# The Banach-Tarski Paradox, the von Neumann-Day conjecture

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# Last lecture

- The Banach-Tarski paradox: Any two bounded subsets in $\mathbb{R}^n$ ($n \geq 3$) with non-empty interior are piecewise $\mathrm{Isom}(\mathbb{R}^n)$–congruent.
- Given $G \curvearrowright X$, two subsets $A, B$ of $X$ are piecewise $G$-congruent if

  $$A = A_1 \sqcup \ldots \sqcup A_k, \quad B = B_1 \sqcup \ldots \sqcup B_k$$

  such that $B_i = g_i A_i$ for every $i \in \{1, 2, \ldots, k\}$.
- Paradoxical subset $Y$ of metric space $X$ (group $G$)= subset piecewise $\mathrm{Isom}(X)$–congruent ($G$–congruent) with several copies of itself.
- The free group of rank 2, $F_2$, is paradoxical.
- $SO(3)$ (and $SO(n)$, $n \geq 3$) contains copies of $F_2$. This and the Axiom of Choice $\Rightarrow$ the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, $n \geq 3$, is $SO(n)$–paradoxical.
- The above and the fact that $\mathbb{S}^{n-1}$ is $SO(n)$-congruent with $\mathbb{S}^{n-1} \setminus C$, $C$ countable $\Rightarrow$ the unit ball in $\mathbb{R}^n$ is $\mathrm{Isom}(\mathbb{R}^n)$–paradoxical.

The Banach-Tarski paradox inspired J. von Neumann to define amenability.

For groups, this property is the negation of being paradoxical.

The initial definition of von Neumann (for groups) was in terms of invariant means.

- We begin with equivalent metric definitions for graphs,
- then move on to groups and add specific definitions using means.

# Cheeger constant

Convention: All graphs $\mathcal{G}$ are connected, unoriented, and have bounded geometry: valency of vertices uniformly bounded.
All edges have length 1.
adjacent vertices = endpoints of one edge.

$F \subset V = V(\mathcal{G})$ set of vertices in a graph $\mathcal{G}$.

vertex-boundary of $F$, $\partial_V F$ = set of vertices in $V \setminus F$ adjacent to vertices in $F$.

Cheeger constant or Expansion Ratio of $\mathcal{G}$:

$$h(\mathcal{G}) = \inf \left\{ \frac{|\partial_V F|}{|F|} \ : \ F \text{ finite non-empty subset of } V \right\} .$$

amenable graph $=$ Cheeger constant zero.

Equivalently, $\exists\ F_n$ non-empty finite in $V$ such that

$$\lim_{n\to\infty} \frac{|\partial_V F_n|}{|F_n|} = 0\,.$$

$(F_n) =$ Følner sequence for the graph.

non-amenable graph$=$ positive Cheeger constant or empty graph.

Finite graphs are amenable: take $F_n = V$.

# Notation

Let $(X, \mathrm{dist})$, $F$ subset of $X$ and $C > 0$:

$$\overline{\mathcal{N}}_C(F) = \{x \in X : \mathrm{dist}(x, F) \leq C\}, \quad \mathcal{N}_C(F) = \{x \in X : \mathrm{dist}(x, F) < C\}.$$

$\mathcal{B}(X) :=$bounded perturbations of the identity, i.e. maps $f : X \to X$ such that

$$\mathrm{dist}(f, id_X) = \sup_{x \in X} \mathrm{dist}(f(x), x) \text{ is finite.}$$

**Lemma**

*In a group with a word metric, $\mathcal{B}(G)$ consists of piecewise right translations: given $f \in \mathcal{B}(G)$ there exist $h_1, \dots, h_n$ in $G$ and a decomposition $G = T_1 \sqcup T_2 \sqcup \dots \sqcup T_n$ such that $f$ restricted to $T_i$ coincides with $R_{h_i}(x) = xh_i$.*

TFAE:

(a) $\mathcal{G}$ is non-amenable.

(b) (expansion condition): $\exists C > 0$ such that for every finite $F \subset V$, $|\overline{\mathcal{N}}_C(F)| \geq 2|F|$.

(c) $\exists\ f \in \mathcal{B}(V)$ such that $\forall v \in V$, $f^{-1}(v)$ contains exactly two elements.

(d) (Gromov's condition) $\exists\ f \in \mathcal{B}(V)$ such that $\forall v \in V$, $f^{-1}(v)$ contains at least two elements.

Consequence: the Cayley graph of $F_2$ with respect to $S = \{a^{\pm 1}, b^{\pm 1}\}$ is non-amenable.

# Slight variation

**Remark**

*Property (b) can be replaced by (b'): for some* $\beta > 1$ *there exists* $C > 0$ *such that* $|\overline{\mathcal{N}}_C(F) \cap V| \geq \beta|F|$. *Indeed*

$$\forall F, |\overline{\mathcal{N}}_C(F)| \geq \alpha|F| \Rightarrow \forall k \in \mathbb{N}, \quad |\overline{\mathcal{N}}_{kC}(F)| \geq \alpha^k|F|.$$

# Reminder Graph theory

Bipartite graph $=$ vertex set $V = Y \sqcup Z$, edges with one endpoint in $X$, one in $Y$.

