psi)}$$

and

$$B_2 = \{\mathbf{x} |\ x_5 = 1, |x_1| = |x_2| = |x_3| = \delta \\ x_4 \ \text{ soln of } P = 0 \text{ that tends to zero as } \psi \to \infty\}$$

Compute:

$$\begin{aligned}\varpi_0(\psi) &= -5\psi \left(\frac{1}{2\pi i}\right)^3 \int_{B_2} \frac{x_5 dx_1 \wedge dx_2 \wedge dx_3}{\partial_4 P(x,\psi)} \\ &= -5\psi \, \frac{1}{(2\pi i)^4} \int_{B_2 \times C_4} \frac{x_5 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{P(x,\psi)}\end{aligned}$$

where C_4 is the circle with $|x_4| = \delta$ in $\mathbb{P}^4$ that winds around $P = 0$.

In the last expression, note that the integrand is independent of x_5.

We introduce

$$1 = \frac{1}{2\pi i}\int_{C_5}\frac{dx_5}{x_5}$$

where C_5 is a circle aroud $P = 0$ of radius $|x_5| = \delta$.
Then we obtain

$$\begin{aligned}\varpi_0(\psi) &= -5\psi\frac{1}{(2\pi i)^5}\int_{B_2\times C_4\times C_5}\frac{dx_1\wedge dx_2\wedge dx_3\wedge dx_4\wedge dx_5}{P(x,\psi)} \\ &= \sum_{n=0}^{\infty}\frac{(5n)!}{n!^5}\frac{1}{(5\psi)^{5n}}\,, \qquad |\psi| > 1\,, \quad 0 < \arg\psi < \frac{2\pi}{5}\end{aligned}$$

So, we have

$$\varpi_0(\lambda) = \int_{B_2}\Omega = \sum_{n=0}^{\infty}a_n\lambda^n\,, \quad |\psi| > 1\,, \quad 0 < \arg\psi < \frac{2\pi}{5}$$

where

$$\lambda = \frac{1}{(5\psi)^{5n}} \qquad a_n = \frac{(5n)!}{n!^5}\in\mathbb{Z}$$

One can find, by analytic continuation

$$\varpi_0(\psi) = -\frac{1}{5}\sum_{m=1}^{\infty}\alpha^{2m}\frac{\Gamma(m/5)}{\Gamma(m)\Gamma^4(1-m/5)}(5\psi)^m$$

valid for $|\psi| < 1$.

We can compute the other periods by finding the differential operator that ϖ_0 satisfies:

$$\mathcal{L}\varpi_0 = 0$$

where $\mathcal{L}$ is a or order $b_3 = 4$.

We find

$$\mathcal{L} = \Theta^4 - 5\lambda\prod_{i=1}^{4}(5\Theta + i)\ , \qquad \Theta = \lambda\frac{d}{d\lambda}$$

Alternatively (Griffiths-Dwork) we can construct a cohomology basis of 3-forms.

Again, let

$$\Omega = \mathrm{Res}_Y\ \frac{\mu}{P}$$

The middle cohomology of Y_ψ is constructed similarly by a residue construction

$$\varphi_\alpha = \mathrm{Res}_Y\frac{\mathbf{x}^{\underline{m}_\alpha}\mu}{P^{1+d_\alpha}}$$

where $\mathbf{x}^{\underline{m}_\alpha} = x_1^{m_{\alpha 1}}\cdots x_5^{m_{\alpha 5}}$ and $5d_\alpha = \deg(\mathbf{x}^{\underline{m}_\alpha})$.

Each φ_α is represented by a rational differential form.

In the case of the quintic, the middle cohomology can be represented by the 3-forms

$$\Omega^{(n)} = n!\ \mathrm{Res}_Y \left((5Q)^n \, \frac{\mu}{P^{n+1}} \right) \ , n = 0, 1, \ldots$$

where $Q = x_1 x_2 x_3 x_4 x_5$.

To find the Picard–Fuchs equation one needs Griffiths' reduction of pole order method.

Illustrate the method:

$$\begin{aligned} \frac{\partial}{\partial x_1}(x_1 \Omega) &= \Omega + x_1 \frac{\partial}{\partial x_1} \Omega \\ &= \mathrm{Res}_Y \, \frac{\mu}{P} - 5\ \mathrm{Res}_Y \left(\frac{\mu}{P^2}(x_1^5 - \psi Q) \right) \end{aligned}$$

So, modulo an exact form, the differential form corresponding to $x_1^5 - \psi Q$ is equivalent to that corresponding to $\mathbf{1}$.

