Advanced Fluid Dynamics lecture 2

Monday, 18 January 2021 11:03

D. Reciprocal Theorem

Consider 2 Stokes flows $(\underline{u}^{(1)}, \underline{\underline{\sigma}}^{(1)})$ and $(\underline{u}^{(2)}, \underline{\underline{\sigma}}^{(2)})$ driven by body forces $\underline{f}^{(1)}$ and $\underline{f}^{(2)}$ and boundary conditions $\underline{u}^{(1)} = \underline{U}^{(1)}$ and $\underline{u}^{(2)} = \underline{U}^{(2)}$ in a volume V.

The ~~rate~~ of working of flow $\underline{u}^{(2)}$ against $\underline{f}^{(1)}$ and $\underline{U}^{(1)}$ equals the rate of working of flow $\underline{u}^{(1)}$ against $\underline{f}^{(2)}$ and $\underline{U}^{(2)}$, so

$$\int_V \underline{f}^{(1)} \cdot \underline{u}^{(2)} \, dV + \int_{\partial V} \underline{U}^{(2)} \cdot \underline{\underline{\sigma}}^{(1)} \cdot \underline{n} \, dS$$

$$= \int_V \underline{f}^{(2)} \cdot \underline{u}^{(1)} \, dV + \int_{\partial V} \underline{U}^{(1)} \cdot \underline{\underline{\sigma}}^{(2)} \cdot \underline{n} \, dS$$

why is this? The LHS is

$$\int_V f_j^{(1)} u_j^{(2)} + \partial_i (\sigma_{ij}^{(1)} u_j^{(2)}) \, dV$$

$$= \int_V \left(f_j^{(1)} + \partial_i \sigma_{ij}^{(1)}\right) u_j^{(2)} + \sigma_{ij}^{(1)} \partial_i u_j^{(2)} \, dV$$

$= \underline{f}^{(1)} + \nabla \cdot \underline{\underline{\sigma}}^{(1)}$

$= 0$

$$= \int_V \sigma_{ij}^{(1)} e_{ij}^{(2)} \, dV$$

$$= 2\mu \int_V e_{ij}^{(1)} e_{ij}^{(2)} \, dV$$

which is symmetric between flow (1) and flow (2).

This is the ~~earlier~~ dissipation result if flow (1) = flow (2).

The reciprocal theorem is the Stokes flow analogue of Green's theorem:

$$\int_V \phi \nabla^2 \psi - \psi \nabla^2 \phi \, dV = \int_{\partial V} \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \, dS$$

rewritten as

$$\int_V \phi \nabla^2 \psi \, dV - \int_{\partial V} \phi \frac{\partial \psi}{\partial n} \, dS$$

$$= \int_V \psi \nabla^2 \phi \, dV - \int_{\partial V} \psi \frac{\partial \phi}{\partial n} \, dS$$

$$= -\int_V \nabla \psi \cdot \nabla \phi \, dV$$

which is symmetric in $\psi \Leftrightarrow \phi$

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Stokes flow around a single sphere

Sphere of radius a rotating with angular velocity $\underline{\Omega}$ in unbounded fluid.

$$\mu \nabla^2 \underline{u} = \nabla p, \quad \nabla \cdot \underline{u} = 0$$

with $\underline{u} = \underline{\Omega} \times \underline{x}$ on $r = a$

$\underline{u}, p \to 0$ as $r \to \infty$

→ divergence $\Rightarrow \nabla^2 p = 0$

The spherically symmetric solution that decays as $r \to \infty$ is

$$\varphi^{(1)} = \frac{1}{r}.$$

Taking spatial derivatives gives the "solid spherical harmonics" that are also solutions of Laplace's equation:

$$\varphi^{(2)}_i = -\frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) = \frac{x_i}{r^3}$$

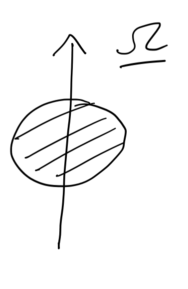
$$\varphi^{(2)}_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) = \frac{\delta_{ij}}{r^3} - 3\frac{x_i x_j}{r^5}$$

(signs chosen for convenience)

$\varphi^{(1)}$ is a scalar, $\varphi^{(2)}_i$ is a vector, $\varphi^{(3)}_{ij}$ is a symmetric, traceless tensor.

