=1=

THM. IF X IRRED. EUCLIDEAN BUILDING (RK. ≥ 2) AND $f : X \to X$ HOMEOM. THEN f IS A HOMOTHETY.

THM. (KLEINER-LEEB) X IRRED. SYMM. SPACE OF RK ≥ 2. EVERY QUASI-ISOMETRY $g : X \to X$ IS S.T. $\exists\ f : X \to X$ HOMOTHETY WITHIN BOUNDED DIST.:

$$\forall x \in X,\ d_X(g(x), f(x)) \leq D,\ D = D(L, C, X).$$

STEP 1 $\forall$ F MAX. FLAT IN X, $\exists$ $F' \subset X$ MAX. FLAT ST.

$$d_{HAUSD}(g(F), F') \leq D = D(L, C).$$

THIS FOLLOWS FROM

THM. GIVEN $\mathcal{F} \subseteq \mathcal{F}(X)$ S.T. EVERY RESCALED ULTRALIMIT IN $\mathcal{F}$ IS AN APARTMENT ($\forall Q_n \in \mathcal{F}$, $\forall g_n \in Q_n$, $\forall \lambda_n \to +\infty$, $\lim_\omega (Q_n, g_n, \frac{1}{\lambda_n} d_X)$ IS AN APART. IN $\mathrm{CONE}_\omega(X, \xi_n, \frac{1}{\lambda_n} d_X)$) $\exists$ $D = D(\mathcal{F})$ s.t. $\forall Q \in \mathcal{F}$, $\exists$ F_Q MAX. FLAT S.T. $d_{HAUS}(Q, F_Q) \leq D$.

IN THE ABOVE, $\mathcal{F} = \{g(F) ; F \text{ MAX. FLAT}\}$.

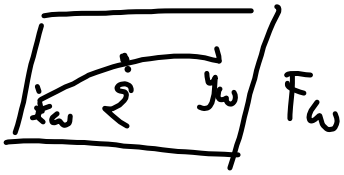
g DEFINES $g_\omega : \mathrm{CONE}_\omega(X, g_n, \lambda_n) \to \mathrm{CONE}_\omega(\dots)$ BI-LIPSCH.

= 2 =

PROOF OF THM. 2 (1) Pick $Q \in \mathcal{F}$, $q \in Q$.

GOAL: FIND F MAX. FLAT S.T $d_{Haus}(Q, F) < \infty$.

CONSIDER CONE$(X; q, n)$, IN IT $\lim_\omega Q = F_\omega$ IS A MAX. FLAT / APART.



CHOOSE A REGULAR GEOD. SEGM. THROUGH $q_\omega = \lim_\omega q \in [x_\omega, y_\omega]$.

$x_\omega = \lim(x_n)$, $y_\omega = \lim(y_n)$, $x_n, y_n \in Q$.

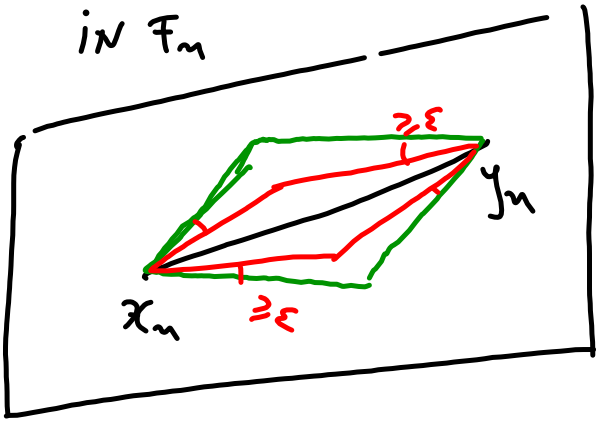
$\theta(\overline{x_\omega y_\omega}) = \theta_0 \in \partial_{INT} \Delta_{mod}$

$\theta_n = \theta(\overline{x_n y_n}) \to \theta_0$, THEN ω-ALMOST SURELY

$\theta_n \in B_{\partial_{mod}}(\theta_0, 2\varepsilon) \subseteq \partial_{int} \Delta_{mod}$

-3-

CONSIDER $[x_n, y_n] \subseteq F_n$ MAX. FLAT UNIQUELY DEF.