Continue now computing:

$$\begin{aligned} &\frac{\partial}{\partial x_2} \left(x_2 \frac{\partial}{\partial x_1} (x_1 \Omega) \right) \\ &\quad = \frac{\partial}{\partial x_2} \left(x_2 \left(\mathrm{Res}_Y \, \frac{\mu}{P} - 5\ \mathrm{Res}_Y \left(\frac{\mu}{P^2}(x_1^5 - \psi Q) \right) \right) \right) \\ &\quad = \mathrm{Res}_Y \, \frac{\mu}{P} - 5\ \mathrm{Res}_Y \left(\frac{\mu}{P^2}(x_1^5 - \psi Q) \right) \\ &\qquad + \mathrm{Res}_Y \, \frac{\mu}{P} - 5\ \mathrm{Res}_Y \left(\frac{\mu}{P^2}(x_2^5 - \psi Q) \right) \\ &\quad + 10\ \mathrm{Res}_Y \left(\frac{\mu}{P^3}(x_2^5 - \psi Q)(x_1^5 - \psi Q) \right) + 5\psi\ \mathrm{Res}_Y \left(\frac{\mu}{P^2} Q \right) \end{aligned}$$

Then

$$\begin{aligned}\text{exact} &= 5\psi\ \mathrm{Res}_Y\left(\frac{\mu}{P^2}Q\right)\\ &\quad + 10\ \mathrm{Res}_Y\left(\frac{\mu}{P^3}(x_2^5-\psi Q)(x_1^5-\psi Q)\right)\end{aligned}$$

After a while we do find the Picard–Fuchs equation.

Exercise: Compute the Picard–Fuchs equation for the family

$$P(x,\psi) = x_1^3 + x_2^3 + x_3^3 - 3\psi x_1 x_2 x_3$$

of cubics in $\mathbb{P}^2$.

So we have the fourth order differential operator

$$\mathcal{L} = \Theta^4 - 5\lambda \prod_{i=1}^{4}(5\Theta + i)\ , \qquad \Theta = \lambda\frac{d}{d\lambda}$$

It is easy to see that $\lambda = 0$ is a regular singular point with all four indices equal to zero. The four periods then have the form

$$\begin{aligned}
\varpi_0(\lambda) &= f_0(\lambda) \qquad \text{(holomorphic)}\\
\varpi_1(\lambda) &= f_0(\lambda)\log\lambda + f_1(\lambda)\\
\varpi_2(\lambda) &= f_0(\lambda)\log^2\lambda + 2f_1(\lambda)\log\lambda + f_2(\lambda)\\
\varpi_3(\lambda) &= f_0(\lambda)\log^3\lambda + 3f_1(\lambda)\log^2\lambda + f_2(\lambda)\log\lambda + f_3(\lambda)
\end{aligned}$$

where the $f_i(\lambda)$ are power series.

To find these solutions: Frobenious

Let

$$\varpi(\lambda, \epsilon) = \sum_{m=0}^{\infty} a_m(\epsilon)\lambda^{m+\epsilon}$$

where ϖ satisfies

$$\mathcal{L}\varpi(\lambda, \epsilon) = \epsilon^4\lambda^\epsilon$$

Then

$$\begin{aligned}\varpi_0(\lambda) &= \varpi(\lambda, 0)\\ \varpi_1(\lambda) &= \frac{\partial}{\partial\epsilon}\varpi(\lambda, \epsilon)\Big|_{\epsilon=0}\end{aligned}$$

etc, and where, for example,

$$f_1(\lambda) = 5\sum_{m=1}^{\infty} a_m(\sigma_{5m} - \sigma_m)\lambda^m \ , \quad \sigma_m = \sum_{n=1}^{m}\frac{1}{n}$$

Indices

$$\mathcal{P}\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ 0 & 1/5 & 0 & \\ 0 & 2/5 & 1 & \psi^{-5} \\ 0 & 3/5 & 2 & \\ 0 & 4/5 & 1 & \end{array}\right\}$$

At $\psi = 1$ we also have a logarithmic divergence.

Near $\psi = 0$ the periods are of the form $\quad \psi^k\ {}_4F_3$.

The last step is to find the periods corresponding to the symplectic basis $\{z^a, \mathcal{G}_a\}$: use *monodromy*.

We know that $\mathcal{G}_2 = \varpi_0 = \int_{B_2} \Omega$. By analytic continuation, one discovers that this period has a logarithm at $\psi = 1$. This is expected because under *monodromy* around $\psi = 1$ one can show that

$$A^2 \to A^2 \qquad B_2 \to B_2 + A^2$$

which in terms of the periods it means

$$z^2 \to z^2 \qquad \mathcal{G}_2 \to \mathcal{G}_2 + z^2$$

One finds

$$z^2 = \int_{A^2} \Omega = c(\varpi_0(\alpha\psi) - \varpi_0(\psi))$$

ETC.

Metric

Recall

$$e^{-K} = i(\bar{z}^a \, \mathcal{G}_a - z^a \bar{\mathcal{G}}_a)$$

where K is the Kähler potential.

We find:

- $\psi \sim 1$ $\qquad g_{\psi\bar{\psi}}(1)$ is finite (has cusps).
- $\psi \to 0$ $\qquad g_{\psi\bar{\psi}} \to \dfrac{25\Gamma^5(4/5)\Gamma^5(2/5)}{\Gamma^5(1/5)\Gamma^5(3/5)}$
- $\psi \to \infty$ $\qquad g_{\psi\bar{\psi}} \to \dfrac{3}{4|\psi|^2 \log^2 |\psi|} + \ldots$

  g is symptotic to a metric of constant negative curvature: $R \to -4/3 + \ldots$.