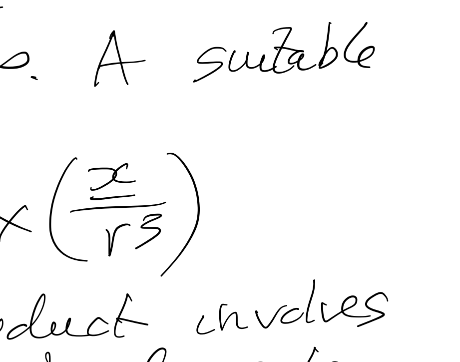
It looks like we could try

$$p(x) = \lambda \Omega_i \varphi^{(2)}_i = \lambda \frac{\underline{\Omega} \cdot \underline{x}}{r^3}.$$



We need a right hand rule to convert the pseudo-vector $\underline{\Omega}$ into a rotation

The ambiguity in the sign convention $\Rightarrow \lambda = 0$



(We could also argue that p should be symmetric in the equatorial plane.)

This leaves $\nabla^2 \underline{u} = 0$ with $\underline{u} \to 0$ as $r \to \infty$. A suitable trial solution is

$$\underline{u}(\underline{x}) = \lambda \underline{\Omega} \times \left(\frac{\underline{x}}{r^3}\right)$$

where the $\times$ product involves the same right hand rule. Choosing λ to fit the BC on $r = a$ gives

$$\underline{u}(\underline{x}) = \left(\frac{a}{r}\right)^3 \underline{\Omega} \times \underline{x}, \quad p(\underline{x}) = 0.$$

$$= O(1/r^2)$$

Translation

Same problem, but now with $\underline{u} = \underline{U}$ on $r = a$

$\underline{U}$ and $\underline{x}$ are both standard vectors, so p can be

$$p(\underline{x}) = \lambda_1 \underline{U} \cdot \frac{\underline{x}}{r^3}.$$

The velocity $\underline{u}$ satisfies

$$\mu \nabla^2 \underline{u} = \nabla p, \quad \nabla \cdot \underline{u} = 0.$$

Decompose $\underline{u} = \underline{u}^{(p)} + \underline{u}^{(h)}$ into a part driven by ∇p and a harmonic part.

$\underline{u}^{(p)} = \frac{p(\underline{x})}{2\mu} \underline{x}$ satisfies $\mu \nabla^2 \underline{u}^{(p)} = \nabla p$.

The remaining harmonic part $\underline{u}^{(h)}$ must also be linear in $\underline{U}$ so try a linear combination of $\varphi^{(1)}$ and $\varphi^{(3)}$:

$$\underline{u}^{(h)} = \lambda_2 \underline{U} \frac{1}{r} + \lambda_3 \underline{U} \cdot \left(\frac{\underline{\underline{I}}}{r^3} - 3\frac{\underline{x}\underline{x}}{r^5}\right).$$

Incompressibility $\Rightarrow$

$$0 = \nabla \cdot \left(\underline{u}^{(p)} + \underline{u}^{(h)}\right) = \left(\frac{\lambda_1}{2\mu} - \lambda_2\right) \frac{\underline{U} \cdot \underline{x}}{r^3}$$

so $\lambda_2 = \frac{\lambda_1}{2\mu}$.

Applying $\underline{u} = \underline{U}$ on $r = a$ gives, with $\underline{x} = a\underline{n}$, $\underline{n}$ the unit normal,

$$\frac{\lambda_1}{2\mu a}\left(\underline{U} + \underline{U} \cdot \underline{n}\,\underline{n}\right) + \frac{\lambda_3}{a^3} \underline{U} \cdot \left(\underline{\underline{I}} - 3\underline{n}\,\underline{n}\right) = \underline{U}$$

coeffs of $\underline{U}$ and coeffs of $\underline{U} \cdot \underline{n}\,\underline{n}$ determine $\lambda_1 = \frac{3}{2}\mu a$, $\lambda_3 = \frac{1}{4}a^3$.

