Available online at www.sciencedirect.com

SCIENCE DIRECT®

Topology 43 (2004) 983–1033

TOPOLOGY

www.elsevier.com/locate/top

Filling in solvable groups and in lattices in semisimple groups

Cornelia Druţu*

UFR de Mathématiques et UMR CNRS no. 8524, Université de Lille I, F-59655 Villeneuve d'Ascq, France

Received 14 September 2000; received in revised form 13 March 2003; accepted 18 November 2003

Abstract

We prove that the filling order is quadratic for a large class of solvable groups and asymptotically quadratic for all $\mathbb{Q}$-rank one lattices in semisimple groups of $\mathbb{R}$-rank at least 3. As a byproduct of auxiliary results we give a shorter proof of the theorem on the nondistorsion of horospheres providing also an estimate of a nondistorsion constant.

MSC: 20F10; 20F65; 22E25; 22E40; 51E24

Keywords: Dehn function; Solvable groups; $\mathbb{Q}$-rank one lattices; Semisimple groups; Asymptotic cone; Horosphere; Euclidean building

1. Introduction

In this paper we give an estimate of the filling order in some particular cases of infinite finitely generated groups and of Lie groups. We may talk about filling area of a loop in Riemannian manifolds, in finitely presented groups and, more generally, in metric spaces. In a metric space we fix a small δ and define, following [20, Section 5.6], the δ-filling area of a loop as the minimal number of small loops of length at most δ ("bricks") one has to put one next to the other in order to obtain a net bounded by the given loop. Usually we choose $\delta = 1$. By means of the filling area one can define the filling function and the filling order in a metric space. The filling order is the order in ℓ of the maximal area needed to fill a loop of length ℓ (see Section 2.1 for definitions and details). With the terminology introduced in Section 2.1, if in a metric space X a function in the same equivalence class as the filling function is smaller that ℓ, ℓ^2 or e^{ℓ}, it is sometimes said that the space X *satisfies a linear, quadratic or exponential isoperimetric inequality*.

* Tel.: +33-32043-6757; fax: +333-20-43-43-02.
E-mail address: drutu@agat.univ-lille1.fr (C. Druţu).

0040-9383/$ - see front matter © 2003 Elsevier Ltd. All rights reserved.
doi:10.1016/j.top.2003.11.004

It is interesting to study these notions for two reasons at least: because a finitely presented group Γ has a solvable word problem if and only if its filling function is (bounded by) a recursive function, and because the filling order is a quasi-isometry invariant [2].

We recall that a *quasi-isometry* between two metric spaces (X_1, d_1) and (X_2, d_2) is a map $q : X_1 \to X_2$ such that

$$\frac{1}{L} d_1(x, y) - c \leqslant d_2(q(x), q(y)) \leqslant L d_1(x, y) + c, \quad \forall x, y \in X_1,$$

for some fixed positive constants c and L, and such that X_2 is at finite Hausdorff distance from the image of q. If such a map exists between X_1 and X_2, the two metric spaces are called *quasi-isometric*. A property invariant up to quasi-isometry is called *a geometric property*.

We shall study two classes of metric spaces: Lie solvable groups endowed with a left invariant Riemannian structure and nonuniform lattices in semisimple groups endowed with a left invariant word metric. The solvable groups are interesting as far as few things are known on their behaviour up to quasi-isometry. In the case of nilpotent groups there is more information. First, a consequence of Gromov's theorem on polynomial growth [19] is that virtual nilpotency is a geometric property in the class of groups (we recall that a group is called *virtually nilpotent* if it has a nilpotent subgroup of finite index). In nilpotent groups, the filling order is at most polynomial of degree $c + 1$, where c is the class of the group [18,20], and it is exactly polynomial of degree $c + 1$ if the group is free nilpotent [5,20]. In the Heisenberg group $\mathscr{H}^3$ it was shown by Thurston in [13] that the filling order is cubic (which implies that $\mathscr{H}^3$ is not automatic). The other Heisenberg groups $\mathscr{H}^{2n+1}$, $n \geqslant 2$, have quadratic filling. This was conjectured by Thurston in [13], Gromov gave an outline of proof in [20, Section 5.1A_4'] and D. Allcock gave the complete proof [1] by means of symplectic geometry. Olshanskii and Sapir [29] later gave a combinatorial proof.

The behaviour is more diversified in the case of solvable groups. To begin with, the property of being virtually solvable is not a geometric property in the class of groups anymore [12]. On the other hand, certain solvable groups are very rigid with respect to quasi-isometry [14–16]. Therefore, the estimate of a quasi-isometry invariant for a class of solvable groups, as the one given in this paper, should be interesting. The filling order is already known for some solvable groups. Thurston has shown that the group *Sol* has exponential filling (so it is not automatic) [13]. Gersten showed that the Baumslag–Solitar groups $BS(1, p)$ have exponential filling order for $p \neq 1$, which in particular implies that they are not automatic, while they are known to be asynchronously automatic. Gromov [20, Section 5.A.9] showed that $\mathbb{R}^n \rtimes \mathbb{R}^{n-1}$, $n \geqslant 3$, has quadratic filling order. Arzhantseva and Osin [3, Theorem 1.4] constructed a sequence of discrete nonpolycyclic solvable groups with filling orders that are at most cubic. We should note here that the discrete solvable groups we deal with in this paper are all polycyclic, as lattices in Lie groups.

We also study nonuniform lattices in semisimple groups. It is known that their filling order is at most exponential [20]. The exact filling order is already known for almost all cases when the ambient semisimple group is of $\mathbb{R}$-rank one (see the comments following Remark 4.2). If the semisimple group has $\mathbb{R}$-rank 2 then the filling order is exponential [23]. We note that this has already been proven by Thurston in [13] in the particular case of $SL_3(\mathbb{Z})$, from which result he deduced that $SL_3(\mathbb{Z})$ is not combable. Also, Thurston stated that the filling order of $SL_n(\mathbb{Z})$, $n \geqslant 4$, is quadratic [17, Remark, p. 86]. In a previous paper we have proved that for some $\mathbb{Q}$-rank one lattices (among which the Hilbert modular groups) and for some solvable groups the filling order is at most “asymptotically cubic” [11]. In this paper we prove the following more general and stronger results.

Theorem 1.1. *Let X be a product of Euclidean buildings and symmetric spaces of noncompact type, X of rank at least 3. Let ρ be a geodesic ray which is not contained in a rank one or in a rank two factor of X.*

(1) *The filling order in the horosphere $H(\rho)$ of X endowed with the length metric is quadratic.*
(2) *Let S be a Lie group acting by isometries, transitively with compact stabilizers on the horosphere $H(\rho)$ of X. In S endowed with any left invariant metric the filling order is quadratic. The same is true for every discrete group Γ acting properly discontinuously cocompactly on $H(\rho)$, Γ endowed with a word metric.*

By length metric we mean the length of the shortest path metric.

The previous result has been obtained independently by Leuzinger and Pittet [24] in the case $X = SL_n(\mathbb{R})/SO(n)$.

For a definition of Euclidean buildings see [22], for a definition of horospheres see Section 2.3. We note that the quadratic estimate on the filling function is sharp. This is because in a product of symmetric spaces of noncompact type and Euclidean buildings which is of rank $r \geqslant 3$, every horosphere contains isometric copies of the Euclidean space $\mathbb{E}^{r-1}$ and there are projections of the horosphere on each of these copies decreasing the distance.

Theorem 1.2. *The filling order in every irreducible $\mathbb{Q}$-rank one lattice of a semisimple group of $\mathbb{R}$-rank at least 3 is at most asymptotically quadratic. That is, for every $\varepsilon > 0$ there exists ℓ_ε such that*

$$A(\ell) \leqslant \ell^{2+\varepsilon}, \quad \forall \ell \geqslant \ell_\varepsilon.$$

The filling order in every $\mathbb{Q}$-rank one lattice in a semisimple group of rank at least 2 is at least quadratic. This is due to the fact that there are maximal flats in the symmetric space of noncompact type associated to the semisimple group on which the lattice acts cocompactly. Thus, Theorem 1.2 gives an “asymptotically sharp” estimate.

Examples where the two theorems apply:

(1.a) Let n be an integer, $n \geqslant 3$, and let $\bar{a}=(a_1, a_2, \dots, a_n)$ be a given fixed vector with at least three nonzero components. Let $\alpha : \mathbb{R}^{n-1} \to GL(n, \mathbb{R})$ be an injective homomorphism whose image consists of all diagonal matrices with diagonal entries $(e^{t_1}, e^{t_2}, \dots, e^{t_n})$ verifying $a_1 t_1 + a_2 t_2 + \cdots + a_n t_n = 0$.

The hypothesis of Theorem 1.1(2), is satisfied by the solvable group $Sol_{2n-1}(\alpha) = \mathbb{R}^n \rtimes_\alpha \mathbb{R}^{n-1}$. More precisely, let us consider $\bar{\alpha} : \mathbb{R}^n \to GL(n, \mathbb{R})$ with $\bar{\alpha}(t_1, t_2, \dots, t_n)$ equal to the diagonal matrix with diagonal entries $(e^{t_1}, e^{t_2}, \dots, e^{t_n})$. We endow the semidirect product $\mathbb{R}^n \rtimes_{\bar{\alpha}} \mathbb{R}^n$ with the left invariant Riemannian structure that coincides with the canonical structure of $\mathbb{R}^{2n}$ at the origin. The group $Sol_{2n-1}(\alpha)$ can be seen as a subgroup of $\mathbb{R}^n \rtimes_{\bar{\alpha}} \mathbb{R}^n$.

The map

$$\Phi : \mathbb{R}^n \rtimes_{\bar{\alpha}} \mathbb{R}^n \to \underbrace{\mathbb{H}^2 \times \cdots \times \mathbb{H}^2}_{n\,\text{times}},$$

$$\Phi((x_1, x_2, \dots, x_n), (t_1, t_2, \dots, t_n)) = ((x_1, e^{t_1}), (x_2, e^{t_2}), \dots, (x_n, e^{t_n})),$$

is an isometry. Here $\mathbb{H}^2$ denotes the hyperbolic plane and each copy of $\mathbb{H}^2$ is endowed with the cartesian coordinates given by the half-plane model. We consider the geodesic ray ρ in $\mathbb{H}^2\times\cdots\times\mathbb{H}^2$ defined by

$$\rho(s)=((0,e^{a_1 s}),(0,e^{a_2 s}),\ldots,(0,e^{a_n s})),\quad s\geqslant 0.$$

The group $Sol_{2n-1}(\alpha)$ can be identified to $H(\rho)$ via the map Φ. The hypothesis that the vector $\bar{a}$ has at least three nonzero components implies that ρ is not contained in a rank one or a rank two factor of $X=\mathbb{H}^2\times\cdots\times\mathbb{H}^2$.

(1.b) Let n be an integer, $n\geqslant 4$, and let $\bar{b}$ be a fixed vector $(b_1,b_2,\ldots,b_n)$ with $\sum_{i=1}^n b_i=0$. We consider the solvable group $S_n(\bar{b})$ of upper triangular matrices of order n with the set of diagonals of the form $\{(e^{t_1},e^{t_2},\ldots,e^{t_n})\mid \sum_{i=1}^n t_i=0,\sum_{i=1}^n b_it_i=0\}$. Let X be the irreducible symmetric space $SL(n,\mathbb{R})/SO(n)$, of rank $n-1$. It can be seen as the set of positive definite quadratic forms on $\mathbb{R}^n$ of determinant one in the canonical base. We consider the geodesic ray ϱ in X defined so that $\varrho(s)$ is the diagonal quadratic form with coefficients $(e^{b_1 s},e^{b_2 s},\ldots,e^{b_n s})$, $s\geqslant 0$. The group $S_n(\bar{b})$ acts simply transitively on $H(\varrho)$.

(2) Theorem 1.2 applies to every irreducible lattice in a semisimple group having $\mathbb{R}$-rank at least 3 and an $\mathbb{R}$-rank one factor (as such a lattice is a $\mathbb{Q}$-rank one lattice [30, Lemma 1.1]). In particular, Theorem 1.2 applies to the Hilbert modular groups $PSL(2,\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of a totally real field K with $[K:\mathbb{Q}]\geqslant 3$.

In [32] and [35] one can find other examples of $\mathbb{Q}$-rank one lattices in semisimple groups of real rank at least 3.

The paper is organised as follows. In Section 2 we recall some basic facts about filling area, asymptotic cones, buildings and $\mathbb{Q}$-rank one lattices in semisimple groups. We recall that every such lattice acts with compact quotient on a space X_0 obtained from the ambient symmetric space $X=G/K$ by deleting a countable family of disjoint open horoballs (see Section 2.4). It follows that the filling order in the lattice is the same as the filling order in X_0.

The solvable groups we consider act on horospheres. Since the projection of the exterior of the corresponding open horoball on the horosphere diminishes distances, one can study the whole exterior of the open horoball, which can also be denoted by X_0, instead of the horosphere.

Thus, both in the case of lattices and of solvable groups it suffices to estimate the filling order in a metric space X_0 obtained from a symmetric space of noncompact type by deleting a family of disjoint open horoballs. The main tool we use to obtain such an estimate is the asymptotic cone. The notion of asymptotic cone has been introduced by Gromov [19], Van Den Dries and Wilkie [34], and consists, philosophically speaking, of giving "an image seen from infinitely far away"of a metric space (see Section 2.1 for definition and properties). To every metric space can be associated a whole class of asymptotic cones (possibly isometric). There are similarities of arbitrary factor "acting" on this class, that is, sending a cone into another (Remark 2.1.1).

We dispose of a result, due to Gromov and Papasoglu, allowing to deduce from an uniform estimate of the filling order in all asymptotic cones an estimate of the filling order in the initial metric space (Section 2.1, Theorem 2.1.2). Thus, instead of considering the space X_0 one can consider its asymptotic cones. Let $\mathbf{K}_0$ be such an asymptotic cone. It is not difficult to prove that each $\mathbf{K}_0$ is obtained from an Euclidean building $\mathbf{K}$ by deleting a family of disjoint open horoballs.

In Section 3 we first place ourselves in an Euclidean building. Essentially, an Euclidean building is a bunch of "flats", that is, of isometric copies of an Euclidean space. By deleting disjoint horoballs one makes polytopic holes into these flats. These polytopic holes can take, up to similarity, only a finite number of shapes. Thus, we may hope to reduce the problem of filling a loop in a space like $\mathbf{K}_0$ to that of filling a loop in an Euclidean space with polytopic holes. We first prove some global and local properties of such a polytopic hole, that is, of the trace of a horoball in a maximal flat. We also provide a way to join two points on a connected horosphere with polygonal lines of length comparable to the distance, in two different cases (see Lemmas 3.3.1 and 3.3.2). As a byproduct of these results we give new proofs of the theorems on the nondistorsion of horospheres in Euclidean buildings and symmetric spaces of noncompact type (Theorems 3.3.3 and 3.3.5 in this paper).

In Section 4 we prove the main theorem, Theorem 4.1, on the filling order in the exterior of a disjoint union of open horoballs. We give here an outline of the proof. First we argue in an Euclidean building $\mathbf{K}$ which is 4-thick and of rank at least 3. We show that a loop contained in the exterior of one open horoball, $\mathbf{K} \setminus Hbo(\rho)$, where ρ is a geodesic ray not parallel to any rank 2 factor, has a quadratic filling area if the loop is included in one apartment (Proposition 4.1.1). Then we show that under the same hypothesis the same conclusion is true for certain loops included in the union of two apartments (Proposition 4.2.2). Then we prove Theorem 4.3, which is a weaker version of Theorem 4.1. This is done as follows. Up to spending a linear filling area, a generic loop can always be assumed to be included in a finite union of apartments, the number of apartments being of the same order as the length ℓ of the loop. In filling the loop, there is a problem when one passes from one apartment to another. More precisely, by means of Proposition 4.2.2 one can show that when passing from one apartment to another, in the process of filling the loop, one might have to spend an area of order ℓ^2. This explains why instead of a quadratic filling area, one obtains a cubic filling area, in this first approach. For certain curves contained in a union of apartments of uniformly bounded cardinal a quadratic filling area is obtained.

In Section 4.4 it is shown that part (b) of Theorem 4.3 implies part (b) of Theorem 4.1 and in particular Theorem 1.1. The proof of the previous implication is done in two steps. Firstly, it is shown that loops composed of a uniformly bounded number of minimising almost polygonal curves have quadratic filling area (see the end of Section 3 for a definition of minimising almost polygonal curves). Secondly an induction procedure is applied.

Section 4.5 contains the proof that Theorem 4.3, (a), implies Theorem 4.1, (a), and Theorem 1.2. According to Theorem 2.1.2, in order to prove Theorem 4.1, (a), it is enough to prove that the filling order is quadratic in all the asymptotic cones of a space X_0 on which a $\mathbb{Q}$-rank one lattice Γ acts properly discontinuously cocompactly. By Theorem 4.3, (a), it is already known that in every asymptotic cone $\mathbf{K}_0$ of X_0 the filling order is at most cubic. It is shown that in reality the filling order in every $\mathbf{K}_0$ is not cubic but quadratic.

To understand the proof of the previous statement we should see first why the filling order in the Euclidean plane is quadratic. In the Euclidean plane, every loop $\mathfrak{C}$ of length ℓ can be filled with at most $(\ell/\lambda_1)^2$ bricks of length λ_1 (one may think of the bricks as being small squares). To fill it with bricks of length $\lambda_2 \ll \lambda_1$, it is enough if we fill the λ_1-bricks with λ_2-bricks, which can be done with $(\lambda_1/\lambda_2)^2$ bricks for each λ_1-brick. In this way the initial loop $\mathfrak{C}$ is filled with at most $(\ell/\lambda_2)^2$ bricks of length λ_2. Thus, the fact that for every λ, however small, we may fill $\mathfrak{C}$ with at most $(\ell/\lambda)^2$ λ-bricks is due to the fact that the quadratic filling is preserved in the small. This should also happen, under replacement of the exponent 2 by the exponent 3, in a space with a cubical filling order. But in the space $\mathbf{K}_0$ the first important remark is that, when one fills a loop of length ℓ with bricks of length 1, one puts $k_2\ell^2$ bricks with uncontrolled shapes and $k_1\ell^3$ bricks which bound small Euclidean squares entirely contained in $\mathbf{K}_0$ (Remark 4.3.2). This means that, for $\lambda < 1$, the λ-filling area of the loop tends to become more and more quadratic as λ becomes smaller and smaller. Since we may choose bricks as small as we want, the quadratic factor will end by dominating the cubic factor. By applying similarities (Remark 2.1.1), since this is a reasoning which is done simultaneously in all asymptotic cones, one can return to bricks of length one and obtain a quadratic filling order.

In the Appendix we provide an isoperimetric inequality for a hypersurface in an Euclidean space composed of points at a fixed distance from a certain polytope $\mathscr{P}$ (Proposition 5.6). From this, by means of Lemma 5.4, we derive an isoperimetric inequality for every polytopic hypersurface whose points are at a distance between $R > 0$ and $aR, a > 1$, from the polytope $\mathscr{P}$. The constants appearing in the isoperimetric inequality in the first case depend only on $\mathscr{P}$ while in the second case they also depend on a. The second result is useful in the proof of our main theorems.

Notations: Throughout the whole paper, in a metric space X, $B(x,r)$ denotes the open ball of center $x \in X$ and radius $r > 0$, $S(x,r)$ denotes its boundary sphere, $\mathscr{N}_r(A) = \{x \in X \mid d(x,A) \leqslant r\}$, $\partial\mathscr{N}_r(A) = \{x \in X \mid d(x,A) = r\}$, $\breve{\mathscr{N}}_r(A) = \{x \in X \mid d(x,A) < r\}$, and $Ext_r(A) = \{x \in X \mid d(x,A) \geqslant r\}$, where $A \subset X$.

We also use, though not systematically, the Vinogradov notation $a \ll b$ or $b \gg a$ for $a \leqslant C \cdot b$, where C is a positive universal constant. We write $a \asymp b$ if $a \ll b$ and $b \ll a$.

2. Preliminaries

2.1. Filling area, filling order, asymptotic cone

The notion of filling area of a loop is well defined in the setting of Riemannian manifolds as well as in finitely presented groups (see for instance [8, Chapter I, Section 8.1.4]). In the sequel we recall the meaning of this notion in geodesic metric spaces, following [20, Section 5.F]. Let X be such a space and $\delta > 0$ a fixed constant. We call "loops" Lipschitz maps $\mathfrak{C}$ from $\mathbb{S}^1$ to X. We call *filling partition of* $\mathfrak{C}$ a pair consisting of a triangulation of the planar unit disk $\mathbb{D}^2$ and of an injective map from the set of vertices of the triangulation to $X, \pi : \mathscr{V} \to X$, where π coincides with $\mathfrak{C}$ on $\mathscr{V} \cap \mathbb{S}^1$. The image of the map π is called *filling disk of* $\mathfrak{C}$. We can join the images of the vertices of each triangle with geodesics (for the vertices which are endpoints of arcs of $\mathbb{S}^1$, we replace the geodesic with the arc of $\mathfrak{C}$ contained between the images of the vertices). We call the geodesic triangles thus obtained *bricks*. The *length of a brick* is the sum of the distances between vertices (for the vertices which are endpoints of arcs of $\mathbb{S}^1$, we replace the distance with the length of the arc of $\mathfrak{C}$ contained between the two images of the vertices). The maximum of the lengths of the bricks in a partition is called *the mesh of the partition*. The partition is called *δ-filling partition of* $\mathfrak{C}$ if its mesh is at most δ. The corresponding filling disk is called *δ-filling disk of* $\mathfrak{C}$. We call *δ-filling area of* $\mathfrak{C}$ the minimal number of triangles in a triangulation associated to a δ-filling partition of $\mathfrak{C}$. We denote it with the double notation $A_\delta(\mathfrak{C}) = P(\mathfrak{C}, \delta)$. If no δ-filling partition of the loop $\mathfrak{C}$ exists, we put $A_\delta(\mathfrak{C}) = +\infty$.

In each of the three cases (Riemannian manifolds, finitely presented groups, geodesic metric spaces) when we have defined a notion of "filling area" for loops, we can now define the *filling function* $A : \mathbb{R}_+^* \to \mathbb{R}_+^* \cup \{+\infty\}$, $A(\ell) :=$ the maximal area needed to fill a loop of length at most ℓ.

In a metric space X, for δ fixed, we use the notation $A_\delta^X(\ell)$ and we call this function the *δ-filling function of X*. Whenever there is no possibility of confusion, we drop the index X. In order to obtain finite valued filling functions, we work in metric spaces for which there exists $\mu > 0$ such that every loop has a μ-filling disk and $A_\mu(\ell) < +\infty$, $\forall \ell > 0$. A metric space satisfying the previous property is called *μ-bounded simply connected.*

Let f_1 and f_2 be two functions of real variable, taking real values. We say that *the order of the function f_1 is at most the order of the function f_2*, and we denote it by $f_1 \prec f_2$, if $f_1(x) \leqslant a f_2(bx+c)+dx+e, \forall x$, where a, b, c, d, e are fixed positive constants. We say that *f_1 and f_2 have the same order*, and we denote it by $f_1 \doteq f_2$, if $f_1 \prec f_2$ and $f_2 \prec f_1$. The relation $\doteq$ is an equivalence relation. The equivalence class of a numerical function with respect to this relation is called *the order of the function*. If a function f has (at most) the same order as the function x, x^2, x^3, x^d or $\exp x$ it is said that *the order of the function f is* (*at most*) *linear*, *quadratic*, *cubic*, *polynomial*, *or exponential*, respectively.

Let X be a μ-bounded simply connected geodesic metric space. For every $\mu \leqslant \delta_1 \leqslant \delta_2$ we have

$$A_{\delta_2}(\ell) \leqslant A_{\delta_1}(\ell) \leqslant A_{\delta_2}(\ell) \cdot A_{\delta_1}(2\delta_2).$$

It follows that all δ-filling functions with $\delta \geqslant \mu$ have the same order. We call it *the filling order of X*. If a geodesic metric space Y is quasi-isometric to X then Y is also bounded simply connected and it has the same filling order as X. If Γ is a finitely presented group, its Cayley graph is μ-bounded simply connected for some μ depending on the presentation and its filling order is the same as the order of the Dehn function. We recall that the Dehn functions corresponding to different presentations of the group have the same order [2].

In the sequel we shall deal only with bounded simply connected geodesic metric spaces. We shall omit to recall this hypothesis each time.

A nonprincipal ultrafilter is a finitely additive measure ω defined on all subsets of $\mathbb{N}$, taking as values 0 and 1 and taking always value 0 on finite sets. Such a measure always exists [10]. For a sequence in a topological space, (a_n), one can define *the ω-limit* as the element a with the property that for every neighbourhood $\mathcal{N}(a)$ of a, the set $\{n \in \mathbb{N} \mid a_n \in \mathcal{N}(a)\}$ has ω-measure 1. We denote a by $\lim_\omega a_n$. If it exists, the ω-limit is unique. Any sequence in a compact space has an ω-limit [7, I.9.1].

Let (X,d) be a metric space. We fix a sequence (x_n) of points in X, which we call *sequence of observation centers*, a sequence of positive numbers (d_n) diverging to infinity, which we call *sequence of scalars*, and a nonprincipal ultrafilter ω. Let $\mathscr{C}$ be the set of sequences (y_n) of points in X with the property that $d(x_n, y_n)/d_n$ is bounded. We define an equivalence relation on $\mathscr{C}$:

$$(y_n) \sim (z_n) \iff \lim_\omega \frac{d(y_n, z_n)}{d_n} = 0.$$

The quotient space of $\mathscr{C}$ with respect to this relation, which we denote $X_\omega(x_n, d_n)$, is called *the asymptotic cone of X* with respect to the observation centers (x_n), the scalars (d_n) and the nonprincipal ultrafilter ω. It is a complete metric space with the metric

$$D([y_n],[z_n]) = \lim_\omega \frac{d(y_n, z_n)}{d_n}.$$

We say that the set $A \subset X_\omega(x_n, d_n)$ is *the limit set of the sets $A_n \subset X$* if

$$A = \{[x_n] \mid x_n \in A_n \ \omega\text{-almost surely}\}.$$

We denote $A = [A_n]$.

In our arguments we shall use the following very simple but important remark.

Remark 2.1.1. The map

$$I_\alpha : X_\omega(x_n, d_n) \to X_\omega\left(x_n, \frac{1}{\alpha} d_n\right), \quad I_\alpha([x_n]) = [x_n]$$

is a similarity of factor α.

There is a relation between the filling order in the asymptotic cones and the filling order in the initial space, established by Papasoglu (see [11, Theorem 2.7]), who adapted an idea of Gromov for this purpose.