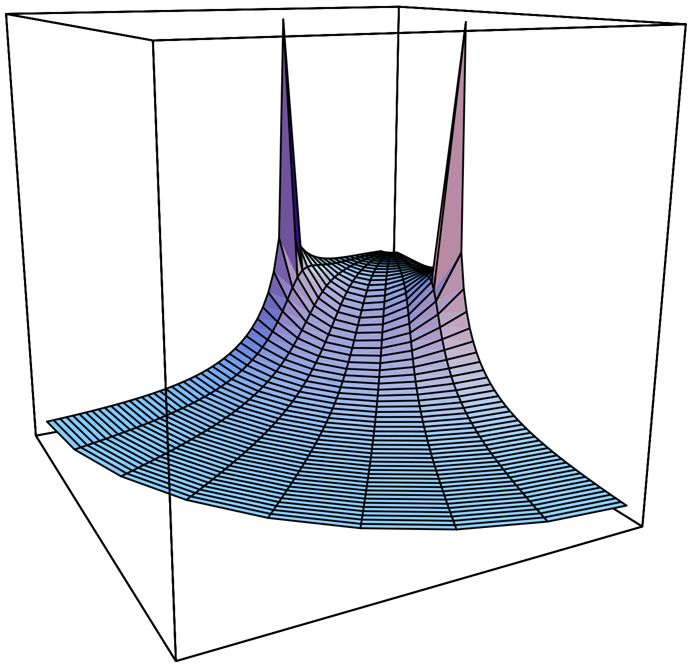


Figure 5: The plot of the metric $g_{\psi\bar{\psi}}$ against ψ in the fundamental region $0 \leq \arg\psi < 2\pi/5$

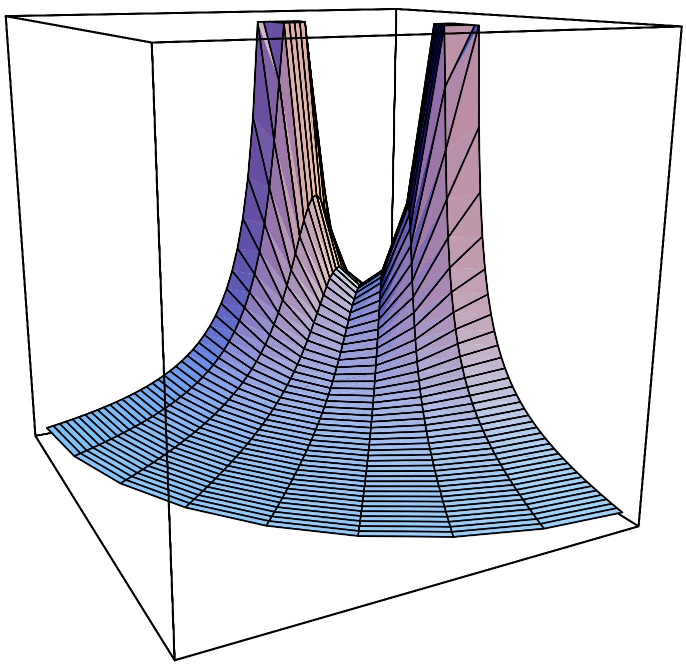


Figure 6: The plot of the curvature $R_{\psi\bar{\psi}}$ against ψ in the fundamental region $0 \leq \arg\psi < 2\pi/5$

Yukawa Couplings

Let
$$W_k = \int_Y \Omega \wedge \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\,\Omega$$
We know all W_0, W_1, W_2 vanish, and that
$$W_3 = \int_Y \Omega \wedge \frac{\mathrm{d}^3}{\mathrm{d}\lambda^3}\,\Omega$$
is the Yukawa coupling.

Claim: Let
$$\frac{\mathrm{d}^4 f}{\mathrm{d}\lambda^4} + \sum_{j=0}^{3} c_j(\lambda)\frac{\mathrm{d}^j f}{\mathrm{d}\lambda^j} = 0$$
be the Picard–Fuchs equation. Then
$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\,W_3 = -\frac{1}{2}c_3(\lambda)W_3(\lambda)$$

Proof: The claim follows by writing W_k in terms of the periods using $\Omega = z^a\alpha_a - \mathcal{G}_a\beta^a$

$\square$

Therefore

$$y_{\psi\psi\psi} = 5 \ \left(\frac{2\pi i}{5}\right)^3 \frac{\psi^2}{1-\psi^5} \frac{1}{\varpi_0^2}$$

Note that as $\psi \to \infty$

$$y_{\psi\psi\psi} \to 5 \ \left(\frac{2\pi i}{5}\right)^3$$

and that as $\psi \to 1$, $y_{\psi\psi\psi} \to \infty$.

Note also that we could have computed $y_{\psi\psi\psi} = \partial_\psi^3 \mathcal{G}$. However the result above is closed.

