The Banach-Tarski Paradox, the von Neumann-Day conjecture

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Last lecture

- Convention: All graphs $\mathcal{G}$ are connected, unoriented, and have bounded geometry: valency of vertices uniformly bounded.
- The following equivalence was proved for a graph $\mathcal{G}$:
  (a) $\mathcal{G}$ is non-amenable (i.e. positive Cheeger constant).
  (b) (expansion condition): $\exists C > 0$ such that for every finite $F \subset V$, $|\overline{\mathcal{N}}_C(F)| \geq 2|F|$.
  (c) $\exists\, f \in \mathcal{B}(V)$ such that $\forall v \in V$, $f^{-1}(v)$ contains exactly two elements.
  (d) (Gromov's condition) $\exists\, f \in \mathcal{B}(V)$ such that $\forall v \in V$, $f^{-1}(v)$ contains at least two elements.
- A non-empty graph $\mathcal{G}$ of sub-exponential growth is amenable.
- R. Brooks Theorem: Let M be a complete connected n-dimensional Riemannian manifold and $\mathcal{G}$ a graph, both of bounded geometry. Assume that M is quasi-isometric to $\mathcal{G}$. Then the Cheeger constant of M is strictly positive if and only if $\mathcal{G}$ is non-amenable.
- If $\mathcal{G}$ and $\mathcal{G}'$ are quasi-isometric graphs then $\mathcal{G}$ is amenable if and only if $\mathcal{G}'$ is.

More on quasi-isometries

Amenability is also relevant for the quasi-isometry versus bi-Lipschitz problem.
To explain the problem, more on quasi-isometries:

A subset N in a metric space X is called

- a net if there exists $\varepsilon > 0$ such that $X = \mathcal{N}_\varepsilon(N)$;
- separated if there exists $\delta > 0$ such that $d(x, y) \geq \delta, \forall x, y \in N$.

Exercise: X, Y are quasi-isometric if and only if there exist $N_X \subset X$ and $N_Y \subset Y$ separated nets and $f : N_X \to N_Y$ bi-Lipschitz bijection, that is, for some $L \geq 1$,

$$\frac{1}{L} d(x, y) \leq d(f(x), f(y)) \leq L d(x, y).$$

Gromov's question

Question (Gromov)

When are two quasi-isometric spaces actually bi-Lipschitz equivalent ?

Examples: Finitely generated groups, separated nets in Euclidean spaces.

Theorem (K. Whyte)

Let $\mathcal{G}_i, i = 1, 2,$ be two non-amenable graphs (of bounded geometry). Then every quasi-isometry $\mathcal{G}_1 \to \mathcal{G}_2$ is at bounded distance from a bi-Lipschitz map.

Other results

Corollary (K. Whyte)

Two non-amenable quasi-isometric groups are bi-Lipschitz equivalent.

Theorem (P. Papasoglu)

Two free groups F_n and F_m, $n, m \geqslant 2$, are bi-Lipschitz equivalent.

Burago-Kleiner, McMullen: examples of separated nets in $\mathbb{R}^2$ not bi-Lipschitz equivalent.

T. Dymarz: examples of amenable groups (lamplighter groups) quasi-isometric, not bi-Lipschitz equivalent.

Hall-Rado Theorem again

Bipartite graph = vertex set $V = Y \sqcup Z$, edges with one endpoint in X, one in Y.

Given two integers $k, l \geq 1$, a perfect (k, l)–matching= a subset of edges such that each vertex in Y is the endpoint of exactly k edges in M, while each vertex in Z is the endpoint of exactly l edges in M.

Theorem (Hall-Rado matching theorem)

A bipartite graph of bounded geometry such that:

- *For every finite subset $A \subset Y$, its vertex-boundary $\partial_V A$ contains at least $k|A|$ elements.*
- *For every finite subset B in Z, its vertex-boundary $\partial_V B$ contains at least $|B|$ elements.*

has a perfect $(k, 1)$–matching.