WE CONSIDER $J_n = \{p \in F_n :$

$\sphericalangle_{x_n}(p, \partial W_{x_n}) \geq \varepsilon$

$\sphericalangle_{y_n}(p, \partial W_{y_n}) \geq \varepsilon$

1.a. $\exists R > 0$ S.T. $D_n \subseteq \mathcal{W}_R(Q)$, $\forall n \in \mathbb{N}$.

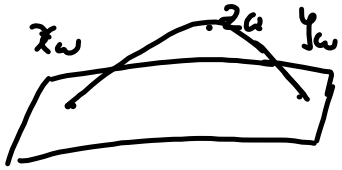
ASSUME NOT TRUE, I.E. $\min\{R ; D_n \subseteq \mathcal{W}_R(Q)\} =: R_n$,

$R_n \to +\infty$ (POSSIBLY ON A SUBSEQ.)

TAKE $z_n \in D_n$ s.t. $d(z_n, Q) = R_n$, TAKE

$\mathrm{CONE}_\omega(X ; z_n, R_n)$. BY HYP. $\lim_\omega Q = F_\omega$ MAX. APART.

$d(z_\omega, F_\omega) = 1$



$\lim D_n = D_\omega \subseteq \lim F_n = F_n'$.

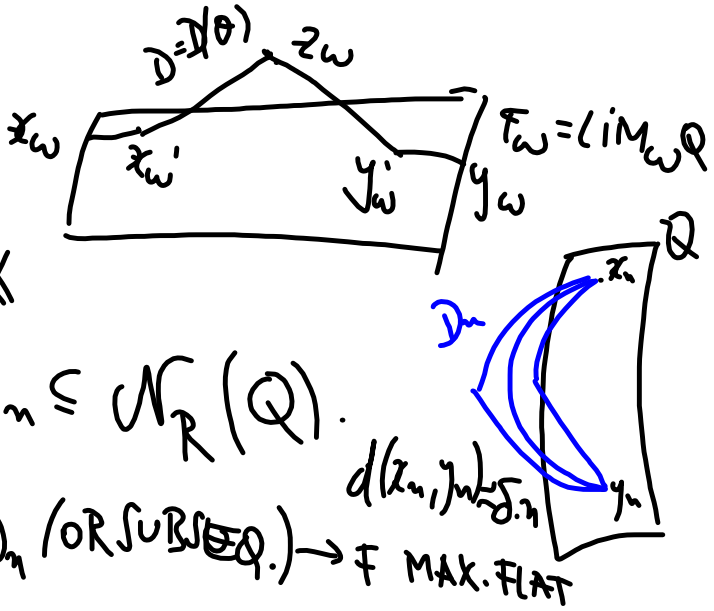
BY CHOICE OF R_n, $D_\omega \subseteq \mathcal{W}_1(F_\omega)$

$\Leftarrow 4 \Leftarrow$

$d(z_n, x_n) \geq R_n \Rightarrow \lim_\omega \frac{d(z_n, x_n)}{R_n} \geq 1$, BUT

COULD BE $\infty \Rightarrow \lim_\omega [z_n, x_n]$ IS EITHER A GEOD. SEGM. OF LENGTH ≥ 1 OR A GEOD. RAY

$\int_1 \subseteq \mathcal{N}_1(F_\omega)$.



COLLAPSING LEMMA:

SAME FOR $\lim_\omega [z_n, y_n]$ ⨯

THUS $\exists R > 0$ S.T. $D_n \subseteq \mathcal{N}_R(Q)$.

APPLY ARZELA-ASCOLI, D_n (OR SUBSEQ.) $\to$ F MAX. FLAT

S.T. $F \subseteq \mathcal{N}_R(Q)$.

1.6 $\exists R' > 0$ S.T. $Q \subseteq \mathcal{N}_{R'}(F)$.