Chapter 7 The Mirror Map and $\mathcal{M}_{KC}(X)$ for the Quintic 3-fold

Section 7.1 Describing $\mathcal{M}^o_{KC}(X)$

Recall $X_t \in \mathbb{P}^4[5]$, where t is the complexified Kähler class parameter

$$B + i\omega = te$$

and e is the generator of $H^2(X, \mathbb{Z})$.

Let $\{w^0, w^1\}$ be homogeneous coordinates on $\mathcal{M}^o_{KC}(X)$ with

$$t = \frac{w^1}{w^0}$$

We have

$$y^o_{ttt} = \int_X e \wedge e \wedge e = 5$$

$$\mathcal{F}^o(X) = -\frac{5}{6}\,\frac{(w^1)^3}{w^0}$$

$$g^o_{t\bar{t}} = \frac{3}{4t_2^2}\,, \qquad t_2 = \Im(t)$$

$$R^o = -\frac{4}{3}$$

String Theory: If (X_t, Y_ψ) is a mirror pair, then

$$\lim_{t_2 \to \infty} \mathcal{M}^o_{KC}(X) \simeq \lim_{\psi \to \infty} \mathcal{M}_{\mathbb{C}S}(Y_\psi)$$

The space $\mathcal{M}^o_{KC}(X)$ corresponds to a "classical limit" of the theory in which t_2 is large (large radius limit) and the mirror symmetry conjecture implies that this limit should coincide with the LCSL.

This gives the first term in the mirror map

$$t \sim -\frac{5}{2\pi i} \log(5\psi) \ , \qquad ie \qquad \lambda \sim \mathrm{e}^{2\pi i t} = q$$

Note that the B-field is defined only modulo $H^2(X, \mathbb{Z})$.The natural coordinate on $\mathcal{M}^o_{KC}(X)$ is q.
Note also that $t_2 \to \infty$ is the limit $\lambda \to 0$.

We also have

$$y_{ttt} = \left(\frac{\partial \psi}{\partial t}\right)^3 y_{\psi\psi\psi}$$

This relation is exact.
The RHS in the LCSL gives

$$y_{ttt} \sim 5 + \dots$$

Lecture 13

String Theory: If (X_t, Y_ψ) is a mirror pair, then

$$\lim_{t_2 \to \infty} \mathcal{M}^o_{KC}(X) \simeq \lim_{\psi \to \infty} \mathcal{M}_{\mathbb{C}S}(Y_\psi)$$

The space $\mathcal{M}^o_{KC}(X)$ corresponds to a "classical limit" of the theory in which t_2 is large (large radius limit) and the mirror symmetry conjecture implies that this limit should coincide with the LCSL.
This gives the first term in the mirror map

$$t \sim -\frac{5}{2\pi i} \log(5\psi) \ , \qquad ie \qquad \lambda \sim \mathrm{e}^{2\pi i t} = q$$

We also have $\quad y_{ttt} = \left(\frac{\partial \psi}{\partial t}\right)^3 y_{\psi\psi\psi} \sim 5 + \dots$

Section 7.2 The Mirror Map

Given an ODE, a point $\lambda = 0$ is a point at which the monodromy is *maximally* unipotent if the monodromy matrix T is unipotent of index 4. That is

$$(T-1)^3 \neq 0 \qquad\qquad (T-1)^4 = 0$$

(The value 4 for the index is the maximum possible in our case).

This is precisely the case for the Picard–Fuchs equation for the mirror quintic 3-fold at $\lambda = 0$

The LCSL is a point of Maximal Unipotent Monodromy.

We will **define** the *mirror map* $t(\psi)$ as follows:

• Require that the monodromy around the LCSL $\lambda = 0$ corresponds to $t \to t+1$.
(recall values of t which differ by an integer give the same physics)
That is

$$t(\lambda) \to t(\lambda) + 1$$

as $\lambda \to e^{2\pi i}\lambda$.

• The fact that the LCSL is a point of maximal unipotent monodromy implies that there are unique cycles γ_0 and γ_1 such that

$$\varpi_0(\lambda) = \int_{\gamma_0} \Omega$$

is single valued near $\lambda = 0$ and

$$\varpi_1(\lambda) = \int_{\gamma_1} \Omega$$

is a period such that the *mirror map*

$$t(\lambda) = \frac{1}{2\pi i} \frac{\varpi_1(\lambda)}{\varpi_0(\lambda)}$$

changes as $t \to t+1$ under monodromy around the LCSL $\lambda = 0$.

For the mirror pair we are discussing:

$$\varpi_0(\lambda) = f_0(\lambda) \qquad \text{(single valued at } \lambda = 0\text{)}$$
$$\varpi_1(\lambda) = f_0(\lambda) \log \lambda + f_1(\lambda)$$

where

$$f_0(\lambda) = \int_{B_2} \Omega = \sum_{n=0}^{\infty} a_n \lambda^n \ , \quad |\psi| > 1$$

$$f_1(\lambda) = 5 \sum_{m=1}^{\infty} a_m (\sigma_{5m} - \sigma_m) \lambda^m \ , \quad \sigma_m = \sum_{n=1}^{m} \frac{1}{n}$$

and

$$a_n = \frac{(5n)!}{n!^5} \in \mathbb{Z}$$

Explicitly we obtain for the mirror map:

$$q \ = \ \exp(2\pi i t) \quad \text{and} \quad t \ = \ \frac{1}{2\pi i} \frac{\varpi_1(\lambda)}{\varpi_0(\lambda)} \ .$$

$$\begin{aligned} \lambda \ = \ & q + 154\, q^2 + 179139\, q^3 + 313195944\, q^4 \\ & + 657313805125\, q^5 + 1531113959577750\, q^6 \\ & + 3815672803541261385\, q^7 \\ & + 9970002717955633142112\, q^8 + \dots \ . \end{aligned}$$

All coefficients are integers.