The complete solution is

$$\underline{u}(\underline{x}) = \frac{3a}{4} \underline{U} \cdot \left(\frac{\underline{\underline{I}}}{r} + \frac{\underline{x}\underline{x}}{r^4}\right) + \frac{a^3}{4} \underline{U} \cdot \left(\frac{\underline{\underline{I}}}{r^3} - 3\frac{\underline{x}\underline{x}}{r^5}\right)$$

$$= \frac{3a}{4} \underline{U} \cdot \left(1 + \frac{a^2}{6}\nabla^2\right)\left(\frac{\underline{\underline{I}}}{r} + \frac{\underline{x}\underline{x}}{r^3}\right)$$

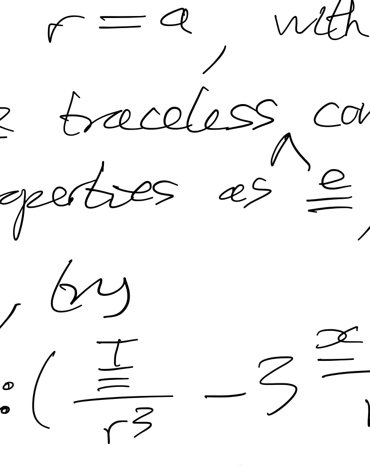
and $p(\underline{x}) = \frac{3}{2}\mu a \underline{U} \cdot \underline{x} \sim 1/r^2$

$\underline{u}$ has a part that decays like $1/r$, and a smaller "finite size" correction that is $O(a^2/r^2)$ smaller.

The response due to a point force, a dipole, comes from taking $a \to 0$ with $\underline{U}a$ finite.

Strain

Like that created by a 4 roller mill, but still consider unbounded fluid & one sphere.



Change the boundary condition to $\underline{u} = \underline{\underline{E}} \cdot \underline{x}$ on $r = a$, with $\underline{\underline{E}}$ a symmetric & traceless, constant tensor. (Same properties as $\underline{\underline{e}}$)

For the pressure, try

$$p(\underline{x}) = \lambda_0 \underline{\underline{E}} : \left(\frac{\underline{\underline{I}}}{r^3} - 3\frac{\underline{x}\underline{x}}{r^5}\right)$$

$\varphi^{(3)}$

but $\underline{\underline{E}} : \underline{\underline{I}} = \mathrm{Tr}\, \underline{\underline{E}} = 0$, leaving

$$p(\underline{x}) = \lambda_1 \frac{\underline{x} \cdot \underline{\underline{E}} \cdot \underline{x}}{r^5}.$$

As before, $\underline{u}^{(p)} = \frac{p\underline{x}}{2\mu}$ satisfies $\mu \nabla^2 \underline{u}^{(p)} = \nabla p$, and add a harmonic function

$$u_i^{(h)} = \lambda_2 E_{ij} \frac{x_j}{r^3} + \lambda_3 E_{jk}\left(\frac{\delta_{ij} x_k + \delta_{ik} x_j}{r^5} - 5\frac{x_i x_j x_k}{r^7}\right)$$

$\varphi^{(4)}$

Incompressibility $\Rightarrow \lambda_2 = 0$, and imposing $\underline{u} = \underline{\underline{E}} \cdot \underline{x}$ on $r = a$ determines

$$\lambda_1 = 5\mu a^3, \quad \lambda_3 = a^5/2 \text{ so}$$

$$u_i = \frac{5}{2} a^3 \frac{x_i x_j x_k}{r^5} E_{jk} + \frac{1}{2} a^5 \left(\frac{\delta_{ij} x_k + \delta_{ik} x_j}{r^5} - 5\frac{x_i x_j x_k}{r^7}\right) E_{jk}$$

decays like $O(1/r^2)$ with an $O(a^2/r^2)$ smaller finite-size correction.

$$p(\underline{x}) = 5\mu a^3 \frac{\underline{x} \cdot \underline{\underline{E}} \cdot \underline{x}}{r^5} \sim \frac{1}{r^3}.$$

We can find the response due to a point stress by taking $a \to 0$ with $a^3 \underline{\underline{E}}$ finite.