Theorem 2.1.2 (Papasoglu). *Let X be a geodesic metric space. If there exists $C > 0$ and $p > 0$ such that in every asymptotic cone of X we have that*

$$A_1(\ell) \leqslant C \cdot \ell^p, \quad \forall \ell \geqslant 1,$$

then there exists $\mu > 0$ such that the space X is μ-bounded simply connected and for every $\varepsilon > 0$ there exists $\ell_\varepsilon > 0$ such that

$$A_\mu^X(\ell) \leqslant \ell^{p+\varepsilon}, \quad \forall \ell \geqslant \ell_\varepsilon.$$

2.2. Spherical and Euclidean buildings

For this section we refer mainly to [22], but also to [10,33].

Before discussing about buildings we introduce some terminology in Euclidean spaces. For a subset A in an Euclidean space $\mathbb{E}^k$ we call *affine span* of A, and we denote it by $\mathit{Span}\, A$, the minimal affine subspace of $\mathbb{E}^k$ containing A. Two polytopes of codimension one are called *parallel* if their affine spans are parallel. A subset A of $\mathbb{E}^k$ is called *relatively open* if it is open in $\mathit{Span}\, A$. The *relative interior* of a subset B of $\mathbb{E}^k$ is its interior in $\mathit{Span}\, B$.

In an Euclidean sphere $\mathbb{S}^k$, we call *spherical span* of a subset σ the trace on the sphere of the affine span of the cone of vertex the origin over σ. We denote it $\mathit{Span}\, \sigma$. We say that two subsets σ and σ' in $\mathbb{S}^k$ are *orthogonal to each other* if $\mathit{Span}\, \sigma$ and $\mathit{Span}\, \sigma'$ are orthogonal. In a spherical building two subsets σ and σ' are *orthogonal* if they are both contained in the same apartment and are orthogonal. We can also define the distance between a point and a convex set in a spherical building each time they are both contained in an apartment as the spherical distance between them in that apartment.

Let Σ be a spherical building. Throughout the whole paper we shall suppose that for all spherical buildings, the associated Weyl group acts on the associated Coxeter complex without fixed points. In this way we rule out the case of spherical buildings having a sphere as a factor and of Euclidean buildings and symmetric spaces of noncompact type having an Euclidean space as a factor.

An apartment is split by each singular hyperplane in it into two halves called *half-apartments*. All simplices in Σ which are not chambers are called *walls*, the codimension one simplices are also called *panels*. Intersections of singular hyperplanes in an apartment are called *singular subspaces*. A chamber is said to be *adjacent* to a singular subspace if their intersection is a wall of the same dimension as the subspace. We also say, in the previous situation, that the singular subspace *supports* the chamber.

We say that a panel *separates* a chamber and a point if there is at least one apartment containing the three of them and in each such apartment the spherical span of the panel separates the point and the interior of the chamber.

Two chambers are said to be *adjacent* if they have a panel in common and disjoint interiors. A *gallery* of chambers is a sequence of chambers such that every two consecutive chambers are adjacent. The number of chambers composing it is called *the length of the gallery*. Given a point and a simplex, *the combinatorial distance* between them is the minimal length of a gallery of chambers such that the first chamber contains the point and the last contains the simplex. For every such gallery of minimal length between the point and the simplex, the last of its chambers is called *the projection of the point on the simplex*.

Lemma 2.2.1. *Let Σ be a spherical building, let x be a point and σ a panel in it such that there is no singular hyperplane containing both. There is a unique projection of x on σ, which is contained in every apartment $\mathscr{A}$ containing x and σ.*

Proof. Let $\mathscr{A}_0$ be an apartment containing x and σ. We choose an interior point y of σ which is not opposite to x. If the geodesic joining x to y is contained in a singular hyperplane $\mathscr{H}$ then x and σ are contained in $\mathscr{H}$. Therefore, the geodesic joining x to y has a point y_0 in the interior of a chamber $\mathscr{W}$ having σ as a panel. Since the second chamber having σ as a panel is separated of x by σ, it follows that $\mathscr{W}$ is the projection of x on σ. Every apartment $\mathscr{A}$ containing x and σ contains by convexity the geodesic joining x to y through y_0, therefore it contains also $\mathscr{W}$. □

For every wall $\mathscr{M}$ in the spherical building we call *star of $\mathscr{M}$*, and we denote it *Star*($\mathscr{M}$), the set of chambers including $\mathscr{M}$. We use the same name and notation for the union of all the chambers including $\mathscr{M}$. A building is called *c-thick* if for every panel $\mathscr{P}$, the cardinal of *Star*($\mathscr{P}$) seen as a set of chambers is at least c.

Every spherical building Σ admits a labelling [9, IV.1, Proposition 1]. With respect to this labelling one can define a projection of the building on the model spherical chamber $p:\Sigma \to \Delta_{\mathrm{mod}}$. Given a subset $\mathscr{C}$ in Σ its image under this projection, $p(\mathscr{C})$, is called *the set of slopes of $\mathscr{C}$*.

The model Coxeter complex of Σ is the Coxeter complex S which is isomorphic to each of its apartments; for every labelling in the spherical building there is a compatible labelling on the model Coxeter complex and a compatible projection $p_{\mathsf{S}}:\mathsf{S} \to \Delta_{\mathrm{mod}}$.

Given an apartment $\mathscr{A}$ in Σ and a chamber $\mathscr{W}$ in it one can always define a map $\mathrm{retr}_{\mathscr{A},\mathscr{W}}:\Sigma \to \mathscr{A}$ which preserves labelling and combinatorial distances to $\mathscr{W}$ and diminishes the other combinatorial distances [33, Section 3.3–3.6]. This map is called *the retraction of Σ onto $\mathscr{A}$ with center $\mathscr{W}$*.

Definition 2.2.2. Let S be a Coxeter complex with a labelling, Δ_{mod} its model spherical chamber and $p_{\mathsf{S}}:\mathsf{S} \to \Delta_{\mathrm{mod}}$ the projection corresponding to the labelling. Let θ be a given point in Δ_{mod}. *The set of orthogonals to θ* is the set of points $q \in \Delta_{\mathrm{mod}}$ such that any preimage of q by p_{S} is at distance $\pi/2$ from a preimage of θ by p_{S}. We denote this set *Ort*(θ).

We note that if S is the model Coxeter complex of a spherical building Σ, endowed with a labelling compatible with the one of Σ, $Ort(\theta)$ also coincides with the set of points $q \in \Delta_{\mathrm{mod}}$ such that any preimage of q by $p : \Sigma \to \Delta_{\mathrm{mod}}$ is at distance $\pi/2$ from a preimage of θ by p.

If $\mathbf{S}$ and $\mathbf{S}'$ are two simplicial complexes, their *join*, denoted by $\mathbf{S} \circ \mathbf{S}'$, is a simplicial complex having as vertex set the disjoint union of the sets of vertices of $\mathbf{S}$ and of $\mathbf{S}'$ and having one simplex $\sigma \circ \sigma'$ for every pair of simplices $\sigma \in \mathbf{S}$ and $\sigma' \in \mathbf{S}'$. We also allow for the possibility that one of the two complexes is empty, and we make the convention that $\mathbf{S} \circ \emptyset = \mathbf{S}$.

There is a geometric interpretation of the join of spherical complexes which goes as follows. Let $\mathbf{S}$ and $\mathbf{S}'$ be spherical simplices in the Euclidean spheres $\mathbb{S}^{k-1} \subset \mathbb{E}^k$ and $\mathbb{S}^{m-1} \subset \mathbb{E}^m$, respectively. The join $\mathbf{S} \circ \mathbf{S}'$ is the spherical simplex in $\mathbb{S}^{k+m-1}$ obtained by embedding $\mathbf{S}$ together with $\mathbb{S}^{k-1}$ into $\mathbb{S}^{k+m-1}$ and likewise $\mathbf{S}'$ together with $\mathbb{S}^{m-1}$, such that the embeddings of $\mathbb{S}^{k-1}$ and $\mathbb{S}^{m-1}$ are orthogonal, and considering the convex hull of $\mathbf{S} \cup \mathbf{S}'$ in $\mathbb{S}^{k+m-1}$. One may also say that $\mathbf{S} \circ \mathbf{S}'$ is obtained from $\mathbf{S}$ and $\mathbf{S}'$ by gluing the endpoints of a quarter of a circle to every pair of points $x \in \mathbf{S}$ and $x' \in \mathbf{S}'$. Since this also makes sense for two spherical complexes and in particular for spherical buildings, we thus get a geometric definition of the join $\Sigma \circ \Sigma'$ of two spherical buildings Σ and Σ'. Kleiner and Leeb [22, Section 3.3] proved the following:

- every decomposition as a join of the model chamber of a spherical building, $\Delta_{\mathrm{mod}} = \Delta_1 \circ \Delta_2 \circ \cdots \circ \Delta_n$, or of the associated Coxeter complex $\mathsf{S} = \mathsf{S}_1 \circ \mathsf{S}_2 \circ \cdots \circ \mathsf{S}_n$ imply a decomposition of the spherical building $\Sigma = \Sigma_1 \circ \Sigma_2 \circ \cdots \circ \Sigma_n$ such that Δ_i and S_i are the model chamber and the associated Coxeter complex of Σ_i, respectively.
- a spherical building Σ is not a join of two nonempty spherical buildings if and only if its model chamber Δ_{mod} has diameter $< \pi/2$ and dihedral angles $\leqslant \pi/2$.

Lemma 2.2.3. *Let Σ be a labelled spherical building and let q be a point in it such that for every decomposition of Σ as a join, $\Sigma = \Sigma_1 \circ \Sigma_2$, q is contained neither in Σ_1 nor in Σ_2. Then in every chamber containing it, q is not orthogonal to any wall of the chamber.*

Proof. Suppose q is contained in the interior of a chamber Δ. If q is orthogonal to a wall of Δ then the diameter of Δ is strictly greater than $\pi/2$. This is impossible, as the diameter of Δ is at most $\pi/2$.

Suppose q is contained in the interior of a wall σ. If q is orthogonal to another wall σ' such that σ and σ' are both included in a chamber Δ then the diameter of Δ and of the model chamber Δ_{mod} is $\pi/2$.

If σ and σ' intersect in a wall, this wall being orthogonal to an interior point of σ it follows that the diameter of σ, hence the diameter of Δ, is $> \pi/2$. Since this is impossible, we may suppose that σ and σ' do not intersect. Let $\mathfrak{m}$ be the convex hull of σ and σ' in Δ. We have $\mathfrak{m} = \sigma \circ \sigma'$.

The building Σ is decomposable as a join because Δ_{mod} has diameter $\pi/2$. The maximal decomposition of Σ as a join, $\Sigma = \Sigma_1 \circ \Sigma_2 \circ \cdots \circ \Sigma_n$, induces a maximal decomposition of Δ as a join, $\Delta = \Delta_1 \circ \Delta_2 \circ \cdots \circ \Delta_n$, which in its turn induces a decomposition of $\mathfrak{m}$, $\mathfrak{m} = \mathfrak{m}_1 \circ \mathfrak{m}_2 \circ \cdots \circ \mathfrak{m}_n$ (where some of the $\mathfrak{m}_i$ might be empty). It follows that $\sigma = \mathfrak{m}_{i_1} \circ \mathfrak{m}_{i_2} \circ \cdots \circ \mathfrak{m}_{i_s}$ and $\sigma' = \mathfrak{m}_{j_1} \circ \mathfrak{m}_{j_2} \circ \cdots \circ \mathfrak{m}_{j_t}$, $\{i_1, i_2, \ldots, i_s\} \sqcup \{j_1, j_2, \ldots, j_t\} = \{1, 2, \ldots, n\}$. This implies that $q \in \sigma \subset \Sigma_{i_1} \circ \Sigma_{i_2} \circ \cdots \circ \Sigma_{i_s}$, which contradicts the hypothesis. $\square$

The following simple remark enlightens us more on the geometry of spherical buildings.

Remark 2.2.4. Let Σ be a labelled spherical building, $p:\Sigma \to \Delta_{\text{mod}}$ the projection on the model chamber induced by the labelling and θ a point in Δ_{mod}. The set $D(\theta)=\{d(x,\sigma)\,|\,x\in\Sigma,\, p(x)=\theta, \sigma$ simplex in $\Sigma\}$ is finite and, when ordered in the increasing order, it contains either three consecutive terms of the form $(\pi/2)-\delta_0, \pi/2, (\pi/2)+\delta_0'$ or two consecutive terms of the form $(\pi/2)-\delta_0, (\pi/2)+\delta_0'$, where δ_0 and δ_0' depend only on θ. If there is no decomposition of Δ_{mod} as $\theta\circ\Delta_1$ then $\delta_0, \delta_0' < \pi/2$.

Notations: Let X be a CAT(0)-space and let $\partial_\infty X$ be its boundary at infinity. For every $x\in X$ and $\alpha\in\partial_\infty X$, we denote by $[x,\alpha)$ the unique ray having x as origin and α as point at infinity. For two geodesic segments or rays $[x,a)$ and $[x,b)$ we denote by $\angle_x(a,b)$ the angle between them in x (see [22] for a definition). We denote by d_T the Tits metric on $\partial_\infty X$. For every point x and every geodesic ray ρ we denote by ρ_x the ray of origin x asymptotic to ρ.

Let now X be a symmetric space of noncompact type or an Euclidean building or a product of a symmetric space of noncompact type with an Euclidean building, X of rank r. In the sequel, for simplicity, we call the apartments in Euclidean buildings also maximal flats. For definitions and results in symmetric space theory and in Euclidean building theory we refer to [8,21,22]. We only recall that

- m-flats are isometric copies of the Euclidean space $\mathbb{E}^m$, $m\leqslant r$; singular m-flats(subspaces) are m-flats which appear as intersections of composing it; we also call singular $(r-1)$-flats singular hyperplanes;
- half-apartments are halves of apartments determined by singular hyperplanes;
- the faces of the Weyl chambers are called walls; the codimension 1 walls are also called panels;
- two Weyl chambers are called adjacent if they have the vertex and a panel in common and disjoint interiors;
- a singular subspace Φ is said to be adjacent to a Weyl chamber W or to support W if $\Phi\cap W$ is a wall of the same dimension as Φ;
- a gallery of Weyl chambers is a finite sequence of Weyl chambers such that every two consecutive Weyl chambers are adjacent; its length is the number of Weyl chambers composing it; a minimal gallery is a gallery of minimal length among the ones which have the same first and last Weyl chambers as itself;
- for every wall M we define $Star(M)$, which we call *the star of* M, as the set of all Weyl chambers having the same vertex as M and including M; we use the same name and notation for the union of all these Weyl chambers;
- Weyl polytopes are polytopes which appear as intersections of half-apartments (they may have dimension smaller than r, as we may intersect opposite half-apartments);
- an Euclidean building is called c-thick if every singular hyperplane is the boundary of at least c half-apartments of disjoint interiors.

The boundary at infinity of X, $\partial_\infty X$, is a spherical building ([29, Chapters 15 and 16; 5, Appendix 5]). The model Coxeter complex and chamber of $\partial_\infty X$ are sometimes called *the model spherical Coxeter complex and chamber of* X.

If X is a c-thick Euclidean building then $\partial_\infty X$ is a c-thick spherical building. If X decomposes as $X = X_1 \times X_2$ then $\partial_\infty X = \partial_\infty X_1 \circ \partial_\infty X_2$. For every maximal flat F in X we denote $F(\infty)$ its boundary at infinity (which is an apartment). We likewise denote $\rho(\infty), W(\infty), \Phi(\infty)$ the boundary at infinity of a ray ρ, a Weyl chamber W, a singular subspace Φ. If two maximal flats F_1, F_2 have $F_1(\infty) = F_2(\infty)$ then $F_1 = F_2$. If $F_1(\infty) \cap F_2(\infty)$ is a half-apartment and X is an Euclidean building then $F_1 \cap F_2$ is a half-apartment. A maximal flat F is said to be *asymptotic to a ray* ρ if $\rho(\infty) \in F(\infty)$.

If X is an Euclidean building, one can define *the space of directions in a point* x as the space of equivalence classes of geodesic segments having x as an endpoint with respect to the equivalence relation "angle zero in x". We denote it by $\Sigma_x X$, and we call its elements *directions in* x. We denote by $\overline{xa}$ the direction corresponding to the geodesic segment or ray $[x, a)$. Given a geodesic ray ρ, we denote by $\overline{\rho_x}$ the direction in x of the ray of origin x asymptotic to ρ. For every convex set $\mathscr{C}$ containing x we define the *set of directions of* $\mathscr{C}$ *in* x, $\mathscr{C}_x$, as the set of directions $\overline{xa}$ corresponding to all $[x, a) \subset \mathscr{C}$.

With respect to the metric induced by the angle, $\Sigma_x X$ becomes a spherical building. If X is a c-thick homogeneous Euclidean building then for every x, $\Sigma_x X$ is a c-thick spherical building.

If a segment $[x, b)$ and a convex set $\mathscr{C}$ are both included, near x, in the same apartment, then the distance between $\overline{xb}$ and $\mathscr{C}_x$ in $\Sigma_x X$ is well defined. We denote it either by $\angle_x(\overline{xb}, \mathscr{C}_x)$ or by $\angle_x(b, \mathscr{C})$.

If X is a homogeneous Euclidean building then for every x, $\Sigma_x X$ has the same model chamber as $\partial_\infty X$, Δ_{mod}, so, with respect to some labelling, one can define a projection $p_x : \Sigma_x \to \Delta_{\mathrm{mod}}$. Moreover, one can choose a labelling on $\Sigma_x X$ compatible with the one on $\partial_\infty X$, that is, such that for every point α in $\partial_\infty X$ we have $p(\alpha) = p_x(\overline{x\alpha})$. We call this common value *the slope of the ray* $[x, \alpha)$. In the sequel we suppose that for every $x \in X$ the spherical building $\Sigma_x X$ is endowed with the labelling compatible with the one of $\partial_\infty X$. This implies that if $[x, y]$ is a nontrivial geodesic segment then for every $a \in [x, y[$, $p_a(\overline{ay}) = p_x(\overline{xy})$. We call this common value *the slope of the segment* $[x, y]$. We note that the slope of the segment $[x, y]$ is in general not the same as the slope of the segment $[y, x]$ (since, generically, two opposite points in a spherical building do not project on the same point of Δ_{mod}, but on two points sent one onto the other by the opposition involution).

If X is a product of symmetric spaces and Euclidean buildings and $\alpha \in \partial_\infty X$, we call *the slope of the ray* $[x, \alpha)$ the image of α by the projection $p : \partial_\infty X \to \Delta_{\mathrm{mod}}$.

We call the ray or segment $[x, \alpha)$ *regular* (*singular*) if its slope is in *Int* Δ_{mod} ($\partial \Delta_{\mathrm{mod}}$). If θ is a slope in X we also call its set of orthogonals *set of orthogonal slopes*.

For a convex set $\mathscr{C}$ in a Euclidean building, its *set of slopes* is the set of all slopes of all nontrivial segments $[x, y] \subset \mathscr{C}$. It is a set invariant with respect to the opposition involution.

According to [22, Proposition 4.3.1] every decomposition of Δ_{mod} as a join, $\Delta_{\mathrm{mod}} = \Delta_1 \circ \Delta_2$, corresponds to a decomposition of X as a product, $X = X_1 \times X_2$, such that Δ_i is the model chamber of the factor X_i, $i = 1, 2$.

We say that a slope θ is *parallel to a factor of* X if Δ_{mod} decomposes nontrivially as a join $\Delta_{\mathrm{mod}} = \Delta_1 \circ \Delta_2$ and $\theta \in \Delta_1$. Using the previous remark one can verify that a slope is parallel to a factor if and only if one(every) segment or ray of slope θ is contained in the copy of a factor of X.

Lemma 2.2.5. *Let D be a half-apartment in an Euclidean building* $\mathbf{K}$ *and let W be a Weyl chamber with a panel in ∂D and with interior disjoint of D. Then there exists an apartment including both D and W.*

Proof. By Proposition 3.27 of Tits [33], there is an apartment A in $\Sigma_x\mathbf{K}$ including both W_x and D_x. The chamber W_x has an opposite chamber W_x' in D_x. Let ρ be a regular ray in W and ρ' the opposite ray in W'. There exists a unique apartment F containing the regular geodesic $\rho \cup \rho'$. It follows that it contains W and W', therefore also ∂D which is the convex hull of the two opposite panels $W \cap \partial D$ and $W' \cap \partial D$. Since D is the convex hull of ∂D and of W' we may conclude. □

In particular, two adjacent Weyl chambers are always included in an apartment, in an Euclidean building.

Definition 2.2.6. Let F be an apartment in an Euclidean building. We say that another apartment F' is a ramification of F if either $F' = F$ or $F \cap F'$ is a half-apartment. If the case $F' = F$ is excluded, F' is called a strict ramification of F.

Corollary 2.2.7. *Let F be an apartment and W' a Weyl chamber adjacent to a Weyl chamber $W \subset F$. Let D be the half-apartment in F containing W such that ∂D contains the panel $W \cap W'$. There exists a ramification F' of F including $D \cup W'$.*

2.3. Horoballs and horospheres

Let X be a CAT(0)-space and ρ a geodesic ray in X. *The Busemann function associated to* ρ is the function $f_\rho : X \to \mathbb{R}, f_\rho(x) = \lim_{t\to\infty} [d(x, \rho(t)) - t]$. This function is well defined and convex. Its level hypersurfaces $H_a(\rho) = \{x \in X \mid f_\rho(x) = a\}$ are called *horospheres*, its level sets $Hb_a(\rho) = \{x \in X \mid f_\rho(x) \leqslant a\}$ are called *closed horoballs* and their interiors, $Hbo_a(\rho) = \{x \in X \mid f_\rho(x) < a\}$, *open horoballs*. We use the notations $H(\rho), Hb(\rho), Hbo(\rho)$ for the horosphere, the closed and open horoball corresponding to the value $a = 0$.

For two asymptotic rays, their Busemann functions differ by a constant. Thus, the families of horoballs and horospheres are the same and we call them horoballs and horospheres *of basepoint* α, where α is the common point at infinity of the two rays.

Remark 2.3.1. Let ρ be a geodesic ray in a complete CAT(0)-space X and let $a < b$ be two real numbers.

(a) There is a natural projection p_{ba} of $H_b(\rho)$ onto $H_a(\rho)$ which is a surjective contraction.
(b) The distance from a point $x \in H_b(\rho)$ to $p_{ba}(x)$ is $b - a$.

Proof. For every $x \in H_b(\rho)$ we consider the ray ρ_x. The point $p_{ba}(x)$ is the intersection of ρ_x with $H_a(\rho)$. For every point $y \in H_a(\rho)$ the ray ρ_y can be extended to a geodesic by the completeness of X, and this geodesic intersects $H_b(\rho)$ in a unique point x. However the extension itself may not be unique. The property (a) follows by the convexity of the distance. □

Lemma 2.3.2. *Let X be a product of symmetric spaces of noncompact type and Euclidean buildings and let $\alpha_1, \alpha_2, \alpha_3$ be three distinct points in $\partial_\infty X$. If there exist three open horoballs Hbo_i of basepoints α_i, $i = 1, 2, 3$, which are mutually disjoint then $\alpha_1, \alpha_2, \alpha_3$ have the same projection on the model chamber Δ_{mod}.*

Proof. The proof is given in the proof of Proposition 5.5 [10], step (b).

2.4. Q-rank 1 lattices

A *lattice* in a Lie group G is a discrete subgroup Γ such that $\Gamma \setminus G$ admits a finite G-invariant measure. We refer to [6], [27] or [31] for a definition of $\mathbb{Q}$-rank 1 lattices in semisimple groups. In the introduction we gave some examples of $\mathbb{Q}$-rank 1 lattices. In the sequel we list the two main properties of $\mathbb{Q}$-rank 1 lattices that we use. The first one relates the word metrics to the induced metric.

Theorem 2.4.1 (Lubotzky, Mozes, Raghunathan [25], [26]). *On every irreducible lattice of a semisimple group of rank at least* 2, *the word metrics and the induced metric are bilipschitz equivalent.*

By means of horoballs one can construct a subspace X_0 of the symmetric space $X = G/K$ on which the lattice Γ acts with compact quotient.

Theorem 2.4.2 (Prasad [30] and Raghunathan [31]). *Let Γ be an irreducible lattice of $\mathbb{Q}$-rank one in a semisimple group G. Then there exists a finite set of geodesic rays $\{\rho_1, \rho_2, \ldots, \rho_k\}$ in the symmetric space $X = G/K$ such that the space $X_0 = X \setminus \bigsqcup_{i=1}^{k} \bigcup_{\gamma \in \Gamma} Hbo(\gamma\rho_i)$ has compact quotient with respect to Γ and such that any two of the horoballs $Hbo(\gamma\rho_i)$ are disjoint or coincide.*

Lemma 2.3.2 implies that $p(\{\gamma\rho_i(\infty) \mid \gamma \in \Gamma, i \in \{1, 2, \ldots, k\}\})$ is only one point which we denote by θ and we call *the associated slope of Γ* (we recall that p is the projection of the boundary at infinity onto the model chamber). We have the following property of the associated slope:

Proposition 2.4.3 (Druţu [10, Proposition 5.7]). *If Γ is an irreducible $\mathbb{Q}$-rank one lattice in a semisimple group G of $\mathbb{R}$-rank at least* 2, *the associated slope*, θ, *is never parallel to a factor of $X = G/K$.*

In particular, if G decomposes into a product of rank one factors, θ is a point in $Int\,\Delta_{\mathrm{mod}}$, that is the rays $\gamma\rho_i$, $i \in \{1, 2, \ldots, k\}$, are regular.

Since the action of Γ on X_0 has compact quotient, Γ with the word metric is quasi-isometric to X_0 with the length metric. Thus, the asymptotic cones of Γ are bilipschitz equivalent to the asymptotic cones of X_0. Theorem 2.4.1 implies that one may consider X_0 with the induced metric instead of the length metric. We study the asymptotic cones of X_0 with the induced metric.

First there is a result on asymptotic cones of symmetric spaces and Euclidean buildings.

Theorem 2.4.4 (Kleiner and Leeb [22]). *Every asymptotic cone of a product X of symmetric spaces of noncompact type and Euclidean buildings, X of rank $r \geqslant 2$, is an Euclidean building $\mathbf{K}$ of rank r which is homogeneous and $\aleph_1$-thick. The apartments of $\mathbf{K}$ appear as limits of sequences of maximal flats in X. The same is true for Weyl chambers and walls, singular subspaces and Weyl polytopes of $\mathbf{K}$. Consequently, $\partial_\infty \mathbf{K}$ and $\partial_\infty X$ have the same model spherical chamber and model Coxeter complex.*

In the sequel, in every asymptotic cone $\mathbf{K}$ of a product X of symmetric spaces and Euclidean buildings we shall consider the labelling on $\partial_\infty \mathbf{K}$ induced by a fixed labelling on $\partial_\infty X$. We denote by P the projection of $\partial_\infty \mathbf{K}$ on Δ_{mod} induced by this labelling and by S the associated Coxeter complex.