What are these integers?

Section 7.3 A Conjecture

$$\mathcal{M}_{\mathbb{C}S}(Y_\psi) \simeq \mathcal{M}_{KC}(X_t)$$

where

$$q \;=\; \exp(2\pi i t) \quad \text{and} \quad t \;=\; \frac{1}{2\pi i}\frac{\varpi_1(\lambda)}{\varpi_0(\lambda)}$$

and

$$\mathcal{G}_Y(\psi) = \mathcal{F}_X(t) = \mathcal{F}^o(t) + \Delta\mathcal{F}(t)$$

$\mathcal{M}_{KC}(X_t)$ is a *deformation* of the special Kähler $\mathcal{M}^o_{KC}(X_t)$ (which has cubic prepotential defined by the triple intersection numbers) which preserves the Special Kähler structure.

Yukawa Coupling

$$\begin{aligned} y_{ttt} \;&=\; 5\,\left(\frac{2\pi i}{5}\right)^3 \frac{\psi^2}{\varpi_0(\psi)^2(1-\psi^5)}\left(\frac{d\psi}{dt}\right)^3 \\ &=\; 5 + \sum_{k=0}^{\infty}\frac{n_k k^3 q^k}{1-q^k} \\ &=\; 5 + 2875q + 4876875q^2 + \ldots \end{aligned}$$

Computation of $y_{\psi\psi\psi}$ together with the mirror map generates these numbers.

How do we interpret this expression?

Section 7.4 Rational Curves

Try to compute y_{ttt} without the use of mirror symmetry.

String Theory: y_{ttt} is a 3-point correlation function which is computed by a path integral

$$y_{ijk} =< V_i, V_j, V_k >= \int \mathcal{D}[\phi] \; V_i V_j V_k \; \mathrm{e}^{-S[\phi]}$$

where

$$\phi \; : \; \Sigma \longrightarrow X$$

Σ is a Riemann surface, the world sheet, which is the 2 dimensional space swept out by a string as it moves in space.

$S[\phi]$ is the action of the string non-linear σ model (the nonlinearity has to do with the curvature of X).

$$\begin{aligned} S[\phi] &= \int \left((B + i\omega)_{\mu\bar{\nu}} \frac{\partial \phi^\mu}{\partial \bar{\sigma}} \frac{\partial \phi^{\bar{\nu}}}{\partial \sigma} + \ldots \right) \mathrm{d}^2\sigma \\ &= \int_\Sigma \phi^*(B + i\omega) + \ldots \\ &= t^i \int_\Sigma \phi^*(e_i) + \ldots \end{aligned}$$

where $B+i\omega = t^i e_i$, and $\{e_i\}$ is a basis for $H^2(X, \mathbb{Z})$.

This depends only on:

1) the class $[B + iw]$

2) homotopy class h of the $Im(\phi)$ in X

So we have

$$< V_i, V_j, V_k >= \sum_h < V_i, V_j, V_k >_h$$

In fact:

$$\begin{aligned} < V_i, V_j, V_k > &= \int \mathcal{D}[\phi]\ V_i V_j V_k\ \mathrm{e}^{-S[\phi]} \\ &= \int \mathcal{D}[\phi]\ V_i V_j V_k\ \mathrm{e}^{-S_{KC}[h]-S_{top}[\phi]} \\ &= \sum_h \mathrm{e}^{-S_{top}[\phi]} \int_h \mathcal{D}[\phi]\ V_i V_j V_k\ \mathrm{e}^{-S_{KC}[h]} \end{aligned}$$

To compute the correlation functions:

1) expand around classical solutions (minima of the action)

2) quantum corrections to the Yukawa coupling (Distler-Greene):

Only from saddle points of the action, that is

$$\delta S = 0 \qquad \text{(stationary points)}$$

These are called instantons.

3) In the topological field theory: this is not an approximation, it is the exact computation

Stationary points of the action: $\delta S = 0$

$$\frac{\partial \phi^\mu}{\partial \bar{\sigma}} = 0$$

These are holomorphic embeddings of

$$\mathbb{P}^1 \longrightarrow X$$

(restricting to $\mathbb{P}^1$)

Then

$\phi(\Sigma)$ could be:

- a point in X
  ($\phi =$ constant, classical contribution)
- and algebraic curve in X
- a multiple cover of an algebraic curve in X

For $h_{11} = 1$,

$$S_{KC}[\phi] = t \int_\Sigma \phi^*(e)$$

Let $\mathcal{L}_k$ be an instanton of degree k.

