1) Show that a potential flow with $\mathbf{u} = \boldsymbol{\nabla}\phi$ minimises the kinetic energy over all incompressible flows in a domain V with prescribed boundary conditions for $\mathbf{u} \cdot \mathbf{n}$ on the boundary $S = \partial V$.

This "minimum energy" theorem complements the minimum dissipation theorem for Stokes flow.

2) Use the reciprocal theorem applied to (a) the flow due to a translating sphere in quiescent fluid, and (b) the flow due a point force (Stokeslet) at an arbitrary point $\mathbf{y}$ outside the sphere, to establish the Faxén law

$$\mathbf{F} = 6\pi\mu a \left[\left(1 + \frac{a^2}{6}\boldsymbol{\nabla}^2 \right) \mathbf{u}^\infty(\mathbf{x}) \Big|_{\mathbf{x}=0} - \mathbf{U} \right] \tag{1}$$

for the drag force $\mathbf{F}$ on a rigid sphere of radius a moving with velocity $\mathbf{U}$ with its centre instantaneously located at $\mathbf{x} = 0$. The flow $\mathbf{u}^\infty$ is the Stokes flow generated by forces outside the sphere that would be present without the sphere.

[You may find it easier to consider the sphere to be located at a general point $\boldsymbol{\xi}$ first. Note also that an arbitrary Stokes flow may be expressed as a superposition of responses due to point forces, and that the flow due to a point force is the limit as $a \to 0$ of the flow due to a moving sphere experiencingthe same drag force.]

3) Suppose that $\phi(\mathbf{x}, t)$ represents the concentration of single spheres (not bead-spring pairs) in a suspension. Suppose that each sphere is subject to a deterministic force derived from a potential $V(\mathbf{x})$, and to Brownian forces. Show that taking the Brownian force on each sphere to be $-k_B T \boldsymbol{\nabla}(\log \phi)$ leads to a Boltzmann distribution $\phi(\mathbf{x}, t) = \phi_0 \exp(-V(\mathbf{x})/(k_B T))$ for the concentration ϕ.

Find the equilibrium distribution ψ for sphere-spring pairs with Hookean springs and Brownian forces in a fluid with no large-scale flow (so $\boldsymbol{\nabla}\mathbf{u} = 0$) and show that the second moment of the separation $\mathbf{R}$ is

$$\langle \mathbf{R}\mathbf{R} \rangle = (k_B T/H)\mathsf{I}. \tag{2}$$

4) Starting from the Fokker–Planck equation for the distribution $\psi(\mathbf{x}, \mathbf{R}, t)$ of bead-spring pairs in lectures, show that the "conformation tensor" $\mathsf{C} = \langle \mathbf{R}\mathbf{R} \rangle = \int \mathbf{R}\mathbf{R}\psi \, \mathrm{d}\mathbf{R}$ evolves according to

$$\partial_t \mathsf{C} + \mathbf{u} \cdot \boldsymbol{\nabla}\mathsf{C} - \mathsf{C} \cdot (\boldsymbol{\nabla}\mathbf{u}) - (\boldsymbol{\nabla}\mathbf{u})^\mathsf{T} \cdot \mathsf{C} = \frac{4k_B T}{\zeta}\mathsf{I} - \frac{4H}{\zeta}\mathsf{C}. \tag{3}$$

5) Find the stress components σ_{ij} for steady shear flow, $\mathbf{u} = u(y)\hat{\mathbf{x}}$ in the standard rheological orientation, for the upper and lower convected Maxwell models, and for the stress evolution equation implied by the Boltzmann equation. Show that the "first normal stress differences" $\sigma_{11} - \sigma_{22}$ are equal, and positive, for the two Maxwell models, but negative for rarefied gases described by the Boltzmann equation.