OTHERWISE $\exists \xi_n \in Q$ s.t. $d(\xi_n, F) = d_n \to +\infty$

TAKE $\mathrm{CONE}(X; \xi_n, d_n)$, $\lim_\omega Q = F'_\omega$ APART.

$\lim_\omega F = F_\omega$ APART. $F \subseteq \mathcal{N}_R(Q) \Rightarrow F_\omega \subseteq F'_\omega$

BUT $\lim \xi_n = \xi_\omega \in F'_\omega$, $d(\xi_\omega, F_\omega) = 1$.

BUT $F_\omega \subseteq F'_\omega$, $F_\omega, F_\omega \simeq \mathbb{R}^k$ $\Rightarrow F_\omega = F'_\omega$ ⨯

$\Rightarrow$

THUS, $\forall Q \in \mathcal{F}$, $\exists F_Q$ MAX. FLAT S.T.

$$d_{HAUS}(Q, F_Q) = \delta_Q < +\infty.$$

Step 2 ASSUME $\exists Q_n$ IN $\mathcal{F}_n$ S.T. $\delta_{Q_n} \to +\infty$

TAKE x_n EITHER IN Q_n S.T. $d(x_n, F_{Q_n}) = \delta_{Q_n}$

OR IN F_{Q_n}, S.T. $d(x_n, Q_n) = \delta_{Q_n}$.

ASSUME THAT 1ST CASE HAPPENS FOR ∞ MANY n.

IN $\mathrm{CONE}(X, x_n, \delta_{Q_n})$: $\lim Q_n = F'_\omega$, $\lim F_{Q_n} = F_\omega$

$$d_{HAUS}(F_\omega, F'_\omega) = 1, \quad \exists x_\omega \in F'_\omega, d(x_\omega, F_\omega) = 1.$$

IMPOSSIBLE IN A E.B. $\ast\ast$.

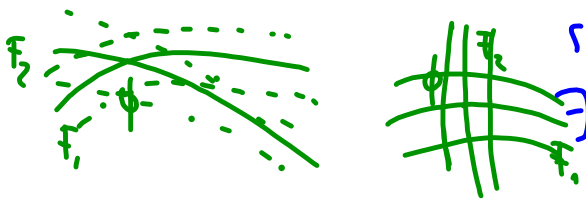
QED TM. 1

WE NOW CONTINUE WITH PF. OF MAIN THM.

LEMMA (GEOM. FEATURE OF SYMM. SP.)

$\forall \phi$ SINGULAR FLAT, $\forall R > 0$, $\exists D = D(R)$ S.T.

$$\phi \subseteq \bigcap_{\substack{\Phi \subseteq F \\ F \text{ MAX.FL.}}} \mathcal{N}_R(F) \subseteq \mathcal{N}_D(\phi)$$



COROLLARY $\exists D' = D(L, C)$

S.T. $\forall F$ MAX. FLAT OR SING. FLAT,

$\exists F'$ —||— (SAME DIM.) S.T.

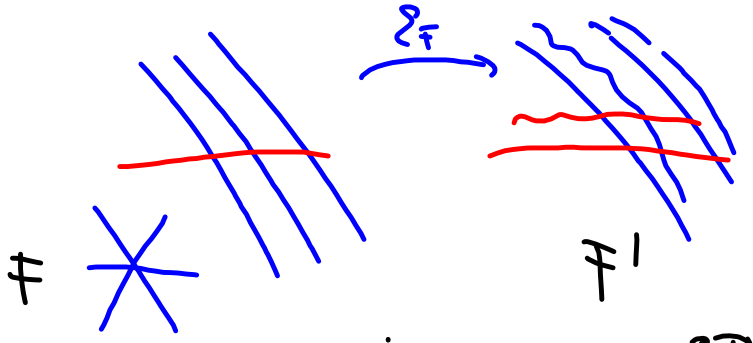
$\mathrm{dist}_H(g(F), F') \leq D'$.

THUS, WE HAVE $\forall F$ MAX. FLAT, $\exists F'$ MAX. FLAT S.T. $\varrho(F)$ IS WITHIN $d_{Haus} \leq D$ OF $F' \rightarrow$

$\rightarrow \varrho_F = proj_{F'} \circ \varrho|_F : F \to F'$ Q.I.