$\mathcal{L}_k =$ a holomorphic embedding of degree k.

That is, $\mathcal{L}_k$ is a holomorphic embedding of

$$\mathbb{P}^1 \longrightarrow X$$

by equations of degree k characterized by $\int_{\mathcal{L}_k} e = k$.

These are also called rational curves of degree k.

Example:

Take $k = 2$ in $X \in \mathbb{P}^4[5]$

$$
\begin{aligned}
\mathbb{P}^1 &\to X \\
(u, v) &\mapsto (u^2, v^2, uv, 0, 0, 0) \\
(u, v) &\mapsto (u^2, v^2, 0, 0, 0)
\end{aligned}
$$

The second is a double cover of

$$(u, v) \mapsto (u, v, 0, 0, 0)$$

Quintic: examples of rational curves of degree 1

$$(u, -\alpha^k u, v, -\alpha^k v, 0)$$

$$
\begin{aligned}
y_{ttt} &= \sum_h < e, e, e >_h \\
&= < e, e, e >_0 + \sum_k < e, e, e >_k
\end{aligned}
$$

We have: $< e, e, e >_0 = y^o_{ttt} \in \mathbb{Z}$.

For the quintic $< e, e, e >_0 = 5$.

The sum over k is the sum over all topologically different ways a $\mathbb{P}^1$ can be embedded in X.

We have:

$$y_{ttt} = y^o_{ttt} + \sum_k N_k k^3 \mathrm{e}^{2\pi i k t}$$

where

N_k is essentially the number of rational curves of degree k but need to handle carefully the multiple covers.

The factor k^3 comes from the insertion of the vertex operators for the observables.

The factor kt in the exponent comes from the action evaluated on $\mathcal{L}_k$:

$$S[\mathcal{L}_k] = \int_{\mathcal{L}_k} (B + iw) = t \int_{\mathcal{L}_k} e = tk$$

Finally

$$y_{ttt} = 5 + \sum_k n_k \frac{k^3 q^k}{1 - q^k}\,, \quad q = \mathrm{e}^{2\pi i t}$$

where the contribution of an m-fold cover of a degree k rational curve has a prefactor $\frac{1}{m^3}$ (Aspinwall + Morrison, 93) and

$n_k =$ number of irreducible curves of degree k. VERY hard to compute.

Mathematical proof: Givental 96; Lian, Liu & Yau, 97.

$$N_1 = n_1$$
$$N_2 = n_1 + 2^3 n_2$$
$$N_3 = n_1 + 2^3 n_2 + 3^3 n_3$$

ETC

$n_1 = 2875$	Clemens 84
$n_2 = 609250$	Katz 86
$n_3 = 317206375$	Ellingsrudd&Stromme 92
ETC	

Lecture 14

Chapter 8

The Strominger-Yau-Zaslow Conjecture

OR Mirror Symmetry is T-duality

Section 8.1 Introduction

Let (X, Y) be a Mirror Pair.

In String Theory a Mirror Pair is one for which

$$\Gamma_X \text{``}=\text{''} \Gamma_Y$$

By this we mean: all correlation functions are **equal**.

But there is more: we have so far ignored "D-branes".

So far:

$\rightarrow$ Parameter spaces of the metric gives striking identities (for example, generating functions for GW invariants).

But also:

$\rightarrow$ Kontsevich: Homological Mirror Symmetry (94)
Fukaya's A_∞ Category of Lagrangian submanifolds on X the same as Bounded derived category of sheaves on Y.

We cannot begin to discuss this, however we will try to discuss the following question.

Suppose claim that (X, Y) is a mirror pair of Calabi-Yau manifolds.

Given X, how do we find Y?

$\rightarrow$ Batyrev:

Let X be a hypersurface in a toric variety described by a lattice polytope Δ_X. This polytope has certain properties which reflect the fact that X is a Calabi-Yau polytope.

Then, the mirror of Y of X is described by the dual lattice polytope dual

$$\Delta_Y = \nabla_X$$

where ∇_X is a lattice polytope with certain properties (which reflect the fact that X is a Calabi-Yau) and ∇ is the dual to Δ.

Another idea came from STY: mirror symmetry is T-duality

Idea:

$$X = T^3 \text{ fibration over } B$$

$$\updownarrow \ \mathrm{T-duality}$$

$$Y = \hat{T}^3 \text{ fibration over } \hat{B}$$

where T^3 and $\hat{T}^3$ are special lagrangian submanifolds of X and Y respectively.

Definition Let X be a Calabi-Yau 3-fold with (ω, Ω). A Special Lagrangial Submanifold $M \subset X$, is a Lagrangian submanifold with respect to ω

$$\omega|_M = 0$$

such that

$$\Im(\Omega) \equiv 0 \quad \text{on } M$$

$$\text{that is} \qquad \Re(\Omega)|_M = Vol(M)$$

In physics: "supersymmetric" 3-brane.

Conjecture

Let (X, Y) be a mirror pair of simply connected CY manifolds.