Lemma

Let $\mathcal{G}$ be a nonamenable graph (of bounded geometry). For each net $V' \subset V = V(\mathcal{G})$, there exists a bijection $f : V' \to V$ which is a bounded perturbation of the inclusion $V' \to V$: there exists $D < \infty$ such that

$$\mathrm{dist}(x, f(x)) \leqslant D$$

for all $x \in V'$.

Assume all valence of the graph $\mathcal{G}$ is at most $m \in \mathbb{N}$.

Assume $V \subset \mathcal{N}_r(V')$.

$\mathcal{G}$ nonamenable $\Rightarrow \exists C > 0$ such that for every finite $\Phi \subset V$,

$$|\overline{\mathcal{N}}_C(\Phi) \cap V| \geq m^{2r} \cdot |\Phi|.$$

Take $D := C + 2r$ and the bipartite graph $Bip_D(V', V)$.

Clearly, for every finite subset $A \subset V'$, $|\partial_V A| \geqslant |A|$.

Let B be a finite subset in V.

$\partial_V B = \overline{\mathcal{N}}_{C+2r}(B) \cap V'$

(1) Let $B' = V' \cap \overline{\mathcal{N}}_r(B)$. Since $B \subset \overline{\mathcal{N}}_r(B')$, $|B| \leqslant m^r |B'|$.

(2) $|\overline{\mathcal{N}}_C(B') \cap V| \geqslant m^{2r}|B'| \geq m^r|B|$.

(3) $|\overline{\mathcal{N}}_{C+r}(B') \cap V'| \geqslant \frac{1}{m^r}|\overline{\mathcal{N}}_C(B') \cap V| \geq |B|$.

(4) $\overline{\mathcal{N}}_{C+r}(B') \cap V' \subset \overline{\mathcal{N}}_{C+2r}(B) \cap V'$

The map f in the Lemma is $(2D+1)$–bi-Lipschitz:

$$\mathrm{dist}(f(a), f(b)) \leq \mathrm{dist}(a, b) + 2D \leq (2D+1)\mathrm{dist}(a, b);$$

$$\mathrm{dist}(a, b) \leq \mathrm{dist}(f(a), f(b)) + 2D \leq (2D+1)\mathrm{dist}(f(a), f(b)).$$

Proof of K. Whyte Theorem:
There exist V_i' separated nets in $V(\mathcal{G}_i)$, $i = 1, 2$, and a bi-Lipschitz bijection $h' : V_1' \to V_2'$.

Lemma implies the existence of bi-Lipschitz bijections

$$f_i : V_i' \to V(\mathcal{G}_i), \quad i = 1, 2,$$

The composition

$$g := f_2 \circ h' \circ f_1^{-1} : V(\mathcal{G}_1) \to V(\mathcal{G}_2)$$

is the required bi-Lipschitz map. □

At the core if this course is the discussion of:

Conjecture (von Neuman-Day conjecture)

Is every finitely generated group either amenable or containing a free non-abelian subgroup ?

Theorem (K. Whyte)

Let $\mathcal{G}$ be an infinite graph (of bounded geometry). The graph $\mathcal{G}$ is non-amenable if and only if there exists a free action of F_2 on $\mathcal{G}$ by bi-Lipschitz maps which are bounded perturbations of the identity.

Amenability for groups

A mean on a set X = a linear functional $m : \ell^\infty(X) \to \mathbb{C}$ s.t.

(M1) if f takes values in $[0, \infty)$ then $m(f) \geq 0$;

(M2) $m(\mathbf{1}_X) = 1$.

TFAE in a group G

1. there exists a mean m on G invariant by left multiplication.
2. there exists a finitely additive probability measure μ on $\mathcal{P}(G)$, the set of all subsets of G, invariant by left multiplication.

A group G is amenable if any of the above is true.

Left, right or both

Proposition

(a) *Invariant by left multiplication (left-invariance) can be replaced by right-invariance.*

(b) *Moreover, both can be replaced by bi-invariance.*

Proof.