Given two integers $k, l \geq 1$, a perfect $(k, l)$–matching$=$ a subset of edges such that each vertex in $Y$ is the endpoint of exactly $k$ edges in $M$, while each vertex in $Z$ is the endpoint of exactly $l$ edges in $M$.

### Theorem (Hall-Rado matching theorem)

*A bipartite graph of bounded geometry such that:*

- *For every finite subset $A \subset Y$, its vertex-boundary $\partial_V A$ contains at least $k|A|$ elements.*
- *For every finite subset $B$ in $Z$, its vertex-boundary $\partial_V B$ contains at least $|B|$ elements.*

*has a perfect $(k, 1)$–matching.*

# Amenability and growth

A growth function of a graph $\mathcal{G}$ with a basepoint $x \in V$

$$\mathfrak{G}_{\mathcal{G},x}(R) := \left|\bar{B}(x,R) \cap V\right| ,$$

where $\bar{B}(x,R)$ is the closed $R$-ball centered at $x$.

Dependence on the choice of x up to asymptotic equivalence.

asymptotic inequality between $f, g : X \to \mathbb{R}$ with $X \subset \mathbb{R}$:
$f \preceq g$ if there exist $a, b > 0$ such that $f(x) \leq ag(bx)$ for every $x \in X$, $x \geq x_0$ for some fixed $x_0$.

$f$ and $g$ are asymptotically equal ( $f \asymp g$ ) if $f \preceq g$ and $g \preceq f$.

**Exercise**

- *If* $f : \mathcal{G} \to \mathcal{G}'$ *is a quasi-isometry then* $\mathfrak{G}_{\mathcal{G},x} \asymp \mathfrak{G}_{\mathcal{G}',f(x)}$.
- $\mathfrak{G}_{\mathcal{G},x} \asymp \mathfrak{G}_{\mathcal{G},x'}$ *for all* $x, x' \in V$ .

Consequence= the growth function of a group well defined up to $\asymp$.

# Amenability and growth II

1. If $G = \mathbb{Z}^k$ then $\mathfrak{G}_G \asymp x^k$.
2. If $G = F_2$ then $\mathfrak{G}_G(n) \asymp e^n$.
3. If $G$ is nilpotent then $\mathfrak{G}_G(n) \asymp n^d$ . (Bass' Theorem)

Construct inductively:

$$C^1 G = G\,,\ C^{n+1} G = [G, C^n G]\,.$$

The lower central series of $G$ is

$$G \geq C^2 G \geq \cdots \geq C^n G \geq C^{n+1} G \geq \ldots$$

$G$ is ($k$-step) nilpotent if there exists $k$ such that $C^{k+1} G = \{1\}$. The minimal such $k$ is the class of $G$.

Examples

1. An abelian group is nilpotent of class 1.
2. The group of upper triangular $n \times n$ matrices with 1 on the diagonal is nilpotent of class $n - 1$.

# Amenability and growth III

1. The growth function of a group is sub-multiplicative:

$$\mathfrak{G}_S(r+t) \leq \mathfrak{G}_S(r)\mathfrak{G}_S(t)\,.$$

2. If $|S| = k$ then $\mathfrak{G}_S(r) \leq k^r$ .

3. the limit

$$\gamma_S = \lim_{n\to\infty} \mathfrak{G}_S(n)^{\frac{1}{n}}\,,$$

exists, called growth constant.

# Amenability and growth IV

If $\gamma_S > 1$ then $G$ is said to be of exponential growth.

If $\gamma_S = 1$ then $G$ is said to be of sub-exponential growth.

A graph $\mathcal{G}$ is of sub-exponential growth if for some basepoint $x_0 \in V$

$$\limsup_{n\to\infty} \frac{\ln \mathfrak{G}_{x_0,X}(n)}{n} = 0\,.$$

For every other basepoint $y_0$, $\mathfrak{G}_{y_0,X}(n) \leq \mathfrak{G}_{x_0,X}\left(n + \mathrm{dist}(x_0, y_0)\right)$ , hence definition independent of the choice of basepoint.

# Amenability and growth V

**Proposition**

*A non-empty graph $\mathcal{G}$ of bounded geometry and sub-exponential growth is amenable: for every $v_0 \in V$ there exists a Følner sequence consisting of metric balls with center $v_0$.*

# Amenability and quasi-isometry I

**Theorem (R. Brooks)**

*Let $M$ be a complete connected n-dimensional Riemannian manifold and $\mathcal{G}$ a graph, both of bounded geometry. Assume that $M$ is quasi-isometric to $\mathcal{G}$. Then the Cheeger constant of $M$ is strictly positive if and only if $\mathcal{G}$ is non-amenable.*

Riemannian manifold of bounded geometry=uniform upper and lower bounds for the sectional curvature

Cheeger constant for $M$: infimum over $h > 0$ such that for every open submanifold $\Omega \subset M$ with compact closure and smooth boundary,

$$Area(\partial\Omega) \geq h\, Vol(\Omega)\,.$$

particular case=when $M$ universal cover of a compact Riemannian manifold $C$ and $\mathcal{G}$ Cayley graph of the fundamental group of $C$.

# Amenability and quasi-isometry II

Theorem (Graph amenability is QI invariant)

*Suppose that $\mathcal{G}$ and $\mathcal{G}'$ are quasi-isometric graphs of bounded geometry. Then $\mathcal{G}$ is amenable if and only if $\mathcal{G}'$ is.*