Then there is a compact 3-manifold B and projections

$$\pi : X \longrightarrow B \qquad\qquad \hat{\pi} : Y \longrightarrow B$$

with fibers T and $\hat{T}$.

Let $p \in B$, with $T_p = \pi^{-1}(p)$ and $\hat{T}_p = \hat{\pi}^{-1}(p)$.

Let $B^\flat = B \setminus D$ (dense, open), where D (the discriminant) is set of points in B for which T_p or $\hat{T}_p$ degenerate. Then: $\forall p \in B^\flat$

• T_p and $\hat{T}_p$ are non-singular SLAG 3-tori in X and Y respectively

• In the large radius limit of X and LCSL of Y, the fibrations are $\underline{T\text{-dual}}$ to each other.

Remark: CY manifolds are not bundles! (Unless it is a T^6 or $T^2 \times K3$) If there were no degenerate fibers then there would be isometries and CY manifolds have no continuous isometries.

What is T-duality?

Problems:

1) T duality is not well defined in its present version (it is not defined for $p \in D$)

2) There are general arguments (Joyce mathDG 0206016) that suggest that, if a fibration exists, then X and Y cannot have the same discriminant. Instead there is a discriminat D and $\hat{D}$ in B but they do not coincide except perhaps in the LRL and LCSL.

Section 8.2 The Physics argument

Let (X, Y) be a Mirror Pair: Γ_X " $=$ " Γ_Y

Γ_X side: "0-branes"

$$\mathcal{M}_X = X$$

regarded as the moduli space of points (0-branes) in X

Γ_Y side: "supersymmetric D3 branes on $\hat{T}$"

$\hat{T} =$ SLAG submanifold in Y with a flat $U(1)$ connection

Let $\mathcal{M}_{\hat{T}}$ be the space of deformations of $\hat{T}$ and the flat connections.

Then, there exists such $\hat{T}$ such that

$$\mathcal{M}_{\hat{T}} = \mathcal{M}_X = X$$

One can show that $\dim \mathcal{M}_{\hat{T}} = 2b_1(\hat{T})$ as both types of deformations are generated by harmonic 1-forms $\hat{T}$. (See Joyce)

Theorem (Mc Lean): Let X be a CY 3-fold and T a compact SLAG 3dim submanifold in Y. Then the moduli space of SLAG deformations of T is smooth of dimension $b_1(T)$.

Since $\dim X = 6$, then we must have $b_1(\hat{T}) = 3$.

Thus, the moduli space of $U(1)$ flat connections on a 3-cycle $\hat{T}$ with $b_1 = 3$ at a point in Y is a 3-torus (which should be a supersymmetric cycle in Y).

We have: $X = \mathcal{M}_{\hat{T}} = 3$ parameter family of SLAG 3-tori and $Y = \mathcal{M}_T$ is its mirror.

Similarly: $Y = \mathcal{M}_T = 3$ parameter family of SLAG 3-tori and $X = \mathcal{M}_{\hat{T}}$ is its mirror.

Section 8.3 T-duality

Let X be a manifold with d commuting isometries, and consider the world sheet action for the SCFT with X as the target space.

Let $\qquad Q = B + i\omega$

(as matrices, that is, $Q_{MN} = g_{MN} + B_{MN}$).

The action is

$$S = \int (Q_{MN}\partial\phi^M\partial\phi^N + \frac{\alpha'}{2}R^{(2)}\varphi(\phi) + ...)d^2\sigma$$

Let

$m, n, \ldots$	isometry directions
$a, b, \ldots$	others

The T-dual action $\hat{S}$ (which has the same form as S) is given by

$$
\begin{aligned}
\hat{Q}_{mn} &= (Q^{-1})_{mn} \\
\hat{Q}_{ab} &= Q_{ab} - Q_{am}(Q^{-1})_{mn}Q_{mb} \\
\hat{Q}_{na} &= (Q^{-1})_{nm}Q_{ma} \\
\hat{Q}_{an} &= -Q_{am}(Q^{-1})_{mn} \\
\hat{\varphi} &= \varphi - \log\det g_{mn}
\end{aligned}
$$

The target space X of S and the target space Y of $\hat{S}$ may be topologically different, however the SCFT are equivalent. Schematically

$$Q = \begin{pmatrix} J & A \\ A^T & L \end{pmatrix} \longrightarrow Q' = \begin{pmatrix} J^{-1} & J^{-1}A \\ -A^TJ^{-1} & L - A^TJ^{-1}A \end{pmatrix}$$

<u>Idea behind T-duality</u>

$$\begin{tikzcd} & M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

M is a G-bundle obtained by promoting the isometries to local symmetries

$$\partial\phi^m \longrightarrow D\phi^m = \partial\phi^m + A^m{}_n\phi^n$$

and adding a new term $\int tr(\Lambda_\alpha F^\alpha)d^2\sigma$, $\alpha = 1, \dots \dim(\mathrm{adj}G)$ where F is the curvature of A and Λ are Lagrange multipliers.

π_1: Integrate out Λ gives $F = 0$ and get back S on X.

π_2: Integrate out A give the dual theory $\hat{S}$ on Y.