S.T. $\forall H$ SINGULAR HYP., $\varrho_F(H)$ IS WITHIN HAUSD DIST. $\leq D'$ FROM H' SINGULAR HYP.

EX. $n=2$



BY ADAPTING THE ARG. IN E.B. WE OBTAIN $\exists$ A HOMOTHETY $h_F : F \to F'$ S.T. $d(\varrho_F(x), h_F(x)) < +\infty$

$\Rightarrow$ THE FACTOR $h_F = \lambda_F \in \mathbb{L}$.

IF $F_1 \cap F_2$ HAVE ∞ INTERS. $\Rightarrow \lambda_{F_1} = \lambda_{F_2}$.

$\forall F_1', F_2'$ MAX. FLATS, $\exists F_1 = F_1', F_2 \ldots, F_n = F_2'$ S.T. $F_i \cap F_{i+1} \infty$

$\Rightarrow \lambda_{F_1'} = \lambda_{F_2'} = \lambda$.

FROM THIS WE DEDUCE THAT IF WE TAKE

$\varrho : (X, d_X) \to (X, \frac{1}{\lambda} d_X)$ THEN

ϱ IS A $(1, C)$-Q.I.

$= 7 =$

TO FINISH THE PROOF WE USE

THM. (J. TITS) GIVEN X SYMM. SP., IRRED., RK. ≥ 2

IF $f : \partial_\infty X \to \partial_\infty X$ IS A SIMPLICIAL ISOMORPHISM

OF SPHERICAL BUILDINGS AND f HOMEO/CONE TOP.

THEN $\exists\, g \in \mathrm{Isom}_0(X) = G$ S.T. $f = \partial_\infty g$.

WE ESTABLISHED A BIJECTION

$$\{\text{MAX. FLATS, SING. FLATS}\} \to \{\dots\}$$

$$F \mapsto F' \text{ S.T. } d_H(\xi(F), F') \leq D.$$

THEN $\exists$ Bij. $\{\text{WEYL CH.}\} \to \{\text{WEYL CH.}\}$

$$W \mapsto W' \text{ S.T. } d_H(\xi(W), W') \leq D$$

+ PRESERVES TYPE OF INCIDENCE.

$\rightsquigarrow$ CONSTRUCT $\partial_\infty \xi : \partial_\infty X \to \partial_\infty X$ SIMPLICIAL ISOM.

+ HOMEOM / CONE TOP.

$\Rightarrow \exists\, g : X \to X$ ISOM. S.T. $\partial_\infty g = \partial_\infty \xi$.

$\forall x \in X$, $x = \bigcap_{\substack{x \in F \\ F \text{ MAX. FL.}}} F \Rightarrow \xi(x) \in \bigcap_{x \in F} \xi(F) \subseteq \bigcap_{F'} \mathcal{N}_D(F')$

$$= \bigcap_{x \in F} \mathcal{N}_D(gF) = g\left(\bigcap_{x \in F} \mathcal{N}_D(F)\right) \underset{\text{LEMMA}}{\subseteq} g\left(\mathcal{N}_D(x)\right) = \mathcal{N}_D(gx).$$

QED

$=\beta=$

THUS $\forall g: X \to X$ p.i. $\exists\, g \in Isom_0(X)$ within BOUNDED DIST.

COROLLARY $\forall\ \Gamma$ f.g. Q.I. To Λ UNIF. LATTICE IN $X \Rightarrow \exists$ SH. EX. SEQ.

$$1 \to \text{FINITE} \longrightarrow \Gamma \longrightarrow \Lambda' \to 1\ ,\quad \Lambda' \leq Isom_0(X)$$

UNIF. LATTICE

SKETCH PF. Γ Q.I. X : $\xi: \Gamma \to X$, $\bar{\xi}: X \to \Gamma$

$\Gamma \curvearrowright \Gamma$ isom. $\rightsquigarrow \Gamma \curvearrowright X$ BY Q.I. $\rightsquigarrow \Gamma \to Isom(X)$

$g \mapsto L_g$ $\quad g \mapsto \xi \cdot L_g \cdot \bar{\xi}$ $\quad g \mapsto f_g$

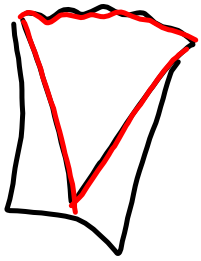
=9=

NON-UNIFORM LATTICES

EX. $\Lambda = SL(n,\mathbb{Z})$, $n \geq 3$.