Concerning the asymptotic cone of a space X_0 obtained from a product of symmetric spaces and Euclidean buildings by deleting disjoint open horoballs, we have the following result.

Theorem 2.4.5 (Druţu [11, Propositions 3.10, 3.11]). *Let X be a* CAT(0) *geodesic metric space and let $\mathbf{K} = X_\omega(x_n, d_n)$ be an asymptotic cone of X.*

(1) *If (ρ_n) is a sequence of geodesic rays in X with $d(x_n, \rho_n)/d_n$ bounded and $\rho = [\rho_n]$ is its limit ray in $\mathbf{K}$, then $H(\rho) = [H(\rho_n)]$ and $Hb(\rho) = [Hb(\rho_n)]$.*

(2) *If $X_0 = X \setminus \bigsqcup_{\rho \in \mathscr{R}} Hbo(\rho)$ and $d(x_n, X_0)/d_n$ is bounded then the limit set of X_0 (which is the same thing as the asymptotic cone of X_0 with the induced metric) is*
$$\mathbf{K}_0 = \mathbf{K} \Big\backslash \bigsqcup_{\rho_\omega \in \mathscr{R}_\omega} Hbo(\rho_\omega), \tag{2.1}$$
where $\mathscr{R}_\omega$ is the set of rays $\rho_\omega = [\rho_n]$, $\rho_n \in \mathscr{R}$.

We note that if X is a product of symmetric spaces and Euclidean buildings, the mutual disjointness of the horoballs $Hbo(\rho)$, $\rho \in \mathscr{R}$, implies by Lemma 2.3.2 that $p(\{\rho(\infty) \mid \rho \in \mathscr{R}\})$ reduces to one point, θ, if card $\mathscr{R} \neq 2$. Then $P(\{\rho_\omega(\infty) \mid \rho_\omega \in \mathscr{R}_\omega\}) = \theta$.

We also need the following result.

Lemma 2.4.6. *Let X be a product of symmetric spaces of noncompact type and Euclidean buildings, X of rank $r \geqslant 2$, and let $\mathbf{K} = X_\omega(x_n, d_n)$ be an asymptotic cone of it. Let F_ω and ρ_ω be an apartment and a geodesic ray in $\mathbf{K}$, F_ω asymptotic to ρ_ω. Let $\rho_\omega = [\rho_n]$, where ρ_n have the same slopes as ρ_ω. Then*

(a) *F_ω can be written as limit set $F_\omega = [F_n]$ with F_n asymptotic to ρ_n ω-almost surely;*

(b) *every geodesic segment $[x, y]$ in $F_\omega \setminus Hbo(\rho_\omega)$ can be written as limit set of segments $[x_n, y_n] \subset F_n \setminus Hbo(\rho_n)$.*

Proof. (a) Suppose first that ρ_ω is regular with slope θ. Then ρ_n are of slope θ. Let $F_\omega = [F_n']$ and let x_n' be the projection of $\rho_n(0)$ onto F_n'. We may replace in the argument each ray ρ_n with the ray asymptotic to ρ_n of origin x_n'. So in the sequel we may suppose that ρ_n has its origin x_n' in F_n'. By hypothesis we have $[\rho_n] \subset [F_n']$, hence there exist rays $\rho_n' \subset F_n'$ of slopes θ and origin x_n' such that if x_n'' is the first point of ρ_n at distance d_n from ρ_n', $\lim_\omega d(x_n, x_n'')/d_n = +\infty$. If such a point

x_n'' does not exist then $\rho_n \subset \mathcal{N}_{d_n}(F_n)$. This implies that $\rho_n \subset F_n$ and we are done. So we suppose that x_n'' always exists. It follows that $\angle_{x_n'}(\bar{\rho}_n, \bar{\rho}_n') \to 0$ as $n \to \infty$. For n sufficiently large, $\rho_n(+\infty)$ becomes opposite to $\rho_n^{op}(+\infty)$ in $\partial_\infty X$, where ρ_n^{op} is the ray opposite to ρ_n' in F_n'. This happens because up to isometry we may suppose that F_n, x_n' and ρ_n' are fixed and then we can use the lower semi-continuity of the Tits metric with respect to the cone topology.

Let F_n be the unique maximal flat containing $\rho_n(+\infty)$ and $\rho_n^{op}(+\infty)$ in its boundary. With an argument up to isometry similar to the previous one we can prove that $d(x_n', F_n)$ is uniformly bounded by a constant M. Let x_n''' be the projection of x_n'' on F_n. Let W_n be the Weyl chamber of vertex x_n''' containing $\rho_n^{op}(\infty)$ in the boundary and let W_n' be the Weyl chamber asymptotic to it of vertex $\mathrm{proj}_{F_n'}(x_n'')$. The Hausdorff distance $d_H(W_n, W_n')$ is at most $d_n + M$ hence in the asymptotic cone $d_H([W_n],[W_n']) \leqslant 1$. On the other hand, $\lim_\omega d(x_n, x_n'')/d_n = \lim_\omega d(x_n, x_n''')/d_n = +\infty$, so $[W_n] = [F_n]$ and $[W_n'] = [F_n']$. It follows that $d_H([F_n],[F_n']) \leqslant 1$ which implies $[F_n] = [F_n'] = F_\omega$.

The case when ρ_ω is singular can be reduced to the previous case by choosing ρ_ω^0 regular in the same Weyl chamber as ρ_ω and asymptotic to F_ω and repeating the previous argument.

(b) The set $F_\omega \backslash Hbo(\rho_\omega)$ as well as ω-almost all sets $F_n \backslash Hbo(\rho_n)$ are half-flats and the conclusion follows easily. $\square$

3. Horospheres in Euclidean buildings: intersections with apartments and nondistorsion

3.1. Intersection of a horoball with an apartment: global properties

We first introduce a notation. For every k-flat Φ (not necessarily singular) and every geodesic ray ρ such that $\inf_{x\in\Phi} f_\rho(x) > -\infty$ we denote

$$Min_\Phi(\rho) = \left\{ y \in \Phi \,\middle|\, f_\rho(y) = \inf_{x\in\Phi} f_\rho(x) \right\}.$$

We recall that f_ρ denotes the Busemann function associated to ρ. We use a similar notation for a half-apartment D instead of the flat Φ.

We describe some global features of intersections between horoballs/horospheres and apartments.

Proposition 3.1.1 (Druţu [10, Proposition 3.1, Lemmata 3.2, 3.8]). *Let* $\mathbf{K}$ *be an Euclidean building,* F *an apartment in it and* $\rho \subset \mathbf{K}$ *a geodesic ray of slope* θ. *Let* $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ *be the points in* $F(\infty)$ *opposite to* $\rho(\infty)$, $k \leqslant q_0$, *where* q_0 *is the number of chambers of the spherical Coxeter complex* S *associated to* $\partial_\infty\mathbf{K}$.

(a) *The flat* F *can be written as* $F = \bigcup_{i=1}^m [F \cap F_i]$, *where*
  - *each* F_i *is an apartment with* $F_i(\infty)$ *containing* $\rho(\infty)$;
  - *each* $F(\infty) \cap F_i(\infty)$ *contains a unique point* α_{j_i} *opposite to* $\rho(\infty)$.

  If $\theta \in \partial\Delta_{\mathrm{mod}}$ *then it is possible that there exist* $i_1 \neq i_2$ *with* $\alpha_{j_{i_1}} = \alpha_{j_{i_2}}$.

(b) *The intersection* $Hb(\rho) \cap F$, *if nonempty, is a convex polytope in* F. *The hypersurface* $H(\rho) \cap F$ *has at most* k *faces.*

(c) *Suppose* $\inf_{x\in F} f_\rho(x) > -\infty$. *The set* $Min_F(\rho)$ *is a Weyl polytope with at most m faces,* $m = m(\mathsf{S})$. *All the slopes of* $Min_F(\rho)$ *are included in* $Ort(\theta)$.

(c_1) *If* $Min_F(\rho)$ *has codimension* 1 *in* F *then it is included in a singular hyperplane* Φ. *There exists* $\epsilon = \epsilon(\mathsf{S})$ *such that*:

- *every two affine spans of codimension one faces of* $Min_F(\rho)$, *if not parallel, make dihedral angles greater that* ϵ *and smaller than* $\pi - \epsilon$;
- *every affine span of a codimension one face of* $Min_F(\rho)$ *makes with every* $H \cap \Phi$, H *singular hyperplane different of* Φ, *dihedral angles greater that* ϵ *and smaller than* $\pi - \epsilon$, *if not parallel to it.*

(c_2) *If* Φ *is a k-flat (not necessarily singular) contained in* F *such that* $\inf_{x\in\Phi} f_\rho(x) > -\infty$ *then* $Min_\Phi(\rho)$ *is the intersection of a Weyl polytope with* Φ.

Proof. We prove the statements that are not proved in the quoted reference. We note that (b) follows from (a) since $Hb(\rho)\cap F$ is a convex set, $Hb(\rho)\cap F = \bigcup_{i=1}^m [Hb(\rho)\cap F\cap F_i]$, $Hb(\rho)\cap F_i$ is a half-apartment and $F\cap F_i$ is a Weyl polytope. In (c) the fact that the number of faces of $Min_F(\rho)$ is uniformly bounded is a consequence of the fact that all faces are supported by intersections of the affine span with singular hyperplanes, and that, by the convexity of $Min_F(\rho)$, there can be at most one face parallel to a fixed face.

Suppose $Min_F(\rho)$ has codimension 1 and is not included in a singular hyperplane. Then it contains a segment $[x, y]$ of regular slope. Let W be a Weyl chamber in F of vertex x including $(x, y]$ in its interior. It follows that $\angle_x(W_x, \overline{\rho_x}) < \pi/2$, which implies that f_ρ takes in W smaller values than $f_\rho(x)$. This contradicts the definition of $Min_F(\rho)$. Hence $Min_F(\rho)$ is contained in a singular hyperplane Φ. Let $\epsilon > 0$ be such that in S every two distinct singular codimension 2 subspaces are at Hausdorff distance at least ϵ and at most $\pi - \epsilon$ one from the other. The second part of the statement in (c_1) then follows from the fact that all affine spans of codimension 1 faces of $Min_F(\rho)$ are codimension 2 singular subspaces in F. □

Corollary 3.1.2. *Let* $\mathbf{K}$ *be an Euclidean building,* $\rho \subset \mathbf{K}$ *a geodesic ray of slope* θ *and* F *an apartment which intersects* $Hb_t(\rho)$. *For every* $s > t$ *the projection in* F *of* $H_s(\rho)\cap F$ *onto* $H_t(\rho)\cap F$ *is a contraction. Moreover, there exists a constant* $a > 1$ *depending only on* θ *and on the spherical Coxeter complex* S *of* $\mathbf{K}$ *such that the distance from a point of* $H_s(\rho)\cap F$ *to its projection onto* $H_t(\rho)\cap F$ *is at most* $a(s-t)$.

Proof. The first statement is a straightforward consequence of Proposition 3.1.1, (b), and of the fact that $Hb_t(\rho) \subset Hb_s(\rho)$.

If θ is parallel to a factor of rank 1 of $\mathbf{K}$ then the second statement is obvious, with $a = 1$. Suppose θ is not parallel to a rank 1 factor of $\mathbf{K}$.

Let x be a point in $H_s(\rho)\cap F$ and y its projection on $H_t(\rho)\cap F$. For every point $z\in Hb_t(\rho)\cap F$, $\angle_y(\overline{yx}, \overline{yz}) \geqslant \pi/2$. Let M be the unique wall or Weyl chamber of vertex y containing the segment $(y,x]$ in its interior. Then M cannot have a segment in common with $H_t(\rho)\cap F$, otherwise the diameter of M_y in the spherical building $\Sigma_y\mathbf{K}$ would be $> \pi/2$. It follows that $\angle_y(M_y, \overline{\rho_y}) > \pi/2$ and therefore, by Remark 2.2.4, that $\angle_y(M_y, \overline{\rho_y}) \geqslant (\pi/2) + \delta_0$, with δ_0 depending only on θ and on S. In particular, $\angle_y(\overline{yx}, \overline{\rho_y}) \geqslant (\pi/2) + \delta_0$, which implies that $s = f_\rho(x) \geqslant f_\rho(y) + d(y,x)\sin\delta_0 = t + d(y,x)\sin\delta_0$. The conclusion follows with $a = 1/\sin\delta_0$. □

Corollary 3.1.3. *Let* $\mathbf{K}$ *be an Euclidean building*, F *an apartment in it and* ρ *a ray of slope* θ. *If* $\inf_{x\in F} f_\rho(x) = -m > -\infty$ *then there exists a constant* $a > 1$ *depending only on the spherical Coxeter complex of* $\mathbf{K}$ *and on* θ *such that*

$$\mathcal{N}_m(Min_F(\rho)) \subset Hb(\rho) \cap F \subset \mathcal{N}_{am}(Min_F(\rho)).$$

Proof. The first inclusion follows from the fact that the Busemann function f_ρ is Lipschitz of constant 1. The second inclusion follows from the previous corollary applied to $H_{-m}(\rho) \cap F$ and each $H_{-s}(\rho) \cap F$ with $0 \leqslant s < m$. $\square$

Corollary 3.1.3 shows that the shape of $Min_F(\rho)$ is important for the shape of $F \cap Hb(\rho)$. Therefore, in the sequel we formulate results about the way $Min_F(\rho)$ changes when we change the apartment F with a ramification of it.

Lemma 3.1.4 (Druţu [10, Lemma 3.12]). *Let* $\mathbf{K}$ *be a* 4*-thick Euclidean building of rank at least* 2, *let* ρ *be a geodesic ray in it, not parallel to any factor, let* F *be an apartment*, D *a half-apartment in* F *and* Φ *the affine span of* $Min_D(\rho)$ *in* F. *Let* D_0 *be the half-apartment in* F *opposite to* D, *and let* D_1, D_2 *be two other half-apartments of boundary* $H = \partial D$ *and interiors mutually disjoint and disjoint of* F.

If the singular hyperplane H *neither contains* Φ *nor is orthogonal to it then* $\inf_{D_i} f_\rho \geqslant \inf_D f_\rho$, $\forall i \in \{0,1,2\}$, *and* $Min_{D_i\cup D}(\rho) = Min_D(\rho)$ *for at least two values of* i *in* $\{0,1,2\}$.

Lemma 3.1.4 implies that in an apartment F, by eventually changing a half-apartment, we can always make sure that $Min_F(\rho)$ stays in a single half-apartment, and by applying the Lemma twice we can even make sure that $Min_F(\rho)$ stays in a strip determined by two parallel hyperplanes. The only thing needed to apply the Lemma is a singular hyperplane which neither contains nor is orthogonal to the affine span of $Min_F(\rho)$. The following results are about the existence of such singular hyperplanes.

Lemma 3.1.5. *Let* X *be a product of symmetric spaces of noncompact type and Euclidean buildings*, X *of rank* $r \geqslant 2$, *let* ρ *be a geodesic ray which is not parallel to any factor of* X, *let* F *be a maximal flat containing* ρ *and let* $\Phi \subset F$ *be a singular flat orthogonal to* ρ. *Every Weyl chamber of* F *has an adjacent singular hyperplane which is neither parallel to* Φ (*or containing* Φ) *nor orthogonal to* Φ.

Proof. Since all Weyl chambers are translations of a finite set of Weyl chambers with common vertex, we may restrict the problem to Weyl chambers having the vertex on Φ. Let W be such a Weyl chamber. If Φ supports W then the result follows from Lemma 3.11 [10]. Suppose Φ does not support W. If the dimension of Φ is k then the number of singular hyperplanes supporting W and containing Φ must be strictly less than $r - k$. So there are at least $k + 1$ hyperplanes supporting W and not containing Φ. They cannot be all orthogonal to Φ because if we choose vectors orthogonal to each of these hyperplanes we obtain a family of at least $k + 1$ linearly independent vectors, so the entire family cannot be contained in Φ. $\square$

Corollary 3.1.6. *Let X and ρ be as in the previous Lemma. If F is a maximal flat not containing ρ, $\mathscr{P}$ is a Weyl polytope in F on which f_ρ is constant and Φ is its affine span, the conclusion of the Lemma* 3.1.5 *still holds.*

Proof. By Proposition 3.1.1, (a), a relatively open subset Ω of the Weyl polytope $\mathscr{P}$ shall be entirely contained into an intersection $F \cap F_i$, where F_i contains a ray ρ_i asymptotic to ρ and $F(\infty) \cap F_i(\infty)$ contains a unique point α opposite to $\rho(\infty)$. Let ρ^{op} be a ray with origin on Ω and $\rho^{op}(\infty) = \alpha$. Then ρ^{op} is included in $F \cap F_i$ and in F_i it is opposite to a ray parallel to ρ_i. Hence it is orthogonal to Ω and to Φ in F, on one hand, and on the other hand, it is not parallel to a factor in X. By applying Lemma 3.1.5 with ρ^{op} instead of ρ we obtain the same conclusion. □

3.2. Intersection of a horoball with an apartment: local properties

We look at an intersection $F \cap H(\rho)$ in the neighbourhood of one of its points x. The first and the second part of the following proposition give a meaning to the directions orthogonal to the codimension one faces of $F \cap H(\rho)$ through x. The third part describes a situation in which $F \cap H(\rho)$ has two codimension one faces through x symmetric with respect to a singular hyperplane through x. This result is essential in the argument of "breaking the faces" described in Lemma 3.3.2.

Proposition 3.2.1. *Let $\mathbf{K}$ be an Euclidean building, $\rho \subset \mathbf{K}$ a geodesic ray of slope θ and F an apartment which intersects $Hb(\rho)$. Let $x \in F \cap H(\rho)$.*

(a) *There exists $\delta > 0$ such that $F \cap B(x,\delta) = \bigcup_{i=1}^{s} [F \cap F_i \cap B(x,\delta)]$, where*
  - *F_i is an apartment and $F_i \cap B(x,\delta)$ contains $\rho_x \cap B(x,\delta)$;*
  - *each $F_i \cap B(x,\delta)$ contains at least one set of the form $W \cap B(x,\delta)$, where W is a Weyl chamber of F of vertex x;*
  - *each set $F \cap F_i \cap B(x,\delta)$ contains only one segment of length δ, $[x,a_i)$, of direction $\overline{xa_i}$ opposite to $\overline{\rho_x}$.*

  If $\theta \in \partial\Delta_{\rm mod}$ then it is possible that there exist $i_1 \neq i_2$ with $a_{i_1} = a_{i_2}$.

(b) *Every codimension one face of $F \cap H(\rho)$ through x is orthogonal to one of the segments $[x,a_i)$. A segment $[x,a_i)$ is orthogonal to a face of $F \cap H(\rho)$ through x if and only if for some component $F \cap F_i \cap B(x,\delta)$ including it either $\angle_x(\overline{\rho_x},(F \cap F_i)_x) < \pi/2$ or $\angle_x(\overline{\rho_x},(F \cap F_i)_x) = \pi/2$ and $(F \cap F_i)_x$ contains a panel orthogonal to $\overline{\rho_x}$.*

(c) *Suppose $F \cap Hbo(\rho) \neq \emptyset$. If x is in the interior of a codimension one face of $F \cap H(\rho)$ then there exists $\delta > 0$ such that $F \cap B(x,\delta)$ contains $\rho_x \cap B(x,\delta)$.*

(d) *Let W^1 and W^2 be two adjacent chambers in F of vertex x, $W^1 \cap W^2 = M$. Suppose that W^1_x and W^2_x are at the same combinatorial distance from $\overline{\rho_x}$ and that $\angle_x(W^t_x,\overline{\rho_x}) < \pi/2$, $t = 1,2$. Suppose no singular hyperplane through $\overline{\rho_x}$ contains M_x. Then there exist $i_1 \neq i_2$ such that $W^t \cap B(x,\delta) \subset F \cap F_{i_t} \cap B(x,\delta)$, $t = 1,2$, and the segments $[x,a_{i_1}]$ and $[x,a_{i_2}]$ are symmetric with respect to Span M.*

Proof. (a) We choose an auxiliary ray ρ^0. If ρ is regular, then $\rho^0 = \rho_x$ while if ρ is singular we choose a Weyl chamber W^0 of vertex x containing ρ_x, and ρ_0 a regular ray in W^0 of origin x. Let $\{\beta_1, \beta_2, \dots, \beta_s\}$ be the points in F_x opposite to $\overline{\rho^0}$ and let A_i be the unique apartment containing β_i and $\overline{\rho^0}$. By Lemma 3.10.2 of Kleiner and Leeb [22], every point in F_x lies in some A_i, so $F_x = \bigcup_{i=1}^{s} [F_x \cap A_i]$. By Lemma 4.2.3 of Kleiner and Leeb [22], $A_i = (F_i)_x$, where F_i is an apartment in the Euclidean building containing x. Lemma 4.1.2, (1), and Sublemma 4.4.1 of Kleiner and Leeb [22] allow to conclude that (a) is true.

(b) First we notice that if $Ort(\theta) \not\subset \partial\Delta_{\mathrm{mod}}$ then any open set in it contains a regular point, while if $Ort(\theta) \subset \partial\Delta_{\mathrm{mod}}$ any open set in it contains a point in the interior of a panel. For every codimension one face $\mathfrak{f}$ of $H(\rho) \cap F$ through x we consider a point $\eta \in \mathfrak{f}_x$, regular if we are in the first case or in the interior of a panel if we are in the second case. The point η is contained in $F_x \cap A_i$ for some $i \in \{1, 2, \dots, s\}$. It follows that an open subset of $\mathfrak{f}$ is included in $F \cap F_i \cap B(x, \delta)$, so it is orthogonal to $[x, a_i)$.

We now prove the equivalence stated in (b). We only prove the direct implication, since the reciprocal is an easy consequence of (a) and of previously used results. Suppose that $[x, a_i)$ is orthogonal to a codimension one face $\mathfrak{f}$ of $H(\rho) \cap F$ through x. We choose a point η as previously in $\mathfrak{f}_x$. In $\Sigma_x\mathbf{K}$ we consider the geodesic between $\overline{xa_i}$ and η, of length $\pi/2$. This geodesic contains a regular point $\eta^0 \neq \eta$ in a chamber $W_x \subset F_x$ intersecting $\mathfrak{f}_x$ either in a relatively open set of regular points or in a panel. On the other hand, W_x is included in some $(F \cap F_j)_x$ and the geodesic from η to η^0 extends to a geodesic of length $\pi/2$ with endpoint the opposite direction $\overline{xa_j}$. But since in F_x the geodesic from η to η^0 extends in a unique way, it follows that $j = i$. Since $(F \cap F_j)_x$ has in common with $\mathfrak{f}_x$ either a relatively open set of regular points or a panel, we may conclude.

(c) Let H be the affine span of the face containing x. If H is not a singular hyperplane then we choose $[a, b]$ a regular segment through $x, x \neq a, b$. In $\Sigma_x\mathbf{K}$, $\overline{\rho_x}$ is contained in a geodesic from $\overline{xa}$ to $\overline{xb}$. It follows that $\overline{\rho_x}$ is contained in F_x and we may conclude by Lemma 4.1.2 of Kleiner and Leeb [22].

Suppose H is a singular hyperplane. We choose W and W' two opposite Weyl chambers in F of vertex x supported by H, and $[x, a] \subset W$, $[x, b] \subset W'$ two opposite regular segments. Suppose $W \cap B(x, \delta) \subset Hb(\rho)$ for a small δ. Then $\angle_x(\overline{\rho_x}, W_x) < \pi/2$. If $\angle_x(\overline{\rho_x}, \overline{xa}) = (\pi/2) - \varepsilon$ and since the opposition involution is an isometry from W_x to W'_x we have $\angle_x(\bar{\rho}_x, \overline{xb}) = (\pi/2) + \varepsilon$. It follows that $\bar{\rho}_x$ is on a geodesic from $\overline{xa}$ to $\overline{xb}$, hence $\bar{\rho}_x \in F_x$.

(d) Statement (a) implies that there exists i_t such that $W^t \cap B(x, \delta) \subset F \cap F_{i_t} \cap B(x, \delta)$, $t = 1, 2$. Since W_x^1 and W_x^2 are at the same combinatorial distance from $\overline{\rho_x}$ and no singular hyperplane through $\overline{\rho_x}$ contains the common panel M_x, according to Lemma 2.2.1 none of W_x^1 or W_x^2 can be the projection of $\overline{\rho_x}$ on M_x and both are separated from $\overline{\rho_x}$ by M_x. Also, $W_x^1 \neq W_x^2$ implies $i_1 \neq i_2$. Moreover, W_x^1 and W_x^2 are at the same combinatorial distance from $(W^0)_x$, with W^0 chosen as in the proof of (a). This implies that $\mathrm{retr}_{(F_{i_1})_x, (W^0)_x}(W_x^2) = W_x^1$.

The hypothesis that $\angle_x(W_x^t, \overline{\rho_x}) < \pi/2$ implies that there exist two points ξ_t in the interiors of W_x^t with $\angle_x(\xi_t, \overline{\rho_x}) < \pi/2$, $t = 1, 2$, and $p_x(\xi_1) = p_x(\xi_2)$. In $(F_{i_t})_x$ the geodesic γ_t joining $\overline{\rho_x}$ to ξ_t extends to a geodesic joining $\overline{\rho_x}$ to $\overline{xa_{i_t}}$. The previous considerations imply that $\mathrm{retr}_{(F_{i_1})_x, (W^0)_x}(\xi_2) = \xi_1$, hence that $\mathrm{retr}_{(F_{i_1})_x, (W^0)_x}(\gamma_2) = \gamma_1$ and in particular that γ_1 and γ_2 intersect M_x in the same point ζ. In F_x the geodesics from ζ to ξ_1 and from ζ to ξ_2 are symmetric with respect to $Span\, M_x$, so the same is true for their prolongations to $\overline{xa_{i_1}}$ and to $\overline{xa_{i_2}}$, respectively. In particular, a_{i_1} and a_{i_2} are symmetric with respect to $Span\, M$, which ends the proof. □

Corollary 3.2.2. *Let* $\mathbf{K}$ *be a* 3*-thick Euclidean building,* x *a point in it and* $\rho \subset \mathbf{K}$ *a geodesic ray of slope* θ. *Let* W *be a Weyl chamber of vertex* x *such that* W_x *does not contain* $\overline{\rho_x}$ *and* $\angle_x(\overline{\rho_x}, W_x) < \pi/2$. *Let* M *be a panel such that* M_x *separates* W_x *and* $\overline{\rho_x}$. *Then there exists a Weyl chamber* $\widehat{W}$ *adjacent to* W, *with* $\widehat{W} \cap W = M$, *such that in any apartment* F *containing both* W *and* $\widehat{W}$ *the point* x *is contained in two faces of* $F \cap H(\rho)$.