(See Rocek&Verlinde 9110053, Cavalcanti&Gualtieri)

Example: $X = U(1)$ bundle over $B = S^2$

$ds^2 = a^2(d\psi + n\cos\theta d\phi)^2 + \lambda^{-1}(d\theta^2 + \sin^2\theta d\phi^2)$

For $n = 0$: $X = S^1 \times S^2$ trivial bundle

For $n = 1$: $X = S^3$ Hopf fibration

For $n > 1$ magnetic monopole with charge n.

Let $B = 0$.

Duality wrt ψ:

$$\hat{g}_{MN} = \begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1}\sin^2\theta \end{pmatrix} \quad \hat{B}_{MN} = \begin{pmatrix} 0 & n\cos\theta & 0 \\ -n\cos\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Dual theory: $Y = S^1 \times S^2$ with $\hat{B} \neq 0$ if $n \neq 0$.

For $n = 0$: $X = {S^1}_{r=a} \times S^2 \longrightarrow Y = S^1_{r=1/a} \times S^2$

Section 8.4 T-duality as Mirror Symmetry for T^2

Let $X = T^2$ with metric g.

Let (x_1, x_2) be real coordinates on X

Complex coordinates: $z = x_1 + \tau x_2$ $\bar{z} = x_1 + \bar{\tau} x_1$

In complex coordinates the metric is

$$(g_{..}) = \frac{\sqrt{g}}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}$$

so

$$\tau = \frac{g_{11}}{(g_{12} + i\sqrt{g})}$$

and the metric is hermitian.

We have

$$Q_{MN}\partial X^M \partial X^N = \\ -\frac{i}{2\tau_2}\left[(B+i\sqrt{g})\partial\bar{z}\bar{\partial}z + (-B+i\sqrt{g})\partial z\bar{\partial}\bar{z}\right]$$

Duality wrt x_1 and we obtain

$$\det\hat{g} = \frac{g_{11}^2}{\det g}$$
$$\hat{\tau} = \frac{\hat{g}_{11}}{(\hat{g}_{12}+i\sqrt{\hat{g}})} = -B + i\sqrt{g}$$
$$\tau = i\sqrt{\hat{g}} - \hat{B}$$

(Hori-Vafa: extension of this idea to topological σ models and mirror symmetry between the A and B models.)

The End

Miniprojecs:

Due date: 1st September

Length: about 20 pages

You can send me by e-mail a project or you can choose. Also, email me for more references.

1) Moduli space of K3 surfaces; mirror symmetry for K3 surfaces.

Describe the moduli space of Ricci-flat metrics in $K3$ surfaces.

Let S be a $K3$ surface.

Harvey& Lawson: there is a complex structure on S compatible with a given Ricci-flat metric such that each SLAG fibration is a T^2 fibration.

Morrison: every CY metric on S admits such fibration (arxiv 9608006)

Fortunately, due to Kodaira, there is a complete classification of singular fibers, each characterized by the conjugacy class of the monodromy action M on $H^1(T^2, \mathbb{Z})$. Then the mirror $\hat{S}$ has $(M^{-1})^T$.

2) Connectedness of moduli spaces of CY 3-folds. Given a CY 3-fold X, can one get to its mirror Y by topology changing transitions (eg conifold transitions)? The singularities allowed are those which are at finite distance (wrt the metric on the moduli space). This is an open question. However, the problem has been considered for example in the context of hypersurfaces in toric varieties. References on request.

3) After Katz (Enumerative Geometry and String Theory). (More references on request)

Define $\bar{M}_{o,n}(X,\beta)$ the space of isomorphism classes of stable maps

$$f : (C, p_1, \ldots, p_n) \longrightarrow X$$

where C is a tree of rational curves, $\beta = f_*[C]$, the points p_i are distinct from each other and they are distinct from nodes of C.

For each i, define the evaluation map

$$ev_i : \bar{M}_{o,n}(X,\beta) \longrightarrow X$$

where

$$ev_i(f : (C, p_1, \ldots, p_n) \to X) = f(p_i) \ .$$

The map ev_i evaluates f at the marked point p_i.

Define Gromov-Witten invariants for $X = \mathbb{P}^N$:

$$< \omega_1, \ldots, \omega_n >_\beta = \int_{\bar{M}_{o,n}(\mathbb{P}^N,\beta)} ev_i^*(\omega_1) \ldots ev_n^*(\omega_n)$$

Do examples (eg $\mathbb{P}^2$, etc).

Define the GW potential and compute it for examples.

For a CY manifold, say the quintic, how does the GW invariant for $n = 3$ relates to the "yukawa" coupling? What is the relation between "rational curves" and GW invariants. How are the higher genus GW invariants defined?

4) It is not known if all Ricci-flat metrics admit SLAG fibrations. It is in fact very hard to even construct SLAGs.

Study the work of Gross and Ruan: Topological Lagrangian fibrations.

For the quintic, say: relation between the prescription by Batyrev and STZ to find the mirror

Zharkov (math AG 9806091): torus fibrations of CY hypersurfaces in toric varieties

Ruan (math AG 99 and 00)

Gross (starting with math AG 99)