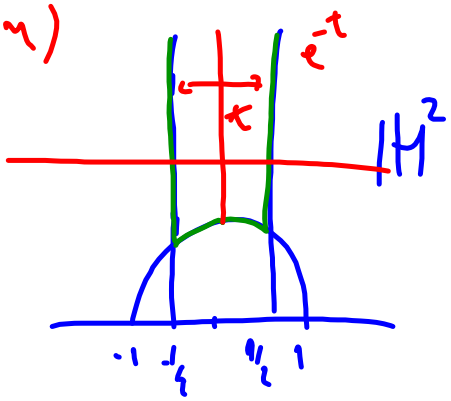
$X = \mathcal{P}_n = SL(n,\mathbb{R})/SO(n)$

$\Lambda \backslash X$ =

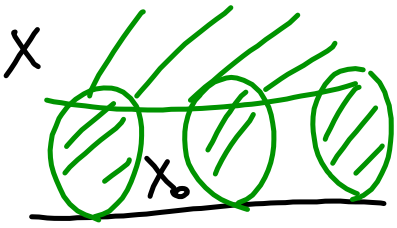


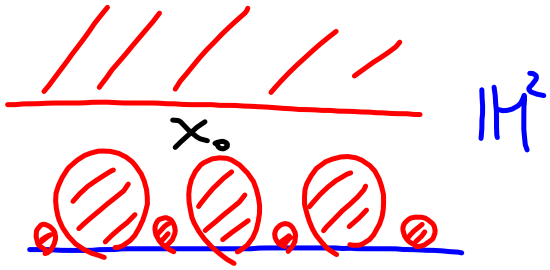
$m = ?$

$\mathbb{H}^2 / SL(2,\mathbb{Z})$

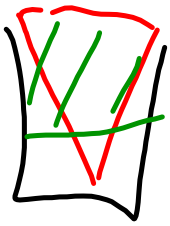


Λ NOT Q.I. TO X. WE CUT CUSP $\rightarrow X_0 \subseteq X$, X_0 Q.I Λ





$\Lambda \backslash X$



$=9=$

THM. (SCHWARTZ – ESKIN)

(1) LET Λ NON-UNIF. LATTICE $\in ISOM_0(X)$, X SYMM., IRRED., RK. 2.

$\forall\ \Gamma$ f.g. QI TO Λ, $\exists$ SH. EK. SEQ.

$1 \to$ FINITE $\to \Gamma \to \Lambda_1 \to 1$, Λ_1 NON-UNIF. LATTICE $\in ISOM_0(X)$

(2) IF $\Lambda, \Lambda_1 \leq ISOM_0(X)$ NON-UNIF. LATTICE, Q.I., THEN $\exists g \in ISOM_0(X)$ S.T. $g\Lambda g^{-1} \cap \Lambda_1 \leq \Lambda_1$ fin. INDEX $g\Lambda g^{-1}$

COR. $\forall \Lambda$ AS ABOVE, $QI(\Lambda) = COMM_G(\Lambda)$. (E.G. $SL(n,\mathbb{Q})$ FOR $SL(n,\mathbb{Z})$).