Proof. Remark 2.2.4 implies that $\angle_x(\overline{\rho_x}, W_x) \leqslant (\pi/2) - \delta_0$, where δ_0 depends only on θ. There exist at least two Weyl chambers adjacent to W containing M and at least one of these two chambers, $\widehat{W}$, has the property that W_x and $\widehat{W}_x$ are at the same combinatorial distance from $\overline{\rho_x}$. Let F be an apartment containing W and $\widehat{W}$. We denote $\widehat{H}$ the affine span of M in F. By Proposition 3.2.1, (d), x is contained into two faces of $F \cap H(\rho)$, orthogonal to two segments $[x, a_1)$ and $[x, a_2)$ symmetric with respect to $\mathit{Span}\, M$. By the properties of M, $\widehat{H}$ does not contain $[x, a_1)$. Also, if $\widehat{H}$ would be orthogonal to $[x, a_1)$ then M_x would be orthogonal to $\overline{xa_1}$ in $\Sigma_x\mathbf{K}$. This would contradict $\angle_x(\overline{\rho_x}, M_x) \leqslant \angle_x(\overline{\rho_x}, W_x) \leqslant (\pi/2) - \delta_0$ and $\angle_x(\overline{\rho_x}, \overline{xa_1}) = \pi$. Therefore, $\widehat{H}$ is not orthogonal to $[x, a_1)$ neither. It follows that $[x, a_1)$ and $[x, a_2)$ are not on the same line, which implies that the two corresponding faces are distinct. $\square$

3.3. Nondistorsion of horospheres

In the proof of Theorem 4.3 we shall need the following two Lemmata. As a byproduct we also obtain new and considerably shorter proofs of the Theorems 1.1 and 1.2 in Druţu [10] on the nondistorsion of horospheres in Euclidean buildings and symmetric spaces.

Both Lemmata deal with the possibility of joining two points x, y in the intersection of an apartment with a horosphere $F \cap H(\rho)$ with a polygonal line included in $F \cap H(\rho)$ and of length comparable to $d(x, y)$. Lemma 3.3.1 states that this can easily be done if x and y are contained into nonparallel codimension one faces of $F \cap H(\rho)$. Lemma 3.3.2 deals with the case when x and y are contained in the interiors of codimension 1 parallel faces. The main idea in the argument is that, by choosing a singular hyperplane skew to the two affine spans of the two faces and by changing one of the two half-apartments bounded by this hyperplane one is able to "break" one of the faces into two different faces. The details we give in the statement of the Lemma on the different possibilities of "breaking" one of the two faces will be necessary further on.

Lemma 3.3.1 (Nonparallel faces). *Let* $\mathbf{K}$ *be an Euclidean building of rank at least* 2 *and of spherical Coxeter complex* S, *and let* ρ *be a geodesic ray in it of slope* θ, *not parallel to a rank one factor. There exists a constant* $C = C(\mathsf{S}, \theta)$ *such that for every apartment* F *intersecting* $Hb(\rho)$, *every two points* $x, y \in F \cap H(\rho)$, *contained into two distinct nonparallel codimension one faces of* $F \cap H(\rho)$, *can be joined with a polygonal line in* $F \cap H(\rho)$ *of length at most* $Cd(x, y)$. *Moreover, the polygonal line may be chosen in a half-plane in* F *having as boundary the line* xy.

In particular, the previous statement applies for every pair of points in $F \cap H(\rho)$ *one of which is contained into two different codimension one faces of* $F \cap H(\rho)$.

We note that the hypothesis of ρ not being parallel to a rank one factor ensures the existence of two nonparallel faces.

Proof. By hypothesis, x and y are contained into two codimension one faces of the convex polytope $F \cap Hb(\rho)$ whose affine spans, H and H', respectively, are not parallel. The set of slopes of both H and H' is $Ort(\theta)$. There is a minimal possible dihedral angle between two distinct and non-parallel hyperplanes of set of slopes $Ort(\theta)$, which we denote by ς. So $\angle(H,H') \geqslant \varsigma$. As $F \cap Hb(\rho)$ is a convex polytope, it is entirely included in a skew quadrant determined by H and H', which we denote by $\mathscr{Q}$. On the other hand, we have $x \in H$ and $y \in H'$. Since $\angle(H,H') \geqslant \varsigma$, there exists a constant C depending only on ς such that x and y may be joined with a polygonal line included in $\partial\mathscr{Q}$ of length at most $Cd(x,y)$. The polygonal line of minimal length will be of the form $[x,z] \cup [z,y]$ with $z \in H \cap H'$. The plane $\mathscr{P}$ determined by the points x,y,z in F intersects $Hb(\rho)$ in a convex polygon entirely included in a sector of vertex z and sides the rays through x and y. It follows that the polygonal line joining x and y in $\mathscr{P} \cap H(\rho)$, contained inside the triangle xyz, has length at most $Cd(x,y)$. □

Notation: Henceforth for a pair of points x, y, in the intersection of an apartment with a horosphere $F \cap H(\rho)$, x, y in nonparallel faces, we shall denote by $\mathbf{L}_{xy}$ a polygonal line joining them in $F \cap H(\rho)$, of length at most $Cd(x,y)$, included in a half-plane of boundary xy, constructed as previously.

Lemma 3.3.2 (Breaking parallel faces). *Let* $\mathbf{K}$, ρ *and* θ *be as in the previous Lemma with the additional hypothesis that* $\mathbf{K}$ *is* 3-*thick and that* ρ *is not parallel to any factor of* $\mathbf{K}$. *Let* F *be an apartment intersecting* $Hb(\rho)$, *and* x *and* y *two points in the interiors of two distinct codimension one faces of* $F \cap H(\rho)$ *with parallel affine spans* H *and* H', *respectively. Let* M_0 *be the unique Weyl chamber or wall of vertex* x *including* $(x,y]$ *in its interior. We define the set* $\mathscr{CN}$ *to be either* $Star(M_0) \cap F$ *if* M_0 *is a wall or* M_0 *if* M_0 *is a Weyl chamber.*

(1) *There exists a constant* $C = C(\mathsf{S},\theta)$ *and a ramification* F' *of* F *containing* M_0 *such that* x *and* y *may be joined in* $F' \cap H(\rho)$ *with a polygonal line of length at most* $Cd(x,y)$.
(2) *The choice of the ramification* F' *can be made as follows*:
  (a) *Suppose* $(M_0)_x$ *does not contain* $\overline{\rho_x}$. *Let* W *be a Weyl chamber in* $\mathscr{CN}$ *such that* W_x *is not the projection of* $\overline{\rho_x}$ *on* $(M_0)_x$ *if* M_0 *is a wall. Then there exists a ramification* F' *of* F *including* W *such that* x *is contained into two codimension one faces of* $F' \cap H(\rho)$.
  (b) *Suppose* $(M_0)_x$ *contains* $\overline{\rho_x}$.
  (b_1) *If the connected component of* $\mathscr{CN} \cap H(\rho)$ *containing* y *has at least two faces then* $F' = F$.
  (b_2) *Suppose the connected component of* $\mathscr{CN} \cap H(\rho)$ *containing* y *has one face. Let* W *be a Weyl chamber in* $\mathscr{CN}$. *For every hyperplane* $\widehat{H}$ *supporting* W *which is neither orthogonal nor coincident with* H, *there exists a ramification* F' *of* F *containing* W *and such that* $\partial(F' \cap F) = \widehat{H}$ *and all the points in* $H(\rho) \cap \widehat{H} \cap W \setminus \{x\}$ *are in two different faces of* $F' \cap H(\rho)$.

Proof. In Fig. 1, the cases (b_1) and (b_2) are represented.

We prove that the choices of the ramification F' proposed in (2) are always possible and that they satisfy (1).

By hypothesis $(x,y) \subset F \cap Hbo(\rho)$. It follows that $\angle_x(\overline{xy}, \overline{\rho_x}) < \pi/2$ which, by Remark 2.2.4, implies that $\angle_x((M_0)_x, \overline{\rho_x}) \leqslant (\pi/2) - \delta_0$, $\delta_0 = \delta_0(\theta,\mathsf{S})$.

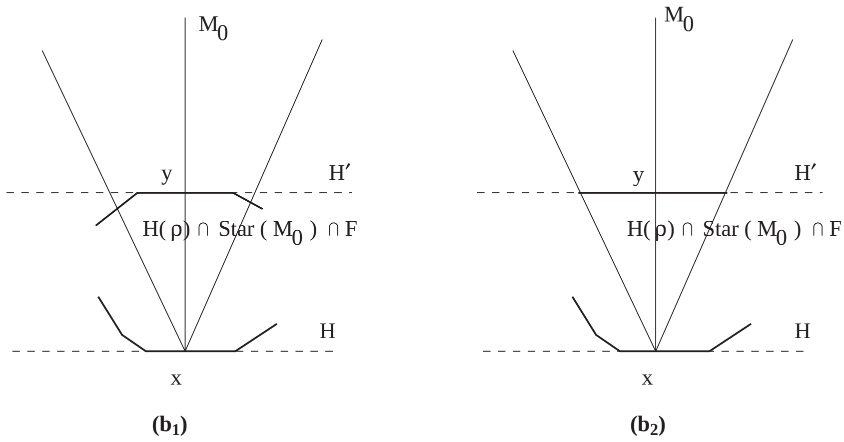

Fig. 1. Types of intersections between the set $\mathscr{C}\mathscr{N}$ and the horoball $H(\rho)$.

(2) (a) The chamber W_x does not contain $\overline{\rho_x}$ and it has a panel M_x separating W_x and $\overline{\rho_x}$. This and the fact that $\angle_x(W_x, \overline{\rho_x}) \leqslant (\pi/2) - \delta_0$ imply, by Corollary 3.2.2 and Lemma 2.2.5, that there exists a ramification F' of F with the desired properties.

(2) (b) The hypothesis that ρ is not parallel to any factor and Lemma 2.2.3 imply that ρ_x is not perpendicular near x to any wall of any Weyl chamber in $\mathscr{C}\mathscr{N}$. In particular, the intersection of H with $\mathscr{C}\mathscr{N}$ reduces to the point x. This implies that the intersection of any Weyl chamber in $\mathscr{C}\mathscr{N}$ with H' is a simplex of diameter $\leqslant \mathbf{c} \cdot d(x, y)$, $\mathbf{c} = \mathbf{c}(\theta, \mathsf{S})$. It follows that $\mathscr{C}\mathscr{N} \cap H'$ is a convex polytope in H' of diameter $\leqslant \mathbf{c}' \cdot d(x, y)$, $\mathbf{c}' = \mathbf{c}'(\theta, \mathsf{S})$. From this and the fact that the connected component of $\mathscr{C}\mathscr{N} \cap H(\rho)$ containing y is included in the truncated polytopic cone determined by $\mathscr{C}\mathscr{N}$ and H' we can deduce that the diameter of the connected component of $\mathscr{C}\mathscr{N} \cap H(\rho)$ containing y, considered with its length metric, is at most $\mathbf{c}'' \cdot d(x, y)$, $\mathbf{c}'' = \mathbf{c}''(\theta, \mathsf{S})$.

(b_1) By hypothesis there exists y_1 in the connected component of $\mathscr{C}\mathscr{N} \cap H(\rho)$ containing y, which is contained into two faces. The previous considerations imply that y_1 may be joined to y with a polygonal line in $F \cap H(\rho)$ of length at most $\mathbf{c}'' \cdot d(x, y)$. Lemma 3.3.1 implies that y_1 may be joined to x with a polygonal line in $F \cap H(\rho)$ of length at most $C(\mathbf{c}'' + 1) \cdot d(x, y)$.

(b_2) In this case the connected component of $\mathscr{C}\mathscr{N} \cap H(\rho)$ containing y is simply $\mathscr{C}\mathscr{N} \cap H'$. Let W and $\widehat{H}$ be as in the statement. The existence of $\widehat{H}$ is guaranteed by Corollary 3.1.6. We denote $M = W \cap \widehat{H}$, $\widehat{D}$ the half-apartment bounded by $\widehat{H}$ including W and $\widehat{D}'$ the opposite half-apartment in F. We prove the statement in (b_2) for an arbitrary point $y_1 \in Int\, M \cap H(\rho) = Int\, M \cap H'$. Suppose y_1 is also in the interior of the face of $F \cap H(\rho)$ of span H', otherwise we are in the case (b_1) and we may take $F' = F$.

By hypothesis $\overline{\rho_x} \in (M_0)_x$ so $\rho_x \cap B(x, \varepsilon) \subset M_0 \cap B(x, \varepsilon)$ for a small $\varepsilon > 0$. Let d be the line in F containing the segment $\rho_x \cap B(x, \varepsilon)$. The fact that $d \cap M_0$, $[x, y]$ and M are contained in the same Weyl chamber implies that the skew quadrant determined by $\widehat{H}$ and H' containing $[x, y]$ has a dihedral angle $\alpha < \pi/2$ (Fig. 2). The same follows for the opposite quadrant, contained in $\widehat{D}'$. Let $\mathscr{D}H'$ and $\mathscr{D}\widehat{H}$ be the half-hyperplanes bounding the latter quadrant. For a small ε_1, $H(\rho) \cap \widehat{D}' \cap B(y_1, \varepsilon_1) = \mathscr{D}H' \cap B(y_1, \varepsilon_1)$.

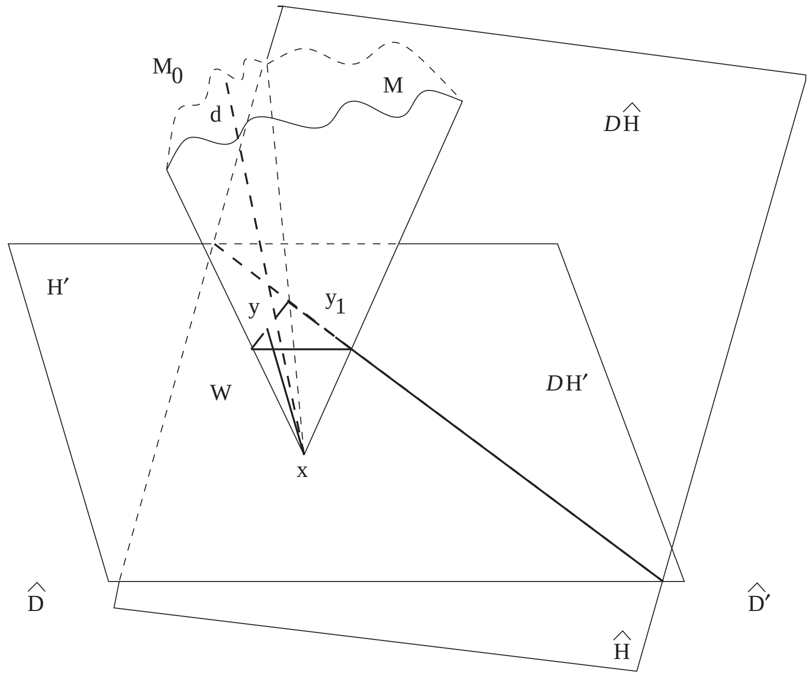

Fig. 2. Quadrants determined by $\widehat{H}$ and H'.

Let $\widehat{D}''$ be another half-apartment bounded by $\widehat{H}$ whose interior is disjoint from F. We consider the apartment $F' = \widehat{D} \cup \widehat{D}''$. Suppose by absurd that y_1 is in the interior of a codimension one face also in $F' \cap H(\rho)$. Let $\mathscr{D}H''$ be the half of the affine span of this face contained in $\widehat{D}''$. As previously $\mathscr{D}H''$ makes with $\mathscr{D}\widehat{H}$ the dihedral angle α and, for a small ε_2, $H(\rho) \cap \widehat{D}'' \cap B(y_1, \varepsilon_2) = \mathscr{D}H'' \cap B(y_1, \varepsilon_2)$.

In the apartment $\widehat{D}' \cup \widehat{D}''$, the quadrant $\mathscr{Q}$ bounded by $\mathscr{D}H'$ and $\mathscr{D}H''$ and containing $\mathscr{D}\widehat{H}$ has dihedral angle $2\alpha < \pi$ and verifies $\partial\mathscr{Q} \cap B(y_1, \varepsilon) \subset H(\rho)$, where $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ and $\partial\mathscr{Q}$ denotes the boundary of $\mathscr{Q}$. The convexity of the Busemann function f_ρ implies that for ε' small enough, $\mathscr{D}\widehat{H} \cap B(y_1, \varepsilon') \subset \mathscr{Q} \cap B(y_1, \varepsilon') \subset Hb(\rho)$. This gives a contradiction in F, in which $Hb(\rho)$ is on the opposite side of H' than $\mathscr{D}\widehat{H}$. $\square$

We recall that if a subset Y in a metric space (X, d_X) is endowed with its own metric d_Y, the metric space (Y, d_Y) is said to be *nondistorted* in (X, d_X) if the set $\{d_Y(y_1, y_2)/d_X(y_1, y_2) \mid y_1, y_2 \in Y, y_1 \neq y_2\}$ has a finite upper bound and a positive lower bound. An upper bound of this set is called *a nondistorsion constant of Y in X*. An upper bound of a set of the form $\{d_Y(y_1, y_2)/d_X(y_1, y_2) \mid y_1, y_2 \in Y, d_X(y_1, y_2) \geqslant D\}$ is called *a nondistorsion constant of Y in X for sufficiently large distances*.

The previous two Lemmata imply the following theorem

Theorem 3.3.3 (Druţu [10, Theorem 1.2]). *Let* $\mathbf{K}$ *be a* 3*-thick Euclidean building of rank at least* 2 *and* ρ *a geodesic ray in it. The following three properties are equivalent*:

(P_1) *The horosphere* $H(\rho)$ *endowed with its length metric is nondistorted in* $\mathbf{K}$;

(P_2) *$H(\rho)$ is connected*;
(P_3) *ρ is not parallel to a rank one factor of* $\mathbf{K}$.

Proof. The implications (P_1) $\Rightarrow$ (P_2) and (P_2) $\Rightarrow$ (P_3) are obvious. We show (P_3) $\Rightarrow$ (P_1). We may suppose that ρ is not parallel to any factor of $\mathbf{K}$. For if $\mathbf{K}=\mathbf{K}_1\times\mathbf{K}_2$ and if there exists a ray r in $\mathbf{K}_1$ and a point x in $\mathbf{K}_2$ such that $\rho(t)=(r(t),x)$, then $H(\rho)=H(r)\times\mathbf{K}_2$, so it will be enough to prove (P_1) for r in $\mathbf{K}_1$, by hypothesis $\mathbf{K}_1$ being of rank at least 2.

If ρ is not parallel to any factor of $\mathbf{K}$ then Lemmata 3.3.1 and 3.3.2, (1), allow to conclude that (P_1) is true. □

Remark 3.3.4. Let $H(\rho)$ be a horosphere in a 3-thick Euclidean building, $\mathbf{K}$, of rank at least 2, and suppose ρ is not parallel to a rank one factor.

(a) Lemmata 3.3.1 and 3.3.2 imply that we may find a nondistorsion constant C_0 of $H(\rho)$ in $\mathbf{K}$ which can be effectively computed given the model spherical Coxeter complex of $\mathbf{K}$ and the slope θ of ρ. More precisely, C_0 can be computed given:
- the minimal dihedral angle between two distinct hyperplanes of sets of slopes $Ort(\theta)$;
- the dihedral angles between one hyperplane H in an apartment F, H of set of slopes $Ort(\theta)$, and each of the singular hyperplanes supporting a Weyl chamber $W\subset F$ which contains a ray orthogonal to H.

(b) Two points $x,y\in H(\rho)\cap F$, where F is an apartment, may be joined either by a polygonal line of type $\mathbf{L}_{xy}$ contained in F or a ramification of it, or by the union of a planar polygonal line in $H(\rho)\cap F$ joining y to a point y_1 with a line $\mathbf{L}_{xy_1}$ contained in F or a ramification of it. This and Proposition 3.1.1, (b), imply in particular that x and y may always be joined by a polygonal line in $H(\rho)$ with at most $2q_0$ edges and of length $\leqslant C_0 d(x,y)$. We denote this polygonal line by $\mathscr{L}_{xy}$.

Theorem 3.3.3 implies the following.

Theorem 3.3.5 (Druţu [10, Theorem 1.3]). *Let X be a product of symmetric spaces of noncompact type and Euclidean buildings, X of rank at least 2 and ρ a geodesic ray in it. The following two properties are equivalent*:

(P_1^*) *The horosphere $H(\rho)$ is nondistorted*;
(P_2^*) *ρ is not parallel to a rank one factor of X.*

In [10, Section 4], there is a proof of the fact that Theorem 3.3.3 implies Theorem 3.3.5.

In the sequel we shall relate the nondistorsion constant of a horosphere $H(\rho)$ in a product X of symmetric spaces and Euclidean buildings to the model spherical Coxeter complex of $\partial_\infty X$ and to the slope θ of ρ.

Let X and ρ be as in the previous theorem, ρ satisfying property (P_2^*). Let x,y be two points in $X\setminus Hbo(\rho)$. We consider a sequence of points $x_0=x,\bar{x}_0,x_1,\bar{x}_1,\dots,x_p,\bar{x}_p=y$ satisfying the following properties:

(1) $0\leqslant p\leqslant 3q_0$ and $d(x,x_i),d(x,\bar{x}_i)\leqslant 2C_0 d(x,y),\ \forall i\in\{0,1,\dots,p\}$;

(2) if $p \geqslant 1$, for every $i \in \{0, 1, \ldots, p-1\}$ there exists a maximal flat F_i such that $[\bar{x}_i, x_{i+1}] \subset F_i \setminus Hbo(\rho)$;
(3) if $p \geqslant 1$, $\Sigma_{i=0}^{p-1} d(\bar{x}_i, x_{i+1}) \leqslant 2C_0 d(x, y)$.

Let $\mathfrak{L}_i$ be a curve of minimal length between x_i and $\bar{x}_i$ in $X \setminus Hbo(\rho)$. The curve joining x and y obtained as $\mathfrak{L}_0 \cup [\bar{x}_0, x_1] \cup \mathfrak{L}_1 \cup [\bar{x}_1, x_2] \cup \cdots \cup \mathfrak{L}_{p-1} \cup [\bar{x}_{p-1}, x_p] \cup \mathfrak{L}_p$ is called an *almost polygonal curve joining x and y*. The points $x_0 = x, \bar{x}_0, x_1, \bar{x}_1, \ldots, x_p, \bar{x}_p = y$ are called *the vertices of the almost polygonal curve*.

We denote

$$\epsilon(x, y) = \inf \left\{ \max_{i \in \{0,1,\ldots,p\}} d(x_i, \bar{x}_i) \mid \{x_i, \bar{x}_i\}_{i \in \{0,1,\ldots,p\}} \text{vertices of an almost polygonal curve} \right\}.$$

We denote

$$\epsilon(d) = \sup\{\epsilon(x, y) \mid x, y \in X \setminus Hbo(\rho), d(x, y) = d\}.$$

The definition of an almost polygonal curve implies that $\epsilon(x, y) \leqslant 4C_0 d(x, y)$ and that $\epsilon(d) \leqslant 4C_0 d$.

Lemma 3.3.6. *We have that $\epsilon(d) = o(d)$.*

Proof. We argue by contradiction and suppose that there exists a sequence of pairs of points $x_n, y_n \in X \setminus Hbo(\rho)$ with $d(x_n, y_n) = d_n$ and $\epsilon(x_n, y_n) > \delta d_n$, $\delta > 0$. Without loss of generality we may suppose that $x_n, y_n \in H(\rho)$. In the asymptotic cone $X_\omega(x_n, d_n)$ let $x_\omega = [x_n]$, $y_\omega = [y_n]$ and $\rho_\omega = [\rho]$. According to Remark 3.3.4, (b), x_ω and y_ω may be joined in $H(\rho_\omega)$ by a polygonal line with at most $2q_0$ segments and of length at most C_0. Let $x_\omega^0 = x_\omega, x_\omega^1, \ldots, x_\omega^p = y_\omega$ be the vertices of this line, $p \leqslant 2q_0$. Each segment $[x_\omega^i, x_\omega^{i+1}]$ is contained in an apartment F_ω^i asymptotic to ρ_ω. By Lemma 2.4.6, we can write $F_\omega^i = [F_n^i]$, where each F_n^i is asymptotic to ρ. The segment $[x_\omega^i, x_\omega^{i+1}]$ is limit of a sequence of segments $[\bar{x}_n^i, x_n^{i+1}] \subset F_n^i \setminus Hbo(\rho)$. The sequence $x_n^0 = x_n, \bar{x}_n^0, x_n^1, \bar{x}_n^1, x_n^2, \ldots, x_n^p, \bar{x}_n^p = y_n$ satisfies the properties (1), (2), (3) in the definition of an almost polygonal curve ω-almost surely. Also $\lim_\omega d(x_n^i, \bar{x}_n^i)/d_n = 0$, $\forall i \in \{0, 1, 2, \ldots, p\}$. This contradicts the fact that $\epsilon(x_n, y_n) > \delta d_n$, $\forall n \in \mathbb{N}$. □

An immediate consequence of Lemma 3.3.6 is an improvement of Theorem 3.3.5. In the theorem only the nondistorsion is stated, without any specification on nondistorsion constants. Let $\mathbf{C}$ be such a constant. Two points x and y in $X \setminus Hbo(\rho)$ with $d(x, y) = d$ can be joined in $X \setminus Hbo(\rho)$ by an almost polygonal curve of length at most $2C_0 d + 3q_0 \mathbf{C} \epsilon(d)$. Lemma 3.3.6 implies that for d sufficiently large the length of the almost polygonal curve is at most $3C_0 d$. Thus $3C_0$ is a nondistorsion constant of $X \setminus Hbo(\rho)$ in X for sufficiently large distances.