The corrresponding positive normal stress difference in the axisymmetric version for cylindrical Couette flow with $\mathbf{u} = u(r)\hat{\boldsymbol{\theta}}$ is responsible for rod climbing in viscoelastic fluids (see chapter 1 of Renardy's book). Particle suspensions also show a negative normal stress difference (see section 7.2.2 of Guazelli and Morris 2012)

[The three stress evolution equations are collected at the start of P. J. Dellar (2014) *Lattice Boltzmann formulation for linear viscoelastic fluids using an abstract second stress*, SIAM J. Sci. Comput. **36** A2507–A2532.]

6) A steady uniaxial extensional flow has the velocity field $\mathbf{u} = (\dot{\gamma}x, -\frac{1}{2}\dot{\gamma}y, -\frac{1}{2}\dot{\gamma}z)$. Show that the equation for the stress in an upper convected Maxwell fluid in this flow has a spatially uniform steady solution of the form

$$\boldsymbol{\sigma}^P = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{22} \end{pmatrix}, \tag{4}$$

and find expressions for the coefficients.

7) Consider a linear Maxwell fluid with stress relaxation time τ and steady-state dynamic viscosity μ filling the half-space $y > 0$. Suppose the boundary $y = 0$ oscillates with tangential velocity $U \sin(\omega t)$. Show that, after an initial transient, the tangential velocity and shear stress in the fluid are given by

$$u(y,t) = U \sin(\omega t - ky) e^{-\kappa y}, \quad \sigma_{xy} = -\rho U \frac{\omega}{\sqrt{k^2+\kappa^2}} \cos\left[\omega t - ky + \tan^{-1}(k/\kappa)\right] e^{-\kappa y}.$$

where

$$k = \left(\frac{\omega}{2\nu}\right)^{1/2} \left(\sqrt{1+\omega^2\tau^2} + \omega\tau\right)^{1/2}, \quad \kappa = \frac{\omega}{2\nu k}, \quad \nu = \frac{\mu}{\rho}$$

Describe qualitatively the initial transient when the tangential velocity at the boundary is $U\Theta(t)$, where the step function $\Theta(t) = 1$ for $t \geq 0$, and $\Theta(t) = 0$ for $t < 0$.

[See I. V. Christov (2010) *Stokes first problem for some non-Newtonian fluids: Results and mistakes*, Mech. Res. Commun. **37** 717–723, and papers by Tanner (1962) and others cited therein.]

8) The A part of the resistance matrix for a very elongated spheroid with semi-axes $a_1 \gg a_2 = a_3$ is the diagonal matrix

$$\mathsf{A} = a_1 \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

with respect to the principal axes of the spheroid, where

$$\lambda_1 = \frac{4\pi}{\log(2a_1/a_2) - 1}, \quad \lambda_2 = \lambda_3 = \frac{8\pi}{\log(2a_1/a_2) - 1}.$$

Suppose this spheroid is sinking under gravity in viscous fluid. Show that the angle γ that its velocity makes with the downward vertical is related to the angle between the $\mathbf{e}_1$ axis of the body and the downward vertical by

$$2\tan(\alpha - \gamma) = \tan\alpha.$$

Hence show that the angle γ can be no larger than approximately 19.47°, whatever the orientation of the body.

[Take both angles α and γ to be between 0 and $\pi/2$.]

9) Consider a suspension of long rigid rods evolving according to Jeffery's equation. Suppose that the macroscopic stress in a suspension of these rods is

$$\overline{\boldsymbol{\sigma}} = -\overline{p}\,\mathsf{I} + \mu\left(2\mathbf{e} + N\,\mathbf{pp}\,(\mathbf{p}\cdot\mathbf{e}\cdot\mathbf{p})\right).$$

Show that the Poiseuille-like flow $\mathbf{u} = u(y)\hat{\boldsymbol{x}}$ driven by a uniform body force f is described by the coupled equations

$$\partial_t\theta = -\frac{1}{2}\left(1 - \beta\cos 2\theta\right)\partial_y u,$$
$$\partial_t u = f + \partial_y\left(\nu(\theta)\partial_y u\right),$$

where

$$\nu(\theta) = \nu_0\left(1 + N\sin^2\theta\cos^2\theta\right)$$

is the effective kinematic viscosity for rods at inclination angle θ. What boundary conditions are required?