FOLLOWS FROM

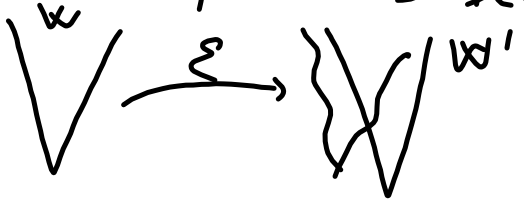
THM. $\forall\ \varphi : X_0 \to X_0$ Q.I. $\exists\ g \in COMM_G(\Lambda)$ s.t.

$$d(\varphi(x), g(x)) \leq D = D(L, C).$$

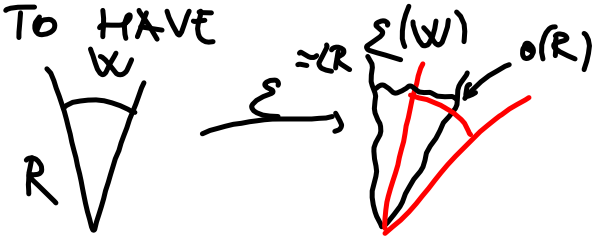
=10=

KEY INGREDIENTS IN PROOF:

- WE AIM AT USING TITS THEOREM: AT CREATING $\partial_\infty g : \partial_\infty X \to \partial_\infty X$
- IN PREVIOUS CASE, WORKED BECAUSE:



ENOUGH TO HAVE



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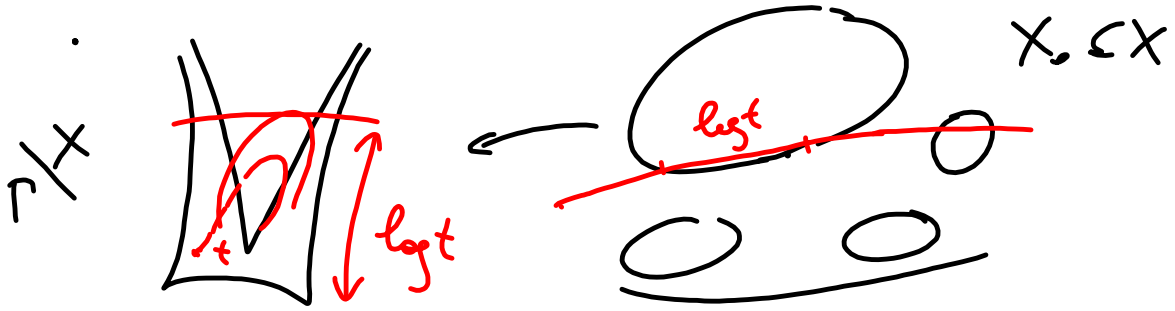
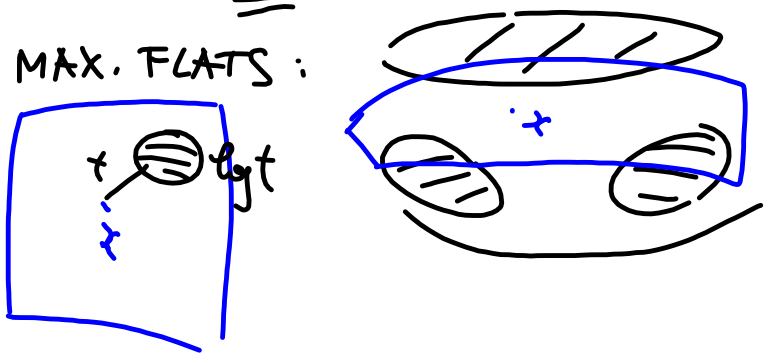
THIS IS ENOUGH BECAUSE :



ERGODIC TH. ⇒ A.E GEOD. RAY DIVERGES IN CUSP LOGARITHMICALLY

IN MAX. FLATS :



USING AS. CONES WE ESTABLISH A BIJ.

{FULL MEAS. SET OF MAX. FLATS} → {—||—}

$F \longrightarrow F'$ S.T.

$\xi(F \cap X_0)$ AND F' HAVE SAME HORIZON

⇒ {FULL MEAS. SET OF WEYL CH.} → {—||—} :

$W \longmapsto W'$ S.T. $\xi(W \cap X_0)$, W' SAME HORIZON

⇒ $\partial_\infty \xi$: {FULL MEAS. SET OF CH. IN $\partial_\infty X$} → {...}

+ UNIF. CONT. ⇒ CAN BE EXT. $\partial_\infty \xi$: ALL CH. Q. ED