Corollary 3.3.7. *Let X be a product of symmetric spaces of noncompact type and Euclidean buildings such that each factor is of rank at least 2. Then there exists a constant $\mathbf{C}_0$ depending only on the model spherical Coxeter complex S of X such that for every geodesic ray ρ in X every two points $x, y \in X \setminus Hbo(\rho)$ sufficiently far away from each other can be joined in $X \setminus Hbo(\rho)$ by an almost polygonal curve of length at most $\mathbf{C}_0 d(x, y)$. In particular, $\mathbf{C}_0$ is a nondistorsion constant of $X \setminus Hbo(\rho)$ and of $H(\rho)$ in X for sufficiently large distances.*

Proof. The conclusion of the argument preceding the statement of the Corollary was that for every geodesic ray ρ in X, every two points $x, y \in X \setminus Hbo(\rho)$ sufficiently far away from each other can be joined in $X \setminus Hbo(\rho)$ by an almost polygonal curve of length at most $3C_0 d(x, y)$. We recall that $C_0 = C_0(\mathsf{S}, \theta)$ where $\theta = P(\rho(\infty))$. The dependence of C_0 on θ is made explicit in Remark 3.3.4, (a). From this dependence it follows that the function associating to each θ in Δ_{mod} the constant C_0 is continuous. In particular, it has a lowest upper bound $\mathbf{C}_0'$. Then $\mathbf{C}_0 = 3\mathbf{C}_0'$ is a nondistorsion constant of $X \setminus Hbo(\rho)$ in X for sufficiently large distances, for every geodesic ray ρ in X. $\square$

Remark 3.3.8. We can generalise all the previous arguments to the case of a space $X_0 = X \setminus \bigsqcup_{\rho \in \mathscr{R}} Hbo(\rho)$ with $\mathscr{R}$ finite and all rays in $\mathscr{R}$ of the same slope θ. The only difference in the definition of an almost polygonal curve is that we must replace condition $0 \leqslant p \leqslant 3q_0$ in (1) by $0 \leqslant p \leqslant 3q_0 \cdot \mathrm{card}\, \mathscr{R}$. Lemma 3.3.6 remains true.

Notations: In the sequel we deal only with spaces $X_0 = X \setminus \bigsqcup_{\rho \in \mathscr{R}} Hbo(\rho)$ with all $\rho \in \mathscr{R}$ of the same slope satisfying property (P_2^*). Suppose $\mathscr{R}$ is finite and x, y are two points in X_0. If the ambient space X is an Euclidean building, we denote by $\mathscr{L}_{xy}$ a curve joining x and y in X_0 obtained from $[x, y]$ by replacing each subsegment $[x', y'] = [x, y] \cap Hb(\rho)$ with a curve $\mathscr{L}_{x'y'}$ as described in Remark 3.3.4, (b). If X has a symmetric space as a factor, we denote by $\mathscr{L}_{xy}$ an almost polygonal curve joining x and y with $\max_i d(x_i, \bar{x}_i) \leqslant 2\epsilon(x, y)$. We call such a curve a *minimising almost polygonal curve joining x and y* (though the appropriate name should probably be "almost minimising almost polygonal curve").

4. Filling in Euclidean buildings and symmetric spaces with deleted open horoballs

In this section we prove the following result:

Theorem 4.1. *Let X be a product of symmetric spaces of noncompact type and Euclidean buildings, X of rank at least* 3, *and let X_0 be a subset which can be written as*

$$X_0 = X \Big\backslash \bigsqcup_{\rho \in \mathscr{R}} Hbo(\rho).$$

Suppose X_0 has the following properties:

(P_1) *for every point x of X_0 there exists a maximal flat $F \subset \mathscr{N}_d(X_0)$ such that $x \in \mathscr{N}_d(F)$, where d is a universal constant*;

(P_2) *all rays ρ in $\mathscr{R}$ have the same slope, and the common slope is not parallel to a rank one factor or to a rank two factor of X. Then*

(a) *the filling order in X_0 is asymptotically quadratic, that is*

$$\forall \varepsilon > 0, \quad \exists \ell_\varepsilon \text{ such that } A_1(\ell) \leqslant \ell^{2+\varepsilon}, \quad \forall \ell \geqslant \ell_\varepsilon;$$

(b) *if the set of rays $\mathscr{R}$ is finite then the filling order in X_0 is quadratic, that is*

$$A_1(\ell) \leqslant C\ell^2, \quad \forall \ell,$$

where the constant C depends on X and on the cardinal of $\mathscr{R}$.

Remark 4.2. It suffices to prove Theorem 4.1 under the hypothesis that the common slope of all rays $\rho \in \mathscr{R}$ is not parallel to any factor.

Proof. Let $X = X_1 \times X_2$ be a decomposition as a product, let $\Delta_{\text{mod}} = \Delta^1_{\text{mod}} \circ \Delta^2_{\text{mod}}$ be the corresponding decomposition of the model chamber of $\partial_\infty X$ and suppose $\theta \in \Delta^1_{\text{mod}}$. Then $X_0 = (X_1 \setminus \bigsqcup_{\rho_1 \in \mathscr{R}_1} Hbo(\rho_1)) \times X_2$. Any loop can be projected onto a loop entirely included into a copy of the factor $X_1 \setminus \bigsqcup_{\rho_1 \in \mathscr{R}_1} Hbo(\rho_1)$, and between the two loops there is a filling cylinder of quadratic surface. Thus, it suffices to prove the result in $X_1 \setminus \bigsqcup_{\rho_1 \in \mathscr{R}_1} Hbo(\rho_1)$. □

The order of the filling in X_0 can be specified in many of the cases when θ is parallel to a factor of rank at most 2. With an argument as in the proof of the previous remark we reduce the problem to the case when X is itself of rank at most 2. The filling order in X_0 is quadratic if $X = \mathbb{H}^n_{\mathbb{R}}$, $n \geqslant 3$, or $X = \mathbb{H}^n_{\mathbb{C}}$, $n \geqslant 3$, or cubic if $X = \mathbb{H}^2_{\mathbb{C}}$ (the last two statements are consequences of results in [1,13, Section 8.1.1; 21, Section 5.A'_4]), exponential if X is a symmetric space of rank two [23], and linear if X is a tree or $X = \mathbb{H}^2_{\mathbb{R}}$.

Before proving Theorem 4.1 we prove the following intermediate result.

Theorem 4.3. *Let* $\mathbf{K}$ *be a* 4*-thick Euclidean building of rank at least* 3 *and let* $\mathbf{K}_0$ *be a subspace of it of the form*

$$\mathbf{K}_0 = \mathbf{K} \Big\backslash \bigsqcup_{\rho \in \mathscr{R}_\omega} Hbo(\rho). \tag{4.1}$$

Suppose $\mathbf{K}_0$ *has the following properties*:

(P_1) *through every point of* $\mathbf{K}_0$ *passes an apartment entirely contained in* $\mathbf{K}_0$;

(P_2) *all rays* $\rho \in \mathscr{R}_\omega$ *have the same slope* θ *which is not parallel to a rank one factor or to a rank two factor of* $\mathbf{K}$.

Then

(a) *the filling order in* $\mathbf{K}_0$ *is at most cubic*, *that is*

$$A_1(\ell) \leqslant C \cdot \ell^3, \quad \forall \ell > 0,$$

where $C = C(\theta, \mathsf{S})$;

(b) *if the set of rays* $\mathscr{R}_\omega$ *is finite then every loop* $\mathfrak{C}$ *in* $\mathbf{K}_0$ *composed of at most* m *segments and of length* ℓ *has*

$$A_1(\mathfrak{C}) \leqslant C \cdot \ell^2,$$

where $C = C(\theta, \mathsf{S}, m)$.

The proof of Theorem 4.3 is done in several steps. First we study loops included in one apartment of the Euclidean building.

Notation: Henceforth for a curve $\mathfrak{L}$ without self-intersection and two points x, y on it we denote $\mathfrak{L}_{xy}$ the arc on $\mathfrak{L}$ of endpoints x and y.

4.1. Loops contained in one apartment

Proposition 4.1.1. *Let* $\mathbf{K}$ *be a* 4-*thick Euclidean building of rank at least* 3, F *an apartment in it,* ρ *a ray of slope* θ *not parallel to a rank two factor and* $\mathfrak{C}:\mathbb{S}^1 \to F \setminus Hbo(\rho)$ *a loop of length* ℓ. *There exists a positive constant* $L = L(\theta, \mathsf{S})$ such that the filling area of the loop $\mathfrak{C}$ in $\mathbf{K} \setminus Hbo(\rho)$ satisfies

$$A_1(\mathfrak{C}) \leqslant L \cdot \ell^2.$$

Proof. We show that we may suppose $\mathfrak{C}(\mathbb{S}^1) \subset F \cap H(\rho)$. If a filling disk obtained by joining a fixed point of $\mathfrak{C}(\mathbb{S}^1)$ with all the other points does not intersect $F \cap Hbo(\rho)$ then we are done. We suppose therefore the contrary which implies that $\mathfrak{C}(\mathbb{S}^1)$ is in the 2ℓ neighbourhood of $F \cap Hb(\rho)$. By Corollary 3.1.2 we may project $\mathfrak{C}$ on $F \cap H(\rho)$ and obtain a curve $\mathfrak{C}' : \mathbb{S}^1 \to F \cap H(\rho)$ of length at most ℓ. Since $\mathfrak{C}(\mathbb{S}^1) \subset \mathcal{N}_{2\ell}(F \cap Hb(\rho))$, the segments along which we project the curve $\mathfrak{C}$ on $\mathfrak{C}'$ form a cylinder in $F \setminus Hbo(\rho)$ with area of order ℓ^2. So it suffices to fill $\mathfrak{C}'$. Thus, we may suppose from the beginning that $\mathfrak{C}(\mathbb{S}^1) \subset F \cap H(\rho)$.

With an argument as in the proof of Remark 4.2 we may reduce to the case when ρ is not parallel to any factor. By hypothesis, we may suppose that the rank of $\mathbf{K}$ is not 2. If $\mathbf{K}$ has rank one then it is an $\mathbb{R}$-tree, every horosphere is totally disconnected in it so every loop in $H(\rho)$ reduces to a point. In the sequel we suppose that $\mathbf{K}$ is of rank at least 3.

In the case when $\inf_{x \in F} f_\rho(x) = -\infty$ the result has been proven in [11], in the proof of Proposition 4.3, in which it appears as case (1). So in the sequel we suppose $\inf_{x \in F} f_\rho(x) = -m > -\infty$. Let Φ be the affine span of $Min_F(\rho)$. The idea of the proof is to change eventually the apartment F with another apartment containing the loop, in which $Min_F(\rho)$ is at a Hausdorff distance of order ℓ from a polytope of codimension 3. More precisely, we prove that by eventually changing the apartment F we may suppose that either $Min_F(\rho)$ is of codimension 3 or $Min_F(\rho)$ is of codimension 2 but contained in a codimension one $\kappa\ell$-strip in Φ or $Min_F(\rho)$ is of codimension 1 but contained in a codimension two $(\epsilon, \kappa\ell)$-strip in Φ, where κ and ϵ are constants depending only on θ and S. See Definition A.5 for the different notions of strips. The previous situation already occurs if Φ has codimension at least 3. In the sequel we suppose Φ has codimension one or two. By Corollary 3.1.6 there exists a singular hyperplane H_1 in F which neither contains nor is orthogonal to Φ. For technical reasons we suppose H_1 intersects $\mathfrak{C}(\mathbb{S}^1)$. For every $x \in \mathfrak{C}(\mathbb{S}^1)$ we consider $H_1(x)$ the singular hyperplane through x parallel to H_1, and we consider $\mathcal{S} = \bigcup_{x \in \mathfrak{C}(\mathbb{S}^1)} H_1(x)$. The strip $\mathcal{S}$ either intersects or does not intersect the relative interior of $Min_F(\rho)$. We show that in both cases we can change F in such a way that $Min_F(\rho)$ either has one dimension less or is contained in a codimension one strip in Φ.

If the strip $\mathcal{S}$ does not intersect the relative interior of $Min_F(\rho)$ then there exists a hyperplane H_1' parallel to H_1 such that

- $\mathfrak{C}(\mathbb{S}^1)$ is contained in one half-apartment D of F bounded by H_1' while $Min_F(\rho)$ is contained in the opposite half-apartment D';
- $H_1' \cap Min_F(\rho)$ has codimension at least 1 in Φ.

By Lemma 3.1.4 there exists a half-apartment D_1 of boundary H_1' and interior disjoint of F such that $Min_{D' \cup D_1}(\rho) = Min_{D'}(\rho)$. Then $Min_{D \cup D_1}(\rho) = Min_{H_1'}(\rho)$ has codimension one in Φ. By replacing the initial flat F with $D \cup D_1$ we may therefore increase the codimension of $Min_F(\rho)$ with 1.

Suppose the strip $\mathscr{S}$ intersects the relative interior of $Min_F(\rho)$. Let H_1^a and H_1^b be the extremal hyperplanes of this strip. The distance between them is at most ℓ. By applying Lemma 3.1.4 twice we may suppose that $Min_F(\rho)$ is entirely contained in the strip. That is, $Min_F(\rho)$ is contained in the strip determined by $H_1^a \cap \Phi$ and $H_1^b \cap \Phi$ in Φ. Since there is a finite number of possibilities for the dimension of Φ and the angle between Φ and H_1, there exists a constant $\kappa = \kappa(\theta, \mathsf{S})$ such that $H_1^a \cap \Phi$ and $H_1^b \cap \Phi$ are at distance at most $\kappa\ell$ from each other. Thus, $Min_F(\rho)$ is contained in a codimension one $\kappa\ell$ -strip in Φ.

If Φ has codimension 2, this finishes the proof. The only case left is when Φ has codimension 1, so it is a singular hyperplane, according to Proposition 3.1.1, (c). To finish the proof in this case it is enough to find another singular hyperplane H_2 in F which neither contains nor is orthogonal to Φ and such that $\Phi \cap H_1 \cap H_2$ has codimension 2 in Φ. The hyperplane H_1 remained in the flat F even after the changes presented previously were performed, so the latter condition makes sense.

Let W be a Weyl chamber adjacent both to H_1 and to $H_1 \cap \Phi$ and let $M = W \cap H_1 \cap \Phi$. If there exists a hyperplane H' supporting W which does not contain M and which is not orthogonal to Φ we take $H_2 = H'$. If not then all hyperplanes supporting W and not containing M are orthogonal to Φ. By Corollary 3.1.6 one of these hyperplanes, H'', is not orthogonal to $Span\, M = H_1 \cap \Phi$. It follows that H'' is not orthogonal to H_1. We consider H_2 the image of H_1 by orthogonal symmetry with respect to H''. Since H_1 is not orthogonal to Φ and does not contain Φ, the same is true for H_2. Also $H_1 \cap H_2$ coincides with $H_1 \cap H''$ so it differs from $H_1 \cap \Phi$, as the latter contains M while H'' does not contain M. Therefore, $H_1 \cap H_2 \cap \Phi$ has codimension 2 in Φ.

By repeating with H_2 the argument done previously with H_1 we obtain a flat F' containing the image of the loop $\mathfrak{C}$ such that $Min_{F'}(\rho)$ is contained in a codimension two $(\epsilon, \kappa\ell)$-strip in Φ.

By Corollary 3.1.3, $\mathscr{N}_m(Min_F(\rho)) \subset Hb(\rho) \cap F \subset \mathscr{N}_{am}(Min_F(\rho))$. Lemma A.4, (b), in the Appendix implies that the projection $\mathbf{p}$ of $H(\rho) \cap F$ onto $\partial\mathscr{N}_m(Min_F(\rho))$ is bilipschitz with respect to the length metrics, the bilipschitz constant depending only on a. According to Proposition A.6 in the Appendix and to Proposition 3.1.1, (c_1), the filling area of the loop $\mathbf{p} \circ \mathfrak{C}$ in $\partial\mathscr{N}_m(Min_F(\rho))$ is at most $L(1+\kappa)\ell^2$. Hence the filling area of $\mathfrak{C}$ in $H(\rho) \cap F$ is bounded by $L'\ell^2$, with L' depending only on the constants a and κ, so on the model spherical Coxeter complex S of $\mathbf{K}$ and on θ. □

4.2. Loops contained in two apartments

We show that loops in the exterior of a horoball which are composed of two arcs, each arc contained in one apartment, have a quadratic filling area outside the horoball. We also have to add a condition on the two apartments: they must have at least a Weyl chamber in common.

First we prove an intermediate result.

Lemma 4.2.1. *Let $\mathbf{K}$ be a 4-thick Euclidean building of rank at least 3, let F, F' be two apartments having a half-apartment D_0 in common and let ρ be a ray of slope θ not parallel to a rank two factor. Let x and y be two points in $D_0 \setminus Hbo(\rho)$ and $\mathfrak{C}_{xy} \subset F \setminus Hbo(\rho)$, $\mathfrak{C}'_{xy} \subset F' \setminus Hbo(\rho)$ two curves without self-intersections joining x and y, of length at most ℓ.*

There exists a positive constant $L = L(\theta, \mathsf{S})$ such that the filling area of the loop $\mathfrak{C} = \mathfrak{C}_{xy} \cup \mathfrak{C}'_{xy}$ in $\mathbf{K} \setminus Hbo(\rho)$ satisfies

$$A_1(\mathfrak{C}) \leqslant L \cdot \ell^2.$$

Proof. *Notations*: For every two points $a, b \in \mathfrak{C}_{xy}$ we denote $\mathfrak{C}_{ab}$ the arc of $\mathfrak{C}_{xy}$ between a and b. We use a similar notation for $\mathfrak{C}'_{xy}$.

Step 1: We show that we can reduce to the case where $x, y \in D_0 \cap H(\rho)$ and $\mathfrak{C}_{xy}$, $\mathfrak{C}'_{xy}$ are the shortest polygonal curves joining x and y in $F \cap H(\rho)$ and $F' \cap H(\rho)$, respectively. If $[x, y] \cap Hb(\rho) = \emptyset$ then we can fill the loop $[x, y] \cup \mathfrak{C}_{xy}$ in $F \setminus Hbo(\rho)$ and the loop $[x, y] \cup \mathfrak{C}'_{xy}$ in $F' \setminus Hbo(\rho)$, respectively, with quadratic areas, and we can conclude. In the sequel we suppose $[x, y] \cap Hb(\rho) \neq \emptyset$. Let $[x', y'] = [x, y] \cap Hb(\rho)$ and let $\bar{\mathfrak{C}}_{x'y'}, \bar{\mathfrak{C}}'_{x'y'}$ be the shortest polygonal curves joining x' and y' in $F \cap H(\rho)$ and $F' \cap H(\rho)$, respectively. The loop $\mathfrak{C}_{xy} \cup [x, x'] \cup \bar{\mathfrak{C}}_{x'y'} \cup [y', y]$ is contained in $F \setminus Hbo(\rho)$ and has length of order ℓ. By Proposition 4.1.1 it can be filled with an area of order ℓ^2. Likewise for the loop $\mathfrak{C}'_{xy} \cup [x, x'] \cup \bar{\mathfrak{C}}'_{x'y'} \cup [y', y]$. So it suffices to fill the loop $\bar{\mathfrak{C}}_{x'y'} \cup \bar{\mathfrak{C}}'_{x'y'}$ with an area of order ℓ^2.

Notations: We denote $H = \partial D_0$. Let D and D' be the half-apartments opposite to D_0 in F and F', respectively.

Step 2: We reduce the problem to the case when $x, y \in H$.

If one of the two arcs $\mathfrak{C}_{xy}, \mathfrak{C}'_{xy}$ is entirely contained in $D_0 \setminus Hbo(\rho)$ then Proposition 4.1.1 allows to conclude. In the sequel we suppose that both $\mathfrak{C}_{xy}$ and $\mathfrak{C}'_{xy}$ have points in $Int\, D$ and in $Int\, D'$, respectively. Let x_1 be the nearest point to x of $\mathfrak{C}_{xy} \cap H$ and y_1 the nearest point to y of $\mathfrak{C}_{xy} \cap H$. Let $\mathfrak{c}$ be the shortest polygonal curve joining x_1 and y_1 in $F' \cap H(\rho)$. Since x_1 and y_1 are already joined in $F' \cap H(\rho)$ by the curve $\mathfrak{C}_{x_1x} \cup \mathfrak{C}'_{xy} \cup \mathfrak{C}_{yy_1}$ whose length is of order ℓ we may deduce the same for the length of $\mathfrak{c}$. According to Proposition 4.1.1 the loop $\mathfrak{c} \cup \mathfrak{C}_{x_1x} \cup \mathfrak{C}'_{xy} \cup \mathfrak{C}_{yy_1}$ can be filled with an area of order ℓ^2. To conclude, it suffices to prove that the loop $\mathfrak{c} \cup \mathfrak{C}_{x_1y_1}$ can be filled with an area of order ℓ^2.

Step 3: We show that by eventually adding some area of order ℓ^2, we may suppose that $\mathfrak{C}_{xy}$ is entirely contained into one half apartment of F bounded by H. The same argument done for $\mathfrak{C}'_{xy}$ and F', and Proposition 4.1.1 imply then the conclusion of the lemma.

Let x_2 be the nearest point to x on $\mathfrak{C}_{xy}$ contained in a codimension one face of $F \cap H(\rho)$ nonparallel to a codimension one face through y. If no such point exists then $\mathfrak{C}_{xy} = [x, y]$ is entirely contained in H. If $x_2 = x$, by eventually adding an area of order ℓ^2, we replace $\mathfrak{C}_{xy}$ with a planar polygonal line $\mathbf{L}_{xy}$ joining x, y as in Lemma 3.3.1, and we are done. We suppose in the sequel that x_2 exists and it is different of x. It follows that x and y are in the interiors of two parallel codimension one faces of $F \cap H(\rho)$ and that $\mathfrak{C}_{xx_2}$ coincides with the segment $[x, x_2]$ entirely contained in the face through x. We replace $\mathfrak{C}_{x_2y}$ with $\mathbf{L}_{x_2y}$, and add some area of order ℓ^2. If $\mathbf{L}_{x_2y}$ intersects H only in y then the curve $\mathfrak{C}_{xy}$ thus modified is included in one half-apartment. Suppose $\mathbf{L}_{x_2y} \cap H = \{x_3, y\}$. The polytopic hypersurface $H \cap H(\rho)$ contains the points x, y, x_3 and it cannot have three distinct parallel codimension one faces. It follows that the point x_3 can be joined either to x or to y with a polygonal line of length of order ℓ. Thus, we replace either $\mathfrak{C}_{xx_3}$ or $\mathfrak{C}_{x_3y}$ with a polygonal line in $H \cap H(\rho)$, by eventually adding some area of order ℓ^2. In both cases we obtain a new curve included in one half-apartment of F of boundary H. □

We now prove the more general statement we need.

Proposition 4.2.2. *Let* $\mathbf{K}$ *be a* 4*-thick Euclidean building of rank at least* 3 *and let* ρ *be a ray of slope* θ *not parallel to a rank two factor. Let* F *and* F' *be two apartments having at least a*

Weyl chamber W_0 in common. Let $[x,y]$ be a regular segment with endpoints in $W_0 \setminus Hbo(\rho)$ and $\mathfrak{C}_{xy} \subset F \setminus Hbo(\rho)$, $\mathfrak{C}'_{xy} \subset F' \setminus Hbo(\rho)$ two curves joining x and y of length at most ℓ. There exists a positive constant $L = L(\theta, \mathsf{S})$ such that the filling area of the loop $\mathfrak{C} = \mathfrak{C}_{xy} \cup \mathfrak{C}'_{xy}$ in $\mathbf{K} \setminus Hbo(\rho)$ satisfies

$$A_1(\mathfrak{C}) \leqslant L \cdot \ell^2.$$

Proof. To simplify we may suppose that W_0 has vertex x. As in Step 1 of the previous proof, we show that we may suppose that $x, y \in W_0 \cap H(\rho), (x,y) \subset Hbo(\rho)$, and that $\mathfrak{C}_{xy}$ and $\mathfrak{C}'_{xy}$ are polygonal lines of minimal length joining x, y in $F \cap H(\rho)$ and $F' \cap H(\rho)$, respectively. This rules out the case when ρ is parallel to a rank one factor.

Let W_0^{op} be the Weyl chamber of vertex x opposite to W_0 in F'. For every minimal gallery of Weyl chambers in F' stretched between W_0 and W_0^{op}, $\overline{W}_0 = W_0, \overline{W}_1, \dots, \overline{W}_{p_0} = W_0^{op}$ there exists by Corollary 2.2.7 a sequence of apartments $\overline{F}_0 = F, \overline{F}_1, \dots, \overline{F}_{p_0} = F'$ such that each $\overline{F}_{i+1}$ is a ramification of $\overline{F}_i$ containing $\overline{W}_0, \overline{W}_1, \dots, \overline{W}_{i+1}$. We show that, by eventually replacing F' with a ramification of it and adding an area of order ℓ^2 we can choose a minimal gallery between W_0 and W_0^{op} in F' in such a way that in each $\overline{F}_i \cap H(\rho)$ the points x and y may be joined by a polygonal line of length of order $d(x,y)$. This and the repeated application of Lemma 4.2.1 will imply the conclusion.

The inclusion $(x,y) \subset Hbo(\rho)$ implies $\angle_x(\overline{\rho_x}, (W_0)_x) < \angle_x(\overline{\rho_x}, \overline{xy}) < \pi/2$. There are several cases, which we denote as in Lemma 3.3.2.

(a) Suppose $(W_0)_x$ does not contain $\overline{\rho_x}$. By Corollary 3.2.2 there exists a Weyl chamber $\widehat{W}$ adjacent to W_0 with the property that in any apartment containing $W_0 \cup \widehat{W}$ x is contained into two faces of the trace of $H(\rho)$ in the apartment. According to Corollary 2.2.7 there exists a ramification F'' of F' including $W_0 \cup \widehat{W}$. We apply Lemma 4.2.1 to the loop composed of $\mathfrak{C}'_{xy}$ and of a polygonal line $\mathbf{L}_{xy} \subset F''$ and we conclude that, up to an additional quadratic filling area, we may suppose from the beginning that F' includes $W_0 \cup \widehat{W}$. We choose the gallery from W_0 to W_0^{op} such that $\overline{W}_1 = \widehat{W}$. Each apartment $\overline{F}_i$ for $i > 0$ includes $W_0 \cup \widehat{W}$ so x is contained into two faces of $\overline{F}_i \cap H(\rho)$. Lemma 3.3.1 implies that in each $\overline{F}_i$ the points x and y may be joined outside $Hbo(\rho)$ by a polygonal line of length of order $d(x,y)$.

(b_1) Suppose $\overline{\rho_x} \in (W_0)_x$ and the connected component of $W_0 \cap H(\rho)$ containing y has at least two faces. We choose an arbitrary minimal gallery of Weyl chambers between W_0 and W_0^{op} in F' and take the corresponding sequence of apartments. Every apartment $\overline{F}_i$ contains W_0 and by the previous hypothesis and Lemma 3.3.2 the points x and y can be joined in $\overline{F}_i \cap H(\rho)$ by a polygonal line of length of order $d(x,y)$.