(a) It suffices to define $\mu_r(A) = \mu(A^{-1})$ and $m_r(f) = m(f_1)$, where $f_1(x) = f(x^{-1})$.

(b) Let μ be a left-invariant f.a.p. measure and μ_r the right-invariant measure in (a). Then for every $A \subseteq X$ define

$$\nu(A) = \int \mu(Ag^{-1}) d\mu_r(g) \, .$$

□

Metric and group amenability

Theorem

Let G be a finitely-generated group. TFAE:

1. *G is amenable;*
2. *one (every) Cayley graph of G is amenable.*

Corollary

A finitely generated group is either paradoxical or amenable.

(1) $\Rightarrow$ (2) If some $\mathrm{Cayley}(G, S)$ is non-amenable then $\exists f \in \mathcal{B}(G)$ with pre-images having 2 elements.

Modulo the equivalence in the Theorem, corollary proven.

A useful tool

We prove (2) $\Rightarrow$ (1): given a Følner sequence on a Cayley graph, construct μ invariant measure on G.

Goal: a new notion of limit for sequences in compact spaces (and later for sequences of spaces and of actions of groups.)

Definition

An ultrafilter on a set I = a finitely additive probability measure $\omega : \mathcal{P}(I) \to \{0, 1\}$.

Example

$\delta_x(A) = 1$ if x in A, 0 otherwise.
Called principal (or atomic) ultrafilter.

Ultralimit

Definition

Consider $f : I \to Y$ topological space.
$y \in Y$ is the ω–limit of f, $\lim_\omega f(i)$, if $\forall U$ neighborhood of y, $\omega(f^{-1}U) = 1$.

Theorem

Assume Y compact and Hausdorff. Each $f : I \to Y$ admits a unique ω-limit.

If $\omega = \delta_x$ then $\lim_\omega f(i) = f(x)$.

Theorem

An ultrafilter is non-principal (non-atomic) if and only if $\omega(F) = 0$ for every F finite.

Existence of ultrafilters

Why do non-principal ultrafilters exist ?

Equivalent definition:
A filter $\mathcal{F}$ on a set I is a collection of subsets of I s.t.:

(F_1) $\emptyset \notin \mathcal{F}$;

(F_2) If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$;

(F_3) If $A \in \mathcal{F}$, $A \subseteq B \subseteq I$, then $B \in \mathcal{F}$.

Example: Complementaries of finite sets in I= the Fréchet filter.

Ultrafilter on I = a maximal element in the ordered set of filters on I with respect to the inclusion.

Non-principal ultrafilter= contains the Fréchet filter.
Exists by Zorn's Lemma.

relation to previous definition: ω is the characteristic function of $\mathcal{U} \subset \mathcal{P}(I)$

Back to the proof

Theorem

Let G be a finitely-generated group. TFAE:

1. *G is amenable;*
2. *one (every) Cayley graph of G is amenable.*

(2)$\Rightarrow$ (1):
A Cayley graph $\mathcal{G}$ is amenable: $\exists$ a Følner sequence $(\Omega_n) \subset G$.

- For every $A \subset G$ define
$$\mu_n(A) = \frac{|A \cap \Omega_n|}{|\Omega_n|}\,.$$
- $|\mu_n(A) - \mu_n(Ag)| \leq \frac{2\partial_V(\Omega_n)}{|\Omega_n|}$ when $g \in S$.
- Let ω be a non-principal ultrafilter on $\mathbb{N}$.
Take $\mu(A) = \omega\text{-lim}\, \mu_n(A)\,.$

Group operations

Proposition

A subgroup of an amenable group is amenable.

Corollary

Any group containing a free non-abelian subgroup is non-amenable.

1. A finite extension of an amenable group is amenable.

2. Let N be a normal subgroup of a group G. The group G is amenable if and only if both N and G/N are amenable.

3. The direct limit G of a directed system $(H_i)_{i \in I}$ of amenable groups H_i, is amenable.

Corollary

A group G is amenable if and only if all finitely generated subgroups of G are amenable.

Corollary

Every solvable group is amenable.