(b_2) Suppose $\overline{\rho_x} \in (W_0)_x$ and the connected component of $W_0 \cap H(\rho)$ containing y has only one face. Lemma 3.3.2, (b_2), implies that in a ramification F'' of F' all the points in the intersection of a panel M of W_0 with $H(\rho)$ are in two different faces of $F'' \cap H(\rho)$. By eventually applying once Lemma 4.2.1, we may suppose that F' has this property itself. We choose the gallery from W_0 to W_0^{op} in F' such that $\overline{W}_1 \cap W_0 = M$. Since each apartment $\overline{F}_i$ for $i > 0$ includes $W_0 \cup \overline{W}_1$, in each $\overline{F}_i$ the points in $Int\, M \cap H(\rho)$ are in two different faces of $\overline{F}_i \cap H(\rho)$. According to Lemma 3.3.2, (b_2), in each intersection $\overline{F}_i \cap H(\rho)$ the points x and y can be joined by a polygonal line of length of order $d(x,y)$. □

4.3. Proof of Theorem 4.3

An important tool in our argument is the set of "good slopes" with respect to the slope of a ray defining a horosphere. These "good slopes" are the slopes which are transverse to the horosphere.

Lemma 4.3.1 (Druţu [11, Lemmata 4.9, 4.10]). (1) *Let Σ be a labelled spherical building, and θ a point in Δ_{mod}. For every small positive number δ_1 there exists a continuum of points $\beta \in \Delta_{\text{mod}}$ such that $d(\beta, Ort(\theta)) > \delta_1$.*

(2) *Let $\mathbf{K}$ be an Euclidean building and ρ a ray of slope θ. If a geodesic segment $[x, y]$ has slope β as in* (1) *then f_ρ decreases or increases on it with a rate at least equal to* $\sin\delta_1$.

If $\theta \in \Delta_{\text{mod}}$ and $\delta_1 > 0$ are fixed, we call slopes $\beta \in \Delta_{\text{mod}}$ verifying the condition in Lemma 4.3.1, (1), δ_1-*good slopes with respect to* θ. Slopes which moreover verify $d(\beta, \partial\Delta_{\text{mod}}) > \delta_1$ are called δ_1-*good regular slopes with respect to* θ. Whenever there is no possibility of confusion we omit θ and δ_1.

Proof of Theorem 4.3. Since the arguments for the proofs of (a) and (b) follow the same lines, we shall present them simultaneously, specifying the differences whenever they occur. We shall refer to the case when $\mathscr{R}_\omega$ is finite as the finite case and to the other case as the general case. In the finite case we consider $\mathfrak{C} : \mathbb{S}^1 \to \mathbf{K}_0$ a Lipschitz loop of length ℓ composed of at most m segments, while in the general case we consider $\mathfrak{C} : \mathbb{S}^1 \to \mathbf{K}_0$ an arbitrary Lipschitz loop of length ℓ.

Step 1: We choose a finite set of points on $\mathfrak{C}(\mathbb{S}^1)$. Let P_0 be a point on $\mathfrak{C}(\mathbb{S}^1)$ which we choose to be endpoint of a segment in the finite case. Let F be an apartment through P_0 entirely contained in $\mathbf{K}_0$ and b a point in $F(\infty)$ such that its projection on Δ_{mod} is a δ_1-good regular slope β with respect to θ. The point b is contained into a unique spherical chamber Δ_0 of $F(\infty)$.

In the general case we fix a small $\lambda > 0$ and choose a finite sequence $P_0, P_1, \dots P_n$ of points on $\mathfrak{C}(\mathbb{S}^1)$, with $n \leqslant 2\ell/\lambda$, which determine a partition of $\mathfrak{C}(\mathbb{S}^1)$ into arcs of length at most λ. We consider the rays $r_k = [P_k, b)$ (Fig. 3, (b)) and apartments F_k containing these rays, where $F_0 = F$. Each F_k has at least a Weyl chamber of boundary at infinity Δ_0 in common with F.

In the finite case let $Q_0 = P_0, Q_1, \dots, Q_j,\ j \leqslant m$, be the endpoints of the segments composing $\mathfrak{C}(\mathbb{S}^1)$. Each segment $[Q_i, Q_{i+1}]$ is contained in an apartment F_i'. By applying Proposition 3.1.1, (a), to the flat F_i' and to the ray $[P_0, b)$ we obtain a partition of $[Q_i, Q_{i+1}]$ into at most q_0 segments, each segment contained in an apartment asymptotic to $[P_0, b)$. In the end we obtain $P_0, P_1, \dots, P_n,\ n \leqslant mq_0$, points on $\mathfrak{C}(\mathbb{S}^1)$ such that each $[P_k, P_{k+1}] \subset \mathfrak{C}(\mathbb{S}^1)$ is contained in an apartment F_k asymptotic to $[P_0, b)$ (Fig. 3,(a)). Both $r_k = [P_k, b)$ and $r_{k+1} = [P_{k+1}, b)$ are contained in F_k.

We know that $d(P_k, F_0) \leqslant d(P_k, P_0) \leqslant \ell$, $\forall k \in \{1, 2, \dots, n\}$. By Lemma 4.6.3, [22], all points M of r_k with $d(M, P_k) \geqslant d(P_k, F_0)/\sin\delta_1$ are contained in F_0. So r_k is included in F_0 at least starting from the point M_k on it with $d(M_k, P_k) = \ell/\sin\delta_1$. We note that $d(P_0, M_k) \leqslant \ell(1 + 1/\sin\delta_1)$. It follows that the hyperplane in F_0 orthogonal to r_0 at distance $\ell(1 + 1/\sin\delta_1)$ from P_0 intersects each r_k in a point N_k. The convexity of the distance implies that $d(N_i, N_j) \leqslant d(P_i, P_j), \forall i \neq j$. The closed polygonal curve $\mathfrak{C}'$ with vertices $N_0, N_1, N_2, \dots, N_n$ has length at most ℓ and is entirely included in F_0 so in $\mathbf{K}_0$. The filling area of $\mathfrak{C}'$ in $\mathbf{K}_0$ is then quadratic. To end the proof it suffices therefore to find a "filling cylinder" between the loops $\mathfrak{C}$ and $\mathfrak{C}'$ of the desired area in $\mathbf{K}_0$. We note that each

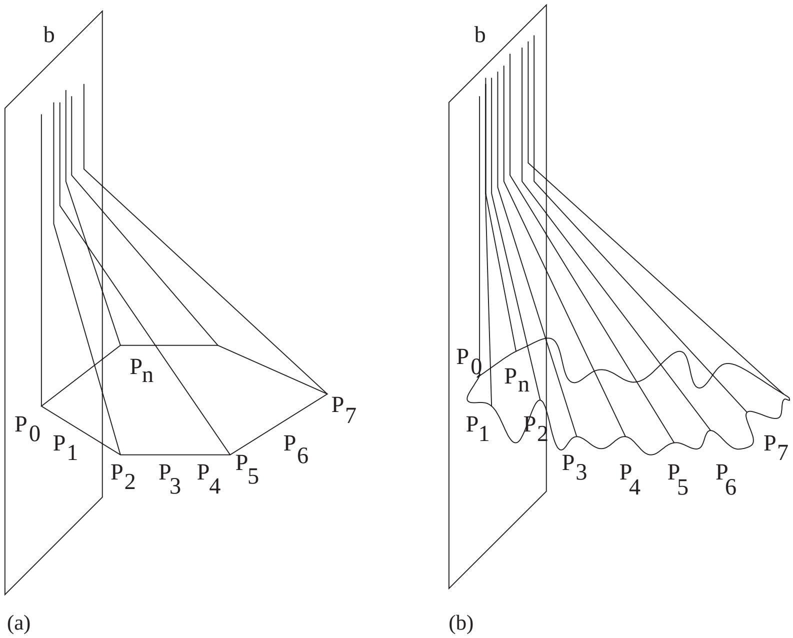

Fig. 3. Construction of the filling cylinder.

segment $[P_k, N_k]$ has length smaller than $d(P_k, P_0) + d(P_0, N_0) + d(N_0, N_k) \leqslant \ell(3 + 1/\sin\delta_1)$ and it has its endpoints in $\mathbf{K}_0$, but it is not necessarily contained in $\mathbf{K}_0$.

Step 2: An auxiliary construction is needed in the general case. In this case $[P_k, N_k]$ and $[P_{k+1}, N_{k+1}]$ are not in the same apartment asymptotic to r_0. We remedy this by constructing for each segment $[P_{k+1}, N_{k+1}]$ a "copy" of it contained in F_k and coinciding with $[P_{k+1}, N_{k+1}]$ on most of its length.

Let R_{k+1} be the first point in which the ray r_{k+1} meets the apartment F_k. By Lemma 4.6.3, [22], $d(P_{k+1}, R_{k+1}) \ll \lambda$. If R_{k+1} is contained in $F_k \cap \mathbf{K}_0$ then we denote it by P'_{k+1}. If it is contained in some $Hbo(\rho) \cap F_k$ then, as f_ρ is Lipschitz of constant 1 and $f_\rho(P_{k+1}) \geqslant 0$ we have $f_\rho(R_{k+1}) \gg -\lambda$. Since the ray r_{k+1} has a good slope, either by continuing along r_{k+1} towards b or by going in F_k in the opposite direction a distance $\ll \lambda$ away, we meet $H(\rho) \cap F_k$. We denote this point of intersection of the ray $r_{k+1} \cap F_k$ (or of its opposite in F_k) with $H(\rho)$ by P'_{k+1}. In both cases we have $d(P_{k+1}, P'_{k+1}) \ll \lambda$, which implies that $d(P_k, P'_{k+1}) \ll \lambda$. To simplify ulterior arguments we make the convention that in the finite case, $P'_{k+1} = P_{k+1}$.

In the sequel we replace simultaneously the pair of segments $[P_k, N_k]$ and $[P'_{k+1}, N_{k+1}]$, both contained in F_k, with curves in $\mathbf{K}_0$ of length of order ℓ. We do this by deforming them in order to avoid each horoball they intersect. We discuss the two cases separately.

Step 3: We first consider the general case. In this case we begin with a global modification: we replace each segment $[x, y]$ appearing as $[P_j, N_j] \cap Hb(\rho)$ or as $[P'_j, N_j] \cap Hb(\rho), j \in \{0, 1, \ldots, n\}$, $[x, y]$ of length at most λ, by a polygonal line $\mathscr{L}_{xy}$, where $\mathscr{L}_{xy}$ has the significance given in the end of Section 3.3.

Next we construct simultaneous deformations of the pairs of segments $[P_k, N_k]$ and $[P'_{k+1}, N_{k+1}]$, thus modified, in order to avoid a horoball $Hb(\rho)$. There are three possible situations:

(a) $Hb(\rho)$ intersects both segments into subsegments $[x, y] \subset [P_k, N_k]$ and $[x', y'] \subset [P'_{k+1}, N_{k+1}]$ of length strictly larger than λ;

(b) one of the intersections $[P_k, N_k] \cap Hb(\rho)$ and $[P'_{k+1}, N_{k+1}] \cap Hb(\rho)$ is empty or of length at most λ, while the other has length strictly larger than λ;

(c) both intersections are either empty or of length at most λ.

The case (c) is already dealt with by the global modification done in the beginning. We consider the case (a). We suppose that x and x' are the endpoints which are nearer than y and y' to P_k and P'_{k+1}, respectively.

The points x and x' cannot be contained in the interiors of two distinct and parallel codimension 1 faces of $F_k \cap H(\rho)$, as the spans of these faces must not separate points in the set $[x, y] \cup [x', y']$, by the convexity of $F_k \cap Hb(\rho)$. Lemma 3.3.1 implies that x and x' can be joined in $F_k \cap H(\rho)$ by a polygonal line $\mathbf{L}_{xx'}$ of length $\ll d(x, x')$. Similarly, y and y' can be joined in $F_k \cap H(\rho)$ by a polygonal line $\mathbf{L}_{yy'}$ of length $\ll d(y, y')$. The distances $d(x, x')$ and $d(y, y')$ are both of order λ.

We have two possibilities.

(1) There exist $x_1 \in \{x, x'\}$ and $y_1 \in \{y, y'\}$ such that x_1 and y_1 are contained into two nonparallel codimension 1 faces of $H(\rho) \cap F_k$.

(2) Every two points $x_1 \in \{x, x'\}$ and $y_1 \in \{y, y'\}$ are contained in the interiors of two parallel codimension 1 faces of $H(\rho) \cap F_k$.

In case (1) according to Lemma 3.3.1 we can join x_1 and y_1 by a polygonal line of length $\ll d(x_1, y_1)$. This and the previous remark on the possibility of joining x, x' and y, y', respectively, imply that in this way we can join both x, y and x', y' by polygonal lines of lengths comparable to the distances in $H(\rho) \cap F_k$.

In case (2) it follows that x, x' are in the interior of the same codimension 1 face, and the same for y, y', and the spans of the two faces are two parallel hyperplanes H and H', respectively. Let W and W' be the Weyl chambers of vertices x and x', respectively, and boundary at infinity Δ_0. The Weyl chamber W contains the segment $(x, y]$ in its interior and so does W' for the segment $(x', y']$. As f_ρ decreases on $[x, y]$ of slope β it follows that $\angle_x(\overline{\rho_x}, W_x) < \angle_x(\overline{\rho_x}, \overline{xy}) < (\pi/2) - \delta_1$.

Proposition 3.2.1, (c), implies that for small $\delta > 0$, $F_k \cap B(x, \delta)$ contains $\rho_x \cap B(x, \delta)$ and $F_k \cap B(x', \delta)$ contains $\rho_{x'} \cap B(x', \delta)$. The chambers W_x and $W'_{x'}$ simultaneously contain or do not contain $\overline{\rho_x}$ and $\overline{\rho_{x'}}$, respectively.

Suppose W_x and $W'_{x'}$ do not contain $\overline{\rho_x}$ and $\overline{\rho_{x'}}$, respectively. If the panel M of W has the property that M_x separates $\overline{\rho_x}$ and W_x, the same is true for the panel M' of W' asymptotic to M, with respect to $\overline{\rho_{x'}}$. Either M or M' has the property that $W \cup W'$ is on one side of its affine span. Suppose it is M and let $\widehat{H}$ be its affine span. By Lemma 3.3.2, (2), (a), there exists a ramification F'_k of F_k with $\partial(F'_k \cap F_k) = \widehat{H}$, containing W (and consequently W') and such that x is in at least two faces of $F'_k \cap H(\rho)$. It follows that, by replacing F_k by F'_k, the point x may be joined with a polygonal line $\mathbf{L}_{xy}$ to y.

Suppose that W_x and $W'_{x'}$ contain $\overline{\rho_x}$ and $\overline{\rho_{x'}}$, respectively. Suppose the connected component of $W \cap H(\rho)$ containing y has at least two faces. The proof of Lemma 3.3.2, (2), (b_1), implies that there exists a point y_1 in $H(\rho) \cap F_k$, which may be joined in $H(\rho) \cap F_k$ to y by a polygonal line of length $\ll d(x,y)$ and to x by a polygonal line $\mathbf{L}_{xy_1}$. The same is true for y' and x' if the connected component of $W'\cap H(\rho)$ containing y' has at least two faces.

Finally, if both $W\cap H(\rho)$ and $W'\cap H(\rho)$ have only one face, its affine span must be H'. We choose a hyperplane $\widehat{H}$ supporting W neither orthogonal nor coincident to H and $\widehat{H}'$ parallel to $\widehat{H}$ and supporting W'. Either $\widehat{H}$ or $\widehat{H}'$ have $W\cup W'$ on one side. Suppose it is $\widehat{H}$. By Lemma 3.3.2, (2), (b_2), there exists a ramification F_k' of F_k including $W\cup W'$ and by $\partial(F_k'\cap F_k)=\widehat{H}$ such that the points in $W\cap\widehat{H}\cap H(\rho)\setminus\{x\}$ are in two different faces of $H(\rho)\cap F_k'$. Therefore, we can join y to one of these points, y_1, by a segment in $H(\rho)\cap W$ of length $\ll d(x,y)$ and we can join y_1 to x in $H(\rho)\cap F_k'$ by a polygonal line $\mathbf{L}_{xy_1}$.

We replace the segments $[x,y]$ and $[x',y']$ by the polygonal lines joining their endpoints in $H(\rho)$ obtained in each of the previous cases. Now we consider the case (b). Suppose that $Hb(\rho)\cap[P_k,N_k]$ is a segment $[x,y]$ of length strictly larger than λ, while $Hb(\rho)\cap[P_{k+1}',N_{k+1}]$ is either empty or a segment $[x',y']$ of length at most λ. The other case is symmetric and treated analogously. Due to the fact that $d(P_k,P_{k+1}')$ and $d(N_k,N_{k+1})$ are of order λ, we have, according to [11, Lemma 4.11], that $d(x,y)\ll\lambda$.

Suppose that $H(\rho)\cap[P_{k+1}',N_{k+1}]$ is a non-trivial segment $[x',y']$. Then x' and y' are already joined by a polygonal line $\mathscr{L}_{x'y'}$ of length $\ll d(x',y')$, according to the modification done in the beginning of the step. As in case (a), we can show that x,x' and y,y' can be joined in $F_k\cap H(\rho)$ by polygonal lines $\mathbf{L}_{xx'}$ and $\mathbf{L}_{yy'}$, respectively, of lengths comparable to the distances.

Assume x' and y' are contained into two parallel codimension one faces of $F_k\cap H(\rho)$. Given H and H' the affine spans of these faces, $F_k\cap H(\rho)$ is contained in the strip determined by H and H'. In particular $d(x,y)\leqslant d(x',y')\leqslant\lambda$, which gives a contradiction. Hence, x' and y' are contained into two non-parallel codimension one faces of $F_k\cap H(\rho)$, so they can be joined in $F_k\cap H(\rho)$ by a polygonal line $\mathbf{L}_{x'y'}$ of length $\ll d(x',y')$. Thus, we can join x and y in $F_k\cap H(\rho)$ by the polygonal line $\mathbf{L}_{xx'}\cup\mathbf{L}_{x'y'}\cup\mathbf{L}_{yy'}$ of length of order $d(x,y)$.

Suppose that $Hb(\rho)$ does not intersect $[P_{k+1}',N_{k+1}]$. Let W be the Weyl chamber of vertex x and boundary at infinity Δ_0. According to Lemmata 3.3.1 and 3.3.2, x and y can be joined by a polygonal line of length $\ll d(x,y)$ either in $F_k\cap H(\rho)$ or in $F_k'\cap H(\rho)$, where F_k' is a ramification of F_k containing W.

We replace $[x,y]$ by the polygonal line joining its endpoints obtained in each of the cases above.

Step 4: We consider the finite case. In this case, we study not only the pair of segments $[P_k,N_k]$ and $[P_{k+1}',N_{k+1}]$, but also the full quadrangle $\mathscr{Q}_k$ having them as opposite edges. By full quadrangle we mean the planar domain bounded by a quadrangle.

We consider a horoball $Hb(\rho)$ intersecting $\mathscr{Q}_k$. The intersection $Hb(\rho)\cap\mathscr{Q}_k$ is a planar convex polygon. We recall that by hypothesis $Hbo(\rho)$ cannot intersect $[P_k,P_{k+1}']$ or $[N_k,N_{k+1}]$. We have three possible situations:

(A) $Hb(\rho)$ intersects both $[P_k,N_k]$ and $[P_{k+1}',N_{k+1}]$ into subsegments $[x,y]$ and $[x',y']$, respectively;

(B) $Hb(\rho)$ intersects only one of the segments $[P_k,N_k]$ and $[P_{k+1}',N_{k+1}]$;

(C) $Hb(\rho)$ intersects none of the segments $[P_k,N_k]$ and $[P_{k+1}',N_{k+1}]$.

The case (A) is dealt with exactly as the case (a) in Step 3. We obtain that x,x' and y,y' can be joined in $F_k\cap H(\rho)$ by polygonal lines $\mathbf{L}_{xx'}$ and $\mathbf{L}_{yy'}$, respectively, of lengths of order ℓ. We also obtain that, by eventually replacing F_k with a ramification of it, we can join a point $x_1 \in \{x, x'\}$ to a point $y_1 \in \{y, y'\}$ by a polygonal line $\mathbf{L}_{x_1 y_1}$ in $F_k \cap H(\rho)$, of length of order ℓ.

We replace $[x, y]$ and $[x', y']$ by the polygonal lines joining their endpoints thus obtained.

(B) Suppose that $Hb(\rho) \cap [P_k, N_k] = [x, y]$, while $Hb(\rho) \cap [P'_{k+1}, N_{k+1}] = \emptyset$. The other case is symmetric. The boundary of $Hb(\rho) \cap \mathscr{Q}_k$ is composed of $[x, y]$ and of a polygonal line $\mathbf{L}_{xy}$ of endpoints x, y and of length of order ℓ. We replace $[x, y]$ by $\mathbf{L}_{xy}$.

(C) In this case $Hb(\rho) \cap \mathscr{Q}_k$ is a convex polygone entirely contained in the interior of $\mathscr{Q}_k$. Since the boundary of $\mathscr{Q}_k$ has length of order ℓ and since the projection onto a convex domain diminishes distances, it follows that $H(\rho) \cap \mathscr{Q}_k$ has length of order ℓ. Proposition 4.1.1 implies that the loop $H(\rho) \cap \mathscr{Q}_k$ can be filled in $\mathbf{K}_0$ with an area of order ℓ^2.

We introduce some notations for both the general and the finite case.

We denote by γ_k the curve between P_k and N_k thus obtained and by γ'_{k+1} the curve between P'_{k+1} and N_{k+1}. We denote by $\mathfrak{C}_k$ the loop composed of the curves γ_k, γ_{k+1}, $\mathscr{L}_{P_k P_{k+1}}$ and $[N_k, N_{k+1}]$, We denote $\mathfrak{C}'_k = \gamma_k \cup \gamma'_{k+1} \cup \mathscr{L}_{P_k P'_{k+1}} \cup [N_k, N_{k+1}]$ and $\mathfrak{C}''_{k+1} = \gamma'_{k+1} \cup \gamma_{k+1} \cup \mathscr{L}_{P_{k+1} P'_{k+1}}$. We note that in the finite case although $P'_{k+1} = P_{k+1}$ the curves γ'_{k+1} and γ_{k+1} do not coincide: one is a curve contained in F_k or in ramifications of it while the other is contained in F_{k+1} or in a ramification of it. In the finite case, $\mathscr{L}_{P_k P_{k+1}} = \mathscr{L}_{P_k P'_{k+1}} = [P_k, P_{k+1}]$ and $\mathscr{L}_{P_{k+1} P'_{k+1}} = \{P_{k+1}\}$. In the general case the shapes of these lines do not really matter. Since the notion of area we work with is discrete all that matters is that the sets $\{P_k, P'_{k+1}, P_{k+1}\}$ have diameters of order λ. In the general case we may replace the loop $\mathfrak{C}(\mathbb{S}^1)$ with the loop $\bigcup_{k=0}^{n-1} \mathscr{L}_{P_k P_{k+1}} \cup \mathscr{L}_{P_n P_0}$. Since the two loops are at a Hausdorff distance of order λ one from the other the replacement can be done up to adding a linear filling area.

We construct the filling cylinder by filling in all the loops $\mathfrak{C}_k$. We do this in two steps: first we fill all the loops $\mathfrak{C}'_k$, then all the loops $\mathfrak{C}''_{k+1}$.

Step 5: We fill the loop $\mathfrak{C}'_k$. First, for every pair of segments $[x, y] = [P_k, N_k] \cap Hb(\rho)$ and $[x', y'] = [P'_{k+1}, N_{k+1}] \cap Hb(\rho)$, which comes from the cases (a) or (b) in Step 3 or from the case (A) in Step 4, we fill the loops obtained by joining x to x', y to y', x to y and x' to y' by polygonal lines. According to the construction in Step 3, (a), and in Step 4, (A), the sub-arc γ_{xy} of γ_k of endpoints x, y and the sub-arc $\gamma'_{x'y'}$ of γ'_{k+1} of endpoints x', y' eventually differ at their ends, where one is eventually obtained from the other by adding either $\mathbf{L}_{xx'}$ or $\mathbf{L}_{yy'}$ or both. Thus, the loop $\gamma_{xy} \cup \mathbf{L}_{xx'} \cup \gamma'_{x'y'} \cup \mathbf{L}_{yy'}$ is reduced to an arc and there is no area needed to fill it. In the case (b) of Step 3, $\gamma_{xy} = \mathbf{L}_{xx'} \cup \mathbf{L}_{x'y'} \cup \mathbf{L}_{yy'}$, while $\gamma'_{x'y'} = \mathscr{L}_{x'y'}$. In this case the loop $\gamma_{xy} \cup \mathbf{L}_{xx'} \cup \gamma'_{x'y'} \cup \mathbf{L}_{yy'}$ is reduced to two arcs and the loop $\mathbf{L}_{x'y'} \cup \mathbf{L}_{x'y'}$, hence it has filling area 1 for λ small enough. Since $d(x, y) > \lambda$, we can write that the filling area in this case is $\ll d(x, y)$.

What remains to be filled is the set of loops of the form $\gamma_{y\bar{x}} \cup \mathbf{L}_{\bar{x}\bar{x}'} \cup \gamma_{y'\bar{x}'} \cup \mathbf{L}_{yy'}$, where y, y' are the upper endpoints of a pair of segments as previously and $\bar{x}, \bar{x}'$ are the lower endpoints of the next pair, $\gamma_{y\bar{x}}$ is the arc of γ_k between y and $\bar{x}$, $\gamma_{y'\bar{x}'}$ is the arc of γ'_{k+1} between y' and $\bar{x}'$.

In the general case the arc $\gamma_{y\bar{x}}$ is at Hausdorff distance of order λ from the segment $[y, \bar{x}]$ and the same is true for the arc $\gamma_{y'\bar{x}'}$ with respect to the segment $[y', \bar{x}']$. On the other hand, the segments $[y, \bar{x}]$ and $[y', \bar{x}']$ are at Hausdorff distance of order λ one from the other. It follows that the loop $\gamma_{y\bar{x}} \cup \mathbf{L}_{\bar{x}\bar{x}'} \cup \gamma_{y'\bar{x}'} \cup \mathbf{L}_{yy'}$ can be filled with an area of order $d(y, \bar{x})$ in $\mathbf{K}_0$.

In the finite case the loop $\gamma_{y\bar{x}} \cup \mathbf{L}_{\bar{x}\bar{x}'} \cup \gamma_{y'\bar{x}'} \cup \mathbf{L}_{yy'}$ has length of order ℓ and it can be filled with an area of order ℓ^2 in $\mathbf{K}_0$, by Step 4, (B) and (C), and the fact that there are finitely many horoballs.

We conclude that to fill the loop $\mathfrak{C}'_k$ we need an area of order $\Sigma d(x, y) + \Sigma d(x', y') + \Sigma d(y, \bar{x})$, so of order ℓ in the general case, and an area of order ℓ^2 in the finite case.

Step 6: Now we fill the loop $\mathfrak{C}''_{k+1}$.

First we consider the finite case. In this case $P'_{k+1}=P_{k+1}$ and the two arcs γ_{k+1} and γ'_{k+1} composing the loop differ only between pairs of points x', y' such that $[x', y'] = [P_{k+1}, N_{k+1}] \cap Hb(\rho)$. The arc of γ'_{k+1} between x' and y', which we denote $\gamma'_{x'y'}$, is in F_k or a ramification of it, while the arc of γ_{k+1} between x' and y', $\gamma_{x'y'}$, is in F_{k+1} or a ramification of it. Both arcs have length of order ℓ. The apartments F_k and F_{k+1} or their respective ramifications have in common a Weyl chamber of boundary Δ_0 and vertex x'. Proposition 4.2.2 implies that the loop $\gamma'_{x'y'} \cup \gamma_{x'y'}$ can be filled with an area of order ℓ^2. Since there is a uniformly bounded number of such loops along $[P_{k+1}, N_{k+1}]$, composing $\mathfrak{C}''_{k+1}$, we conclude that $\mathfrak{C}''_{k+1}$ can be filled with an area of order ℓ^2.

In the general case γ_{k+1} and γ'_{k+1} differ between pairs of points x', y' such that $[x', y']=[P'_{k+1}, N_{k+1}] \cap Hb(\rho) = [P_{k+1}, N_{k+1}] \cap Hb(\rho)$ and $d(x', y') > \lambda$, and they may also differ near P'_{k+1} and P_{k+1}, respectively. For a pair of points x', y' as previously, we reason as in the finite case by means of Proposition 4.2.2. The only difference is that the lengths of $\gamma_{x'y'}$ and of $\gamma'_{x'y'}$ are of order $d(x', y')$ so the area needed to fill $\gamma'_{x'y'} \cup \gamma_{x'y'}$ is of order $d(x', y')^2$.

We analyse what happens near P'_{k+1} and P_{k+1}. We recall that we denoted by R_{k+1} the first point in which the ray r_{k+1} meets the apartment F_k. We have that $d(P_{k+1}, R_{k+1}) \ll \lambda$. When we chose P'_{k+1} in Step 2 we had three cases:

(1) R_{k+1} is contained in $F_k \cap \mathbf{K}_0$ and $P'_{k+1} = R_{k+1}$;
(2) R_{k+1} is contained in some $Hbo(\rho) \cap F_k$ and $P'_{k+1} \in [R_{k+1}, N_{k+1}] \cap H(\rho)$;
(3) R_{k+1} is contained in some $Hbo(\rho) \cap F_k$ and $R_{k+1} \in [P'_{k+1}, N_{k+1}]$; in this case $R_{k+1} \in [P'_{k+1}, y'_0] \cap [x'_0, y'_0]$ where the segments $[P'_{k+1}, y'_0]$ and $[x'_0, y'_0]$ are the intersections of $Hb(\rho)$ with $[P'_{k+1}, N_{k+1}]$ and $[P_{k+1}, N_{k+1}]$, respectively.

In the cases (1) and (2) the curve γ_{k+1} contains with respect to γ'_{k+1} an extra-arc of length $\ll \lambda$ which we may ignore.

In the case (3), γ'_{k+1} and γ_{k+1} differ between P'_{k+1}, y'_0 and x'_0, y'_0, respectively, and γ_{k+1} contains the extra-arc between P_{k+1} and x'_0 which we may likewise ignore, as being of order λ. Let $\gamma_{P'_{k+1}, y'_0}$ be the curve which replaces the segment $[P'_{k+1}, y'_0]$ in γ'_{k+1} and $\gamma_{x'_0, y'_0}$ the curve which replaces the segment $[x'_0, y'_0]$ in γ_{k+1}. The endpoints of these two curves do not coincide. We consider the horosphere $H_{-c_0\lambda}(\rho)$ containing R_{k+1}. Let y''_0 be the second intersection point of $[R_{k+1}, N_{k+1}]$ with $H_{-c_0\lambda}(\rho)$.

By Corollary 3.1.2 the curves $\gamma_{P'_{k+1}, y'_0}$ and $\gamma_{x'_0, y'_0}$ project onto two curves $\mathfrak{c}_1$ and $\mathfrak{c}_2$ of smaller length, contained in $H_{-c_0\lambda}(\rho) \cap F_k$ and $H_{-c_0\lambda}(\rho) \cap F_{k+1}$, respectively, at Hausdorff distance $\ll \lambda$ from the initial curves. By eventually extending $\mathfrak{c}_i$, $i=1,2$, with arcs of length $\ll \lambda$ one may suppose that both have R_{k+1} and y''_0 as endpoints. By Proposition 4.2.2 one needs a λ-filling area of order $d(R_{k+1}, y''_0)^2$ to fill $\mathfrak{c}_1 \cup \mathfrak{c}_2$. By the properties of the horospheres formulated in Section 2.3, the λ-filling area previously found for $\mathfrak{c}_1 \cup \mathfrak{c}_2$ gives a δ-filling area for $\gamma_{P'_{k+1}, y'_0} \cup \gamma_{x'_0, y'_0} \cup \mathscr{L}_{x'_0, P'_{k+1}}$, with $\delta \ll \lambda$. For λ sufficiently small δ is smaller than 1.

In the end we obtain that in the general case the loop $\mathfrak{C}''_{k+1}$ can be filled with an area of order $\Sigma_{x', y'} d(x', y')^2 + d(P_{k+1}, y'_0)^2 \leqslant \ell^2$.

Step 7: We conclude that to fill the loops $\mathfrak{C}'_k$ and $\mathfrak{C}''_{k+1}$, so to fill the loop $\mathfrak{C}_k$, we need an area of order ℓ^2. By summing over $\{1, 2, \ldots, n\}$ we obtain a filling area of order $n\ell^2$ for the initial loop. In the finite case this gives an area of order ℓ^2 while in the general case this gives an area of order ℓ^3. □

Remark 4.3.2. The cubic order in the general case comes from the fact that to fill the loops $\mathfrak{C}_k''$ an area of order ℓ^2 is needed. On the other hand, since the filling disk of each $\mathfrak{C}_k''$ is obtained by means of Proposition 4.2.2, the bricks of length at most 1 composing it are boundaries of small Euclidean triangles entirely contained in a polytopic surface $H(\rho)\cap F$.

Thus, for a generic loop of length ℓ we obtain a 1-filling disk composed of $k_1\ell^3$ boundaries of small Euclidean triangles entirely contained in polytopic surfaces of type $H(\rho)\cap F$ and of $k_2\ell^2$ bricks on which nothing special can be said.

4.4. Quadratic filling order in solvable groups

By means of Theorem 4.3, (b), we prove Theorem 4.1, (b). A consequence of it is Theorem 1.1. First we prove an intermediate result.

Proposition 4.4.1. *Let X be a product of symmetric spaces of noncompact type and Euclidean buildings, X of rank at least* 3, *and let X_0 be a subset of it which can be written as*

$$X_0 = X \Big\backslash \bigsqcup_{\rho\in\mathscr{R}} Hbo(\rho).$$

Suppose that the set of rays $\mathscr{R}$ is finite and suppose X_0 has the properties (P_1) *and* (P_2) *formulated in Theorem* 4.1. *For every $m \geqslant 4$ there exists a constant $\mathfrak{K}$ depending on m, on X, on the cardinal of $\mathscr{R}$, on the constant d appearing in* (P_1) *and on the slope θ appearing in* (P_2), *such that for every loop $\mathfrak{C}$* in *X_0 of length ℓ composed of at most m minimising almost polygonal curves*,

$$A_1^{X_0}(\mathfrak{C}) \leqslant \mathfrak{K}\ell^2. \tag{4.2}$$

Proof. It suffices to prove that (4.2) is satisfied for loops $\mathfrak{C}$ of length at least ℓ_0 for ℓ_0 sufficiently large. We reason by contradiction. Suppose that in X there exists a sequence of subsets $X_0^n = X \setminus \bigsqcup_{\rho\in\mathscr{R}_n} Hbo(\rho)$ with card $\mathscr{R}_n \leqslant N$ and θ the common slope of all rays in $\mathscr{R}_n$, such that X_0^n has properties (P_1), with a constant d independent of n, and (P_2) and in each X_0^n there exists a loop $\mathfrak{C}_n$ of length $\ell_n \geqslant \ell_0$ composed of at most m minimising almost polygonal curves, with

$$A_1^{X_0^n}(\mathfrak{C}_n) \geqslant n\ell_n^2. \tag{4.3}$$

In each X_0^n we consider a loop $\mathfrak{C}_n$ of minimal length with the previous properties. Inequality (4.3) implies that ℓ_n must diverge to $+\infty$. Let x_n be a point on the image of $\mathfrak{C}_n$ and let $\mathbf{K}=X_\omega(x_n,\ell_n/10)$ and $\mathbf{K}_0=[X_0^n]$. We may write $\mathbf{K}_0=\mathbf{K}\setminus\bigsqcup_{\rho_\omega\in\mathscr{R}_\omega} Hbo(\rho_\omega)$, where all rays ρ_ω have slope θ. Since card $\mathscr{R}_n \leqslant N, \forall n\in\mathbb{N}$, and since ω chooses one out of a finite number of possibilities, card $\mathscr{R}_\omega \leqslant N$. The space $\mathbf{K}_0$ has properties (P_1) and (P_2) formulated in Theorem 4.3. The limit set of the sequence of loops $\mathfrak{C}_n$ is a loop $\mathfrak{C}$ of length 10 composed of at most $3q_0mN$ segments. According to Theorem 4.3, (b), $\mathfrak{C}$ can be filled with an area of at most $100C$. Moreover, each of the $100C$ bricks composing the filling disk is an Euclidean triangle contained either in the exterior of a set of polytopes in a maximal flat, $F\setminus\bigsqcup_{\rho_\omega\in\mathscr{R}_\omega} Hbo(\rho_\omega)$, or in a face of a polytope $F'\cap H(\rho_\omega), \rho_\omega\in\mathscr{R}_\omega$, that is in a hyperplane $F\cap H(\rho_\omega)$, where F is asymptotic to ρ_ω. Thus, each of the bricks $B_i,\ i\in\{1,2,\dots,100C\}$, composing the filling disk is the limit of a sequence of Euclidean triangles B_i^n contained either in flat sets of type $F_i^n \setminus \bigsqcup_{\rho\in\mathscr{R}_n} Hbo(\rho)$ or in intersections $F_i^n \cap H(\rho_i^n)$ with $\rho_i^n \in \mathscr{R}_n$ and F_i^n asymptotic to ρ_i^n.

Let E_i and E_j be the edges along which two bricks B_i and B_j are glued one to the other. We recall that all edges have length at most $\lambda \leqslant 1$. Let E_i^n and E_j^n be the sequences of edges of B_i^n and B_j^n, respectively, such that $[E_i^n] = E_i = E_j = [E_j^n]$. The Hausdorff distance δ_{ij}^n between E_i^n and E_j^n has the property that $\lim_\omega \delta_{ij}^n/\ell_n = 0$. By joining with minimising almost polygonal curves the pairs of endpoints of E_i^n and E_j^n which coincide in the limit we obtain a loop, $\mathfrak{C}_{ij}^n$. We can divide the loop $\mathfrak{C}_{ij}^n$ into approximately $\lambda\ell_n/\delta_{ij}^n$ loops with lengths of order δ_{ij}^n composed of two minimising almost polygonal curves and of two subsegments of E_i^n and E_j^n, respectively. Since ℓ_n was the minimal length of a loop satisfying (4.3), each of these loops has filling area $\ll n(\delta_{ij}^n)^2$. It follows that $A_1^{X_0^n}(\mathfrak{C}_{ij}^n) \ll n\ell_n\delta_{ij}^n$.

We fill $\mathfrak{C}_n$ in X_0^n by filling each of the bricks $B_i^n \subset F_i^n \setminus \bigsqcup_{\rho\in\mathscr{R}_n} Hbo(\rho)$ and $B_i^n \subset F_i^n \cap H(\rho_i^n)$ and each of the loops $\mathfrak{C}_{ij}^n$. We obtain a filling area $A_1^{X_0^n}(\mathfrak{C}_n) \leqslant 100C \cdot \kappa(\ell_n^2/100) + 200C\kappa' n\ell_n\delta_{ij}^n$. It follows that $n\ell_n^2 \leqslant C \cdot \kappa\ell_n^2 + 200C\kappa' n\ell_n\delta_{ij}^n$. If we divide the inequality by $n\ell_n^2$ and we consider the ω-limit, we obtain $1 \leqslant 0$, a contradiction. □

Proof of Theorem 4.1, (b). We proceed by induction. First we need some constants. According to Theorem 3.3.5, X_0 is nondistorted in X. Let $\mathbf{C}$ be a nondistorsion constant of X_0 in X. A minimising almost polygonal curve joining two points x, y in X_0 has length at most $2C_0 d(x,y) + 6q_0 N\mathbf{C}\epsilon(x,y)$, where $N=$ card $\mathscr{R}$. The inequality $\epsilon(x,y) \leqslant 4C_0 d(x,y)$ implies that its length is at most $\mathfrak{k}d(x,y)$ with $\mathfrak{k} = 2C_0 + 24q_0 N\mathbf{C}C_0$. Let b be an integer which is very large compared to $\mathfrak{k}$. Then there exists an integer M between $b(\mathfrak{k}+1)$ and $b^2/2$. Let $\mathfrak{K}$ be the constant provided by Proposition 4.4.1 for loops composed of at most M minimising almost polygonal curves. Let $C = \max\{2\mathfrak{K}\mathfrak{k}^2, 2M, 1\}$. We show by induction on n the following statement:

(I_n) If $1 \leqslant \ell \leqslant b^n$ then $A_1(\ell) \leqslant C\ell^2$.

(I_0) is satisfied because $C \geqslant 1$. Suppose (I_n) is satisfied and let us prove (I_{n+1}). Let $\mathfrak{C}$ be a loop of length $\ell \in (b^n, b^{n+1}]$. We divide the loop $\mathfrak{C}$ into M arcs of equal length and we join the endpoints of these arcs by almost polygonal curves. We obtain M loops $\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_M$, of lengths at most $(\ell/M)(1+\mathfrak{k})$ and one loop $\mathfrak{C}_0$ of length at most $\mathfrak{k}\ell$ composed of M minimising almost polygonal curves. By Proposition 4.4.1, $A_1(\mathfrak{C}_0) \leqslant \mathfrak{K}\mathfrak{k}^2\ell^2$.

Since $M \geqslant b(\mathfrak{k}+1)$, $(1+\mathfrak{k})/M \leqslant 1/b$, hence each $\mathfrak{c}_i$ has length at most ℓ/b. If $n=0$ then it follows that $A_1(\mathfrak{c}_i) = 1$ and that $A_1(\mathfrak{C}) \leqslant M + \mathfrak{K}\mathfrak{k}^2\ell^2 \leqslant C\ell^2$. If $n \geqslant 1$ then by (I_n), $A_1(\mathfrak{c}_i) \leqslant C\ell^2/b^2, \forall i$. It follows that $A_1(\mathfrak{C}) \leqslant CM(\ell^2/b^2) + \mathfrak{K}\mathfrak{k}^2\ell^2 \leqslant C\ell^2$. □

Remark 4.4.2. In order to obtain Theorem 1.1 in the Introduction from Theorem 4.1, (b), we only need to verify that if the space X_0 is the exterior of an open horoball, then it verifies property (P_1). This is proved in [11, proof of Corollary 4.16].

4.5. Asymptotically quadratic filling order in lattices

In this section we prove Theorem 4.1, (a). A consequence of it is Theorem 1.2, since a space X_0 on which a $\mathbb{Q}$-rank one lattice acts cocompactly, endowed with the induced metric, has asymptotically quadratic filling order. We note that such a space X_0 satisfies properties (P_1) and (P_2) formulated in Theorem 4.1 (see [10, Propositions 5.5 and 5.7] and [28, Lemma 8.3]).

Proof of Theorem 4.1, (a). By Theorem 4.3, (a), in every asymptotic cone $\mathbf{K}_0$ of X_0 we have the inequality $A_1(\ell) \leqslant k\ell^3$, $\forall \ell > 0$, where $k = k(X_0)$. We show that the filling order is actually quadratic in every asymptotic cone $\mathbf{K}_0$. We have similarities between asymptotic cones (Remark 2.1.1) which allow to deduce that in every asymptotic cone $A_\lambda(\ell) \leqslant k(\ell/\lambda)^3, \forall \lambda > 0$. We recall that $A_1(\ell) \leqslant k\ell^3$ means that we need at most $k\ell^3$ bricks of length at most 1 to fill a loop of length ℓ. By construction of the filling disk, $k_1\ell^3$ of these bricks bound small Euclidean triangles entirely contained into intersections of horospheres with apartments, while at most $k_2\ell^2$ bricks have shapes on which we know nothing. If a small loop $\mathfrak{c}_1$ of length less than 1 bounds an Euclidean triangle entirely contained into the intersection of a horosphere with an apartment, then its λ-filling area in $\mathbf{K}_0$ satisfies $A_\lambda(\mathfrak{c}_1) \leqslant k(1/\lambda^2), \forall \lambda \in [0, 1[$. If a loop $\mathfrak{c}_2$ has length at most one and is arbitrary, at least $A_\lambda(\mathfrak{c}_2) \leqslant k(1/\lambda^3)$. Thus, for a generic loop $\mathfrak{C}$ of length at most ℓ we have

$$A_\lambda(\mathfrak{C}) \leqslant k_1\ell^3 \cdot k\,\frac{1}{\lambda^2} + k_2\ell^2 \cdot k\,\frac{1}{\lambda^3}. \tag{4.4}$$

If we replace λ by ℓ/M, we obtain

$$P\left(\ell, \frac{\ell}{M}\right) \leqslant k_1'\ell M^2 + \frac{k_2'}{\ell}M^3.$$

If one takes $\sqrt{M} \leqslant \ell \leqslant 2\sqrt{M}$, one obtains

$$P\left(\ell, \frac{\ell}{M}\right) \leqslant CM^{2.5}. \tag{4.5}$$

But since, by similarities and changing cone (Remark 2.1.1), one can modify the length, the relation (4.5) holds for every length ℓ and every M in every asymptotic cone. We conclude that the filling order is at most 2.5 in all asymptotic cones.

In the same way one can show that the filling order is quadratic in all asymptotic cones. Suppose that the minimal order of filling common to all asymptotic cones is $2+\varepsilon$. Then in all asymptotic cones we have $A_1(\ell) \leqslant k\ell^{2+\varepsilon}$. By means of similarities we conclude that in all asymptotic cones $A_\lambda(\ell) \leqslant k(\frac{\ell}{\lambda})^{2+\varepsilon}$. Then we can modify the inequality (4.4) and write that for a generic loop $\mathfrak{C}$ of length at most ℓ

$$A_\lambda(\mathfrak{C}) \leqslant k_1\ell^3 \cdot k\,\frac{1}{\lambda^2} + k_2\ell^2 \cdot k\,\frac{1}{\lambda^{2+\varepsilon}},$$

which implies that in all asymptotic cones we have

$$P\left(\ell, \frac{\ell}{M}\right) \leqslant k_1'\ell M^2 + \frac{k_2'}{\ell^\varepsilon}M^{2+\varepsilon}.$$

For lengths $\ell \in [M^{\varepsilon/2}, 2M^{\varepsilon/2}]$, we obtain

$$P\left(\ell, \frac{\ell}{M}\right) \leqslant CM^{2+\varepsilon-(\varepsilon^2/2)}.$$

The previous inequality can be generalised to all lengths, by similarities and changing cone. It finally gives a filling order $2+\varepsilon-\varepsilon^2/2$ in all asymptotic cones. This contradicts the minimality of the order $2+\varepsilon$.

Thus, we get a quadratic filling order in all asymptotic cones of X_0, hence an asymptotically quadratic filling order in X_0, by Theorem 2.1.2. □

Acknowledgements

The main part of this paper was written during the author's stay at Max-Planck-Institut für Mathematik in Bonn. I would like to thank this institution for its hospitality. I would also like to thank Mark Sapir for having shown me the induction trick, which I used in the proof of the implication Theorem 4.3, (b) ⇒ Theorem 4.1, (b).

Appendix A.

We prove several useful results of Euclidean geometry.

Notations: Let Φ be an affine subspace of dimension k in the Euclidean space $\mathbb{E}^n$. For every $p\in\Phi$ we denote $\Phi_p^{\perp}$ the subspace orthogonal to Φ through p of dimension $n-k$.

If Ψ is an affine subspace of Φ and $p\in\Psi$, we denote $\Psi_{p,\Phi}^{\perp}$ the intersection $\Psi_p^{\perp}\cap\Phi$.

We consider $\mathscr{P}$ a convex polytope in the Euclidean space $\mathbb{E}^n$, $n\geqslant 2$, Φ its affine span and k the common dimension of $\mathscr{P}$ and Φ. We suppose $k\leqslant n-1$.

For every $x_0\in\mathscr{P}$ we denote

$$\mathscr{K}_{x_0}(\mathscr{P})=\{x\in\mathbb{E}^n \mid d(x,\mathscr{P})=d(x,x_0)\}.$$

For a face $\mathfrak{f}$ of $\mathscr{P}$ we denote by *Int* $\mathfrak{f}$ the relative interior of $\mathfrak{f}$ if $\dim\mathfrak{f}\geqslant 1$. We make the convention *Int* $\mathfrak{f}=\mathfrak{f}$ if $\mathfrak{f}$ is a vertex.

Lemma A.1. (1) *For every point $x_0\in Int\,\mathfrak{f}$, $\mathscr{K}_{x_0}(\mathscr{P})$ is a convex polytopic cone in $(Span\,\mathfrak{f})_{x_0}^{\perp}$. It coincides with the convex hull of $\Phi_{x_0}^{\perp}$ and of a convex polytopic cone $\mathscr{K}_{x_0}^{\Phi}(\mathscr{P})$* contained in $(Span\,\mathfrak{f})_{x_0,\Phi}^{\perp}$.

(2) *For every two points x_0, y_0 in $Int\,\mathfrak{f}$, $\mathscr{K}_{y_0}(\mathscr{P})$ is the image of $\mathscr{K}_{x_0}(\mathscr{P})$ by the translation of vector $\overrightarrow{x_0y_0}$.*

Proof. If $x\neq x_0$ the fact that $d(x,\mathscr{P})=d(x,x_0)$ is equivalent to the fact that for every $y\in\mathscr{P}\setminus\{x_0\}$, $\angle_{x_0}(x,y)\geqslant\pi/2$. If $x_0\in Int\,\mathfrak{f}$ then $\angle_{x_0}(x,y)=\pi/2$ for every $y\in\mathfrak{f}\setminus\{x_0\}$. Therefore, we may write

$$\mathscr{K}_{x_0}(\mathscr{P})=\{x\in(Span\,\mathfrak{f})_{x_0}^{\perp}\mid\langle\overrightarrow{x_0x},\overrightarrow{x_0y}\rangle\leqslant 0,\ \ \forall y\in\mathscr{P}\setminus\{x_0\}\},\tag{5.1}$$

where $\langle\vec{u},\vec{v}\rangle$ denotes the scalar product of the vectors $\vec{u}$ and $\vec{v}$.

For two points x_0, y_0 in $Int\,\mathfrak{f}$, there exists $\varepsilon>0$ such that $\mathscr{P}\cap B(y_0,\varepsilon)$ is the image of $\mathscr{P}\cap B(x_0,\varepsilon)$ by translation of vector $\overrightarrow{x_0y_0}$. This and relation (5.1) imply (2).

For every $x\in\mathscr{K}_{x_0}(\mathscr{P})$, $x\neq x_0$, we can decompose $\overrightarrow{x_0x}$ in a unique way as $\overrightarrow{x_0x'}+\overrightarrow{x_0x''}, x'\in\Phi, x''\in\Phi_{x_0}^{\perp}$.

Then $\langle\overrightarrow{x_0x},\overrightarrow{x_0y}\rangle\leqslant 0\Leftrightarrow\langle\overrightarrow{x_0x'},\overrightarrow{x_0y}\rangle\leqslant 0, \forall y\in\mathscr{P}\setminus\{x_0\}$. Let $\mathscr{K}_{x_0}^{\Phi}(\mathscr{P})$ be the set of points $x'\in\Phi$ satisfying the latter condition. To end the proof of (1) it remains to show that $\mathscr{K}_{x_0}^{\Phi}(\mathscr{P})$ is a polytopic cone in $(Span\,\mathfrak{f})_{x_0,\Phi}^{\perp}$.

Suppose $\mathfrak{f}$ is a vertex p of $\mathscr{P}$, in which case $x_0 = p$ and $(Span\,\mathfrak{f})^{\perp}_{p,\Phi} = \Phi$. In a neighbourhood of p, $\mathscr{P}$ coincides with the convex hull of the edges of endpoint p. Let $y_1, y_2, \dots, y_r$ be the endpoints distinct of p of all the edges through p. We can write

$$\mathscr{K}^{\Phi}_{p}(\mathscr{P}) = \{x \in \Phi \mid \langle \overrightarrow{px}, \overrightarrow{py_i} \rangle \leqslant 0, \quad \forall i \in \{1,2,\dots,r\}\}. \tag{5.2}$$

This is a convex polytopic cone.

Suppose $\mathfrak{f}$ has dimension at least 1. Then $x_0 \in Int\,\mathfrak{f}$ implies that $\mathscr{K}^{\Phi}_{x_0}(\mathscr{P}) \subset (Span\,\mathfrak{f})^{\perp}_{x_0,\Phi}$. In a neighbourhood of x_0, $\mathscr{P}$ coincides with the convex hull of $Span\,\mathfrak{f}$ and $\mathscr{P} \cap (Span\,\mathfrak{f})^{\perp}_{x_0,\Phi}$. It follows that

$$\mathscr{K}^{\Phi}_{x_0}(\mathscr{P}) = \{x \in (Span\,\mathfrak{f})^{\perp}_{x_0,\Phi} | \langle \overrightarrow{x_0x}, \overrightarrow{x_0y} \rangle \leqslant 0, \forall y \in \mathscr{P} \cap (Span\,\mathfrak{f})^{\perp}_{x_0,\Phi}, y \neq x_0\}.$$

The point x_0 is a vertex of the convex polytope $\mathscr{P} \cap (Span\,\mathfrak{f})^{\perp}_{x_0,\Phi}$. This and the argument done in the case when $\mathfrak{f}$ was a vertex allow to conclude that $\mathscr{K}^{\Phi}_{x_0}(\mathscr{P})$ is a convex polytopic cone. $\square$

Remark A.2. (1) If $x_0 \in Int\,\mathscr{P}$ then $\mathscr{K}_{x_0}(\mathscr{P}) = \Phi^{\perp}_{x_0}$.

(2) For every $x_0 \in \mathscr{P}$ we have $\partial \mathscr{N}_R(\mathscr{P}) \cap \mathscr{K}_{x_0}(\mathscr{P}) = S(x_0, R) \cap \mathscr{K}_{x_0}(\mathscr{P})$.

(3) Let $\mathfrak{f}$ be a face of $\mathscr{P}$ or $\mathscr{P}$ itself. The closure of the set $\bigcup_{x_0 \in Int\,\mathfrak{f}} \mathscr{K}_{x_0}(\mathscr{P})$ is the set

$$\mathscr{K}_{\mathfrak{f}}(\mathscr{P}) = \{x \in \mathbb{E}^n \mid d(x,\mathscr{P}) = d(x,\mathfrak{f}) = d(x,x_0), [x,x_0] \subset (Span\,\mathfrak{f})^{\perp}_{x_0}\}.$$

Notation: Let $x_0 \in \mathscr{P}$. For every face $\mathfrak{f}$ containing x_0 we denote

$$\mathscr{K}^{\mathfrak{f}}_{x_0}(\mathscr{P}) = \mathscr{K}_{x_0}(\mathscr{P}) \cap \mathscr{K}_{\mathfrak{f}}(\mathscr{P}).$$

Lemma A.3. Suppose either $k \leqslant n-2$ or $k = n-1$ and $\mathscr{P} \neq \Phi$. Suppose $\mathscr{P}$ has m faces. Let the hypersurface $\partial \mathscr{N}_R(\mathscr{P}), R > 0$, be endowed with the length metric d_{ℓ}. Two arbitrary points x, y in $\partial \mathscr{N}_R(\mathscr{P})$ can be joined with a curve $\mathfrak{C}$ *in* $\partial \mathscr{N}_R(\mathscr{P})$ such that

- length $\mathfrak{C} \asymp d_{\ell}(x, y)$;
- $\mathfrak{C} = \mathfrak{C}_1 \cup \mathfrak{C}_2 \cup \cdots \cup \mathfrak{C}_k$, *with* $k \leqslant m+1$, such that each $\mathfrak{C}_i$ is contained in $\mathscr{K}_{\mathfrak{f}}(\mathscr{P}) \cap \partial \mathscr{N}_R(\mathscr{P})$ for some face $\mathfrak{f} = \mathfrak{f}(i)$ (possibly $\mathfrak{f} = \mathscr{P}$) and each $\mathfrak{C}_i$ is composed of at most two segments parallel to $Span\,\mathfrak{f}$ and one arc of circle contained in a set of the form $\mathscr{K}_{x_0}(\mathscr{P}) \cap \partial \mathscr{N}_R(\mathscr{P})$, where $x_0 \in \mathfrak{f}$.

Proof. Let $\mathfrak{g}$ be a geodesic joining x and y in $\partial \mathscr{N}_R(\mathscr{P})$. We consider a minimal set of faces $\mathfrak{f}_1, \mathfrak{f}_2, \dots, \mathfrak{f}_k$ of $\mathscr{P}$ (where $\mathscr{P}$ itself is considered a face) such that $\mathfrak{g} \subset \bigcup_{i=1}^{k} \mathscr{K}_{\mathfrak{f}_i}(\mathscr{P})$. We fix the face $\mathfrak{f}_i$. Let p and q be the first and, respectively, the last point of $\mathfrak{g}$ contained in $\mathscr{K}_{\mathfrak{f}_i}(\mathscr{P})$. Let p_0 and q_0 be the projections of p and q on $\mathscr{P}$. If $p = q$ we can ignore the face $\mathfrak{f}_i$, which contradicts the minimality of the set of faces. Hence $p \neq q$. Let p' be the image of p by the translation of vector $\overrightarrow{p_0q_0}$. There are two cases: either $\mathfrak{f}_i$ is of codimension at least 2 in $\mathbb{E}^n$ or $\mathfrak{f}_i = \mathscr{P}$ is of codimension one.

Suppose $\mathfrak{f}_i$ is of codimension at least 2. Then p and q can be joined in $\partial \mathscr{N}_R(\mathscr{P})$ by the curve $\mathfrak{C}_i$ composed of the segment $[p, p']$ and of the arc of circle joining p' and q in $S(q_0, R) \cap \mathscr{K}_{q_0}(\mathscr{P})$. The length of this curve is $\ll d(p,q)$, so $\ll d_{\ell}(p,q)$.

Suppose $\mathfrak{f}_i = \mathscr{P}$ is of codimension one. If Φ is not separating p and q then $p' = q$ and we can join p and q by $\mathfrak{C}_i = [p,q]$. Suppose Φ separates p and q. Let $z \in \partial \mathscr{P}$ be such that $d(p_0, z) + d(z, q_0) =$

$\inf_{t\in\partial\mathscr{P}}[d(p_0,t)+d(t,q_0)]$. Let p'' and q'' be the respective images of p and q by the translations of vectors $\overrightarrow{p_0z}$ and $\overrightarrow{q_0z}$. Then p and q can be joined in $\partial\mathscr{N}_R(\mathscr{P})$ by the curve $\mathfrak{C}_i$ composed of the segments $[p,p''],[q,q'']$ and a half circle of length πR joining p'' and q'' in $\mathscr{K}_z(\mathscr{P})\cap S(z,R)$. In order to prove that the curve $\mathfrak{C}_i$ thus obtained has length $\asymp d_\ell(p,q)$ it suffices to prove that $d_\ell(p,q)\geqslant d(p_0,z)+d(z,q_0)$. Every curve joining p and q in $\partial\mathscr{N}_R(\mathscr{P})$ of length $d_\ell(p,q)$ has at least one point ζ in Φ. Then $d_\ell(p,q)\geqslant d(p,\zeta)+d(\zeta,q)\geqslant d(p_0,\zeta')+d(\zeta',q_0)\geqslant d(p_0,z)+d(z,q_0)$, where ζ' is the projection of ζ on $\mathscr{P}$.

In all cases the curve previously described, joining p and q in $\partial\mathscr{N}_R(\mathscr{P})$, has length $\asymp d_\ell(p,q)$ and it is composed of at most two segments and one arc of circle.

We repeat the previous argument for each face $\mathfrak{f}_i$. We obtain in the end a curve $\mathfrak{C}$ joining x and y, of length $\ll d_\ell(x,y)$, which decomposes as $\mathfrak{C}=\mathfrak{C}_1\cup\mathfrak{C}_2\cup\cdots\cup\mathfrak{C}_k$, where each $\mathfrak{C}_i$ corresponds to a face $\mathfrak{f}_i$, and $k\leqslant m+1$. □

Lemma A.4. *Suppose the polytope $\mathscr{P}$ is either of codimension at least 2 or of codimension 1 and different of Φ. Let $a\geqslant 1$ be a fixed constant.*

1. *The projection of $\partial\mathscr{N}_{aR}(\mathscr{P})$ onto $\partial\mathscr{N}_R(\mathscr{P})$ is bilipschitz with respect to the length metrics, the constant of the bilipschitz equivalence depending only on a.*
2. *Let $\mathfrak{R}$ be a convex polytope in $\mathbb{E}^n$. If*

$$\mathscr{N}_R(\mathscr{P})\subset\mathfrak{R}\subset\mathscr{N}_{aR}(\mathscr{P})$$

then the projection of the hypersurface $\partial\mathfrak{R}$ onto $\partial\mathscr{N}_R(\mathscr{P})$ is bilipschitz with respect to the length metrics, the bilipschitz constant depending only on a.

Proof. (a) Let x,y be two points in $\partial\mathscr{N}_{aR}(\mathscr{P})$ and let x',y' be their respective projections on $\partial\mathscr{N}_R(\mathscr{P})$. Since $\mathscr{N}_R(\mathscr{P})$ is a convex set, $d_\ell(x,y)\geqslant d_\ell(x',y')$. In order to prove the converse inequality, we join x' and y' in $\partial\mathscr{N}_R(\mathscr{P})$ by a curve $\mathfrak{C}=\mathfrak{C}_1\cup\mathfrak{C}_2\cup\cdots\cup\mathfrak{C}_k$ as in Lemma A.3. For every arc $\mathfrak{C}_i$ we construct the arc $\mathfrak{C}_i'$ by considering for each point $x_i\in\mathfrak{C}_i$ its projection x_i^0 on $\mathscr{P}$ and the intersection x_i' of the ray of origin x_i^0 through x_i with $\partial\mathscr{N}_{aR}(\mathscr{P})$. The length of $\mathfrak{C}_i'$ is at most the length of $\mathfrak{C}_i$ multiplied by a. Then $d_\ell(x,y)\leqslant\sum_{i=1}^k \text{length}(\mathfrak{C}_i')\leqslant a\sum_{i=1}^k \text{length}\,(\mathfrak{C}_i)=a\cdot\text{length}(\mathfrak{C})\ll a d_\ell(x',y')$.

(b) Let x and y be two distinct points on $\partial\mathfrak{R}$ and let x' and y' be their respective projections on $\partial\mathscr{N}_R(\mathscr{P})$. Obviously, $d_\ell(x,y)\geqslant d_\ell(x',y')$. We note that x' is on the segment $[x,x_0]$, where x_0 is the projection of x on $\mathscr{P}$. Let x'' be the intersection point of $\partial\mathscr{N}_{aR}(\mathscr{P})$ with the ray through x of origin x_0. The point y'' is obtained from y in the same way. By (a) $d_\ell(x'',y'')\ll a d_\ell(x',y')$. In the sequel we show that $d_\ell(x,y)\ll d_\ell(x'',y'')$. The following remark is essential. By the convexity of $\mathfrak{R}$, every hyperplane H in $\mathbb{E}^n$ containing a codimension 1 face of $\partial\mathfrak{R}$ has the property that the distance from $\mathscr{P}$ to H is at least R. In particular, for every point $\alpha_0\in\mathscr{P}$ and every nontrivial segment $[\alpha,\beta]\subset\partial\mathfrak{R}$, the distance from α_0 to the line $\alpha\beta$ is at least R.

We join x'' and y'' in $\partial\mathscr{N}_{aR}(\mathscr{P})$ as in Lemma A.3, by a curve $\mathfrak{C}=\mathfrak{C}_1\cup\mathfrak{C}_2\cup\cdots\cup\mathfrak{C}_k$ of length $\ll d_\ell(x'',y'')$. Each $\mathfrak{C}_i$ is joining two points p_i and q_i, whose projections on $\mathscr{P}$, p_i^0 and q_i^0, are contained in a face $\mathfrak{f}_i$ of $\mathscr{P}$, and such that $[p_i,p_i^0]$ and $[q_i,q_i^0]$ are orthogonal to $\mathfrak{f}_i$. We note that $q_i\equiv p_{i+1},\forall i$. Let $\bar p_i=[p_i,p_i^0]\cap\partial\mathfrak{R}$ and $\bar q_i=[q_i,q_i^0]\cap\partial\mathfrak{R}$. It suffices to prove that $d_\ell(\bar p_i,\bar q_i)\ll$ length $(\mathfrak{C}_i)$ for every i.

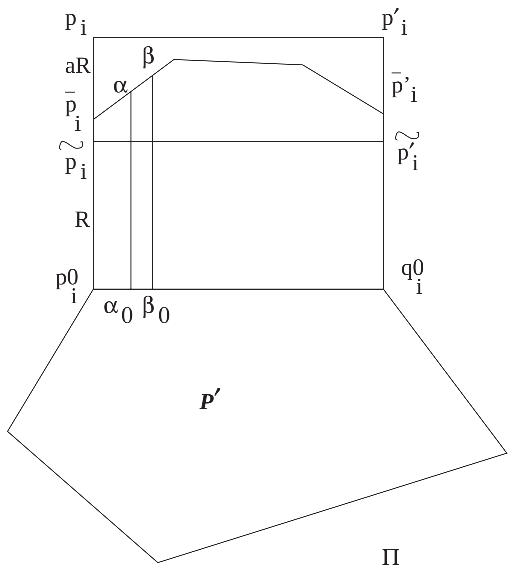

Fig. 4. The plane Π.

Suppose $\mathfrak{f}_i$ is of codimension at least 2. Then $\mathfrak{C}_i$ is composed of a segment $[p_i, p_i']$ and of an arc of circle between p_i' and q_i. Let $\bar{p}_i'=[p_i', q_i^0]\cap\partial\mathfrak{N}$, $\tilde{p}_i=[p_i, p_i^0]\cap\partial\mathcal{N}_R(\mathscr{P})$ and $\tilde{p}_i'=[p_i', q_i^0]\cap\partial\mathcal{N}_R(\mathscr{P})$. The plane Π determined by p_i, p_i', p_i^0 and q_i^0 intersects $\mathscr{P}$ either in a convex polygon $\mathscr{P}'$ having $[p_i^0, q_i^0]$ in the boundary or in $[p_i^0, q_i^0]$. The intersection $\Pi\cap\partial\mathcal{N}_{aR}(\mathscr{P})$ includes the segment $[p_i, p_i']$, $\Pi\cap\partial\mathcal{N}_R(\mathscr{P})$ includes the segment $[\tilde{p}_i, \tilde{p}_i']$ and the intersection $\Pi\cap\mathfrak{N}$ is a polygon containing $\bar{p}_i$ and $\bar{p}_i'$ in its boundary (Fig. 4).

Let $[\alpha,\beta]$ be a segment in the polygonal line $\Pi\cap\partial\mathfrak{N}$ which is also included in the quadrangle $\tilde{p}_i-\tilde{p}_i'-p_i'-p_i$. Let $[\alpha_0,\beta_0]$ be its projection onto $[p_i^0, q_i^0]$. Since the distance from α_0 to the line $\alpha\beta$ is at least R, while $d(\alpha_0,\alpha)\leqslant aR$, it follows that $\angle_\alpha(\alpha_0,\beta)\geqslant \arcsin\frac{1}{a}$, hence that $d(\alpha,\beta)\leqslant ad(\alpha_0,\beta_0)$. This implies that $d_\ell(\bar{p}_i, \bar{p}_i')\leqslant ad(p_i, p_i') = ad_\ell(p_i, p_i')$.

Let $\tilde{q}_i=[q_i^0, q_i]\cap\partial\mathcal{N}_R(\mathscr{P})$. The plane Π' determined by p_i', q_i^0 and q_i intersects $\mathscr{P}$ in a convex polygon and its face $\mathfrak{f}_i$ in the point q_i^0. The intersections $\Pi'\cap\partial\mathcal{N}_{aR}(\mathscr{P})$ and $\Pi'\cap\partial\mathcal{N}_R(\mathscr{P})$ contain the arcs of the circles of center q_i^0 and radius aR and R, respectively, joining p_i', q_i and $\tilde{p}_i', \tilde{q}_i$, respectively (Fig. 5). The fact that $d_\ell(\bar{p}_i', \bar{q}_i)\leqslant cd_\ell(p_i', q_i)$, with c a constant depending on a, follows from [10, Lemma 3.5].

We conclude that $d_\ell(\bar{p}_i,\bar{q}_i)\leqslant d_\ell(\bar{p}_i,\bar{p}_i')+d_\ell(\bar{p}_i',\bar{q}_i)\leqslant ad_\ell(p_i,p_i')+cd_\ell(p_i',q_i)\leqslant\varkappa\cdot\text{length }\mathfrak{C}_i$, where $\varkappa=\max(a,c)$.

Suppose $\mathfrak{f}_i\equiv\mathscr{P}$ is of codimension 1. In this case $\mathfrak{C}_i$ consists either of the segment $[p_i, q_i]$ or of the union of two segments parallel to Φ with an arc of circle of length πaR in a set of the form $\mathscr{K}_z(\mathscr{P})\cap S(z, aR)$. In the first situation, we conclude by looking at the plane determined by p_i, p_i^0, q_i, q_i^0 and repeating the first part of the previous argument. In the second situation we split $\mathfrak{C}_i$ into its three components. For each of the two segments we repeat the first part of the previous argument, for the arc of circle we repeat the second part of the previous argument. □

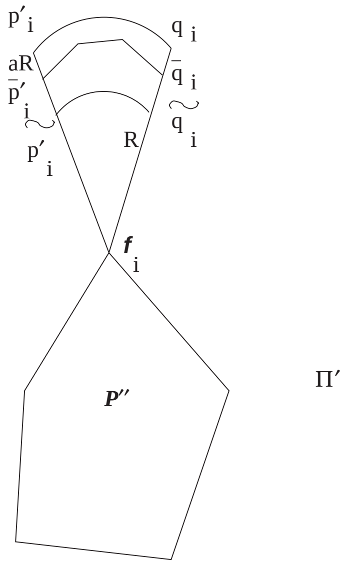

Fig. 5. The plane Π'.

Now we prove the key Euclidean geometry result on filling in hypersurfaces. We use the following terminology. Let Φ be an affine subspace in $\mathbb{E}^n$ and let $F:\mathbb{E}^n \to \mathbb{R}$ be a linear form such that Φ is not parallel to $\ker F$.

Definition A.5. We call codimension one d-strip in Φ a set of the form $\{x\in\Phi \mid a\leqslant F(x)\leqslant b\}$ such that its boundary hyperplanes in Φ are distance d apart. We call codimension two (ϵ, d)-strip in Φ the intersection of two codimension one d-strips such that all the dihedral angles between the boundary hyperplanes of one strip and the boundary hyperplanes of the other are greater than ϵ.

Proposition A.6. *Suppose that* $n\geqslant 3$.

(1) *If* $\mathscr{P}$ *has codimension at least* 3 *then there exists an universal constant* $L>0$ *such that for every* $R>0$ *the filling function in* $\partial\mathscr{N}_R(\mathscr{P})$ *satisfies*

$$A_1(\ell)\leqslant L\cdot \ell^2,\quad \forall \ell>0.$$

(2) *If* $\mathscr{P}$ *has codimension at least* 2 *and it is contained in a codimension one* δ*-strip in* Φ *then there exists an universal constant* $L>0$ *such that for every* $R>0$ *the filling function in* $\partial\mathscr{N}_R(\mathscr{P})$ *satisfies*

$$A_1(\ell)\leqslant L\cdot(\ell^2+\ell\delta),\quad \forall \ell>0.$$

(3) *Suppose* $\mathscr{P}$ *has codimension one and it is contained in a codimension two* (ϵ,δ)*-strip in* Φ*. Suppose moreover that*

- *every two nonparallel affine spans of codimension one faces of* $\mathscr{P}$ *make dihedral angles greater than* ϵ *and smaller than* $\pi-\epsilon$;
- *every affine span of a codimension one face of* $\mathscr{P}$ *makes with a hyperplane bounding the* (ϵ,δ)*-strip dihedral angles greater than* ϵ *and smaller than* $\pi-\epsilon$*, if it is not parallel to it.*

Then the same conclusion as in (2) *occurs.*

The first step in the proof is the following lemma.

Lemma A.7. *Let* $\mathfrak{C}:\mathbb{S}^1\to\partial\mathscr{N}_R(\mathscr{P})$ *be a loop of length* $\ell<R$*. Then its filling area in* $\partial\mathscr{N}_R(\mathscr{P})$ *satisfies*

$$A_1(\mathfrak{C})\leqslant L\ell^2,$$

where L is an universal constant.

Proof. Let $x\in\mathfrak{C}(\mathbb{S}^1)$ and x_0 its projection on $\mathscr{P}$. Then $\angle_{x_0}(x,y)\geqslant\pi/2,\forall y\in\mathscr{P}\setminus\{x_0\}$. Let H and H' be the hyperplanes orthogonal to $[x,x_0]$ through x_0 and x, respectively. The hyperplane H separates x and $\mathscr{P}$ and $d(x,H)=R$. Since $\ell<R$ it follows that $\mathfrak{C}(\mathbb{S}^1)\cap H=\emptyset$. We project the curve $\mathfrak{C}$ onto a curve $\mathfrak{C}'$ in H'. Let for every $y\in\mathfrak{C}(\mathbb{S}^1)$ the point y' be its projection on H'. For every $a\in\mathscr{P}$, $\angle_y(y',a)\geqslant\frac{\pi}{2}$, therefore $d(y',a)\geqslant d(y,a)$. It follows that the filling cylinder between $\mathfrak{C}$ and $\mathfrak{C}'$ of area $\ll\ell^2$ composed of the segments $[y,y'], y\in\mathfrak{C}(\mathbb{S}^1)$, is contained in $Ext_R(\mathscr{P})$. The hyperplane H' is also contained in $Ext_R(\mathscr{P})$ and we can fill $\mathfrak{C}'$ in it with an area $\ll\ell^2$. Hence $\mathfrak{C}$ can be filled in $Ext_R(\mathscr{P})$ with an area $\ll\ell^2$. By projecting on $\partial\mathscr{N}_R(\mathscr{P})$ we conclude. □

Proof of Proposition A.6. Let $\mathfrak{C}:\mathbb{S}^1\to\partial\mathscr{N}_R(\mathscr{P})$ be a loop of length ℓ. According to Lemma A.7 we may suppose $\ell\geqslant R$.

(1) By eventually slightly perturbing $\mathfrak{C}(\mathbb{S}^1)$ we may suppose $\mathfrak{C}(\mathbb{S}^1)\cap\Phi=\emptyset$. This is true in every $\mathscr{K}_{\mathfrak{f}}(\mathscr{P})\cap\partial\mathscr{N}_R(\mathscr{P})$, where $\mathfrak{f}$ is a face of $\mathscr{P}$, therefore it is true on $\partial\mathscr{N}_R(\mathscr{P})$.

For every $x\in\mathfrak{C}(\mathbb{S}^1)$ let x_0 be its projection on $\mathscr{P}$, x_0' its projection on $\Phi_{x_0}^{\perp}$ and x' the intersection between the ray of origin x_0 through x_0' and $S(x_0,R)$. We consider the arc of circle $\mathfrak{a}_{xx'}$ joining x and x' in $\mathscr{K}_{x_0}(\mathscr{P})\cap S(x_0,R)$. The curve $\mathfrak{C}':\mathbb{S}^1\to\partial\mathscr{N}_R(\mathscr{P})$, $\mathfrak{C}'(\theta)=x'$ for θ such that $\mathfrak{C}(\theta)=x$, has length at most ℓ. If $x,y\in\mathfrak{C}(\mathbb{S}^1)\cap\mathscr{K}_{\mathfrak{f}}(\mathscr{P})$ then the Hausdorff distance between $\mathfrak{a}_{xx'}$ and $\mathfrak{a}_{yy'}$ is at most $d(x,y)$. Since $\bigcup_{\mathfrak{f}}\mathscr{K}_{\mathfrak{f}}(\mathscr{P})$ covers $\partial\mathscr{N}_R(\mathscr{P})$, we conclude that the arcs of circle $\mathfrak{a}_{xx'}$ with x varying on $\mathfrak{C}(\mathbb{S}^1)$, compose a "filling cylinder" between $\mathfrak{C}$ and $\mathfrak{C}'$ of area $\ll\ell R\ll\ell^2$.

Let now x' be a point on $\mathfrak{C}'(\mathbb{S}^1)$ and x_0 its projection on $\mathscr{P}$. We have $[x',x_0]\subset\Phi_{x_0}^{\perp}$. For every $y'\in\mathfrak{C}'(\mathbb{S}^1)$ of projection y_0 on $\mathscr{P}$ we consider y'' its image by translation of vector $\overrightarrow{y_0x_0}$. The curve $\mathfrak{C}''$ composed of the points y'' has length at most ℓ and the segments $[y',y'']$ compose a filling cylinder of area $\ll\ell^2$ between $\mathfrak{C}'$ and $\mathfrak{C}''$. The curve $\mathfrak{C}''$ is contained in $\Phi_{x_0}^{\perp}\cap S(x_0,R)$, a sphere of dimension at least 2, and it can be filled in this sphere with an area $\ll\ell^2$.

(2) We may again suppose, modulo a slight perturbation of $\mathfrak{C}$, that $\mathfrak{C}(\mathbb{S}^1)\cap\Phi=\emptyset$. As in (1) we can construct a filling cylinder of area $\ll\ell^2$ between the curve $\mathfrak{C}$ and a curve $\mathfrak{C}''$ of length at most ℓ contained in $\Phi_{x_0}^{\perp}\cap S(x_0,R)$, for some point $x_0\in\mathscr{P}$. The polytope $\mathscr{P}$ is contained in a

codimension one δ-strip $\mathscr{S}$ bounded by two hyperplanes H_1 and H_2 in Φ. Let $x_1 \in H_1$ at distance at most δ from x_0. We translate the curve $\mathfrak{C}''$ with the vector $\overrightarrow{x_0x_1}$. We obtain a curve $\mathfrak{C}'''$ of the same length in $\Phi_{x_1}^{\perp} \cap S(x_1, R)$. The filling cylinder between $\mathfrak{C}''$ and $\mathfrak{C}'''$ composed of segments parallel and of equal length to $[x_0, x_1]$ has area at most $\ell\delta$. The curve $\mathfrak{C}'''$ is contained in $\partial\mathscr{N}_R(\mathscr{S})$. Since $\mathscr{N}_R(\mathscr{P}) \subset \mathscr{N}_R(\mathscr{S})$, it suffices to fill $\mathfrak{C}'''$ in $\partial\mathscr{N}_R(\mathscr{S})$ with a quadratic area to end the argument. Such a filling can be constructed in $\mathscr{K}_{H_1}(\mathscr{S}) \cap \partial\mathscr{N}_R(\mathscr{S})$.

(3) In this case $\mathscr{K}_{\mathscr{P}}(\mathscr{P}) \cap \partial\mathscr{N}_R(\mathscr{P})$ has two connected components isometric to $\mathscr{P}$, which we denote Π_1 and Π_2. We fix $\vartheta > 0$ small and we consider $\mathscr{M}_\vartheta$ the set of points $y \in \mathscr{N}_R(\mathscr{P})$ such that $\angle_{y_0}(y, \Phi) \leqslant \vartheta$, where y_0 is the projection of y on $\mathscr{P}$. We denote $\breve{\mathscr{M}}_\vartheta$ the set of points $y \in \mathscr{N}_R(\mathscr{P})$ such that $\angle_{y_0}(y, \Phi) < \vartheta$. We also consider the set of points $y \in \mathscr{N}_R(\mathscr{P})$ such that $\angle_{y_0}(y, \Phi) \geqslant \vartheta$ and we denote Π_i^ϑ its connected component containing Π_i, $i = 1, 2$, $\partial\Pi_i^\vartheta = \Pi_i^\vartheta \cap \mathscr{M}_\vartheta$ and $\breve{\Pi}_i^\vartheta = \Pi_i^\vartheta \setminus \mathscr{M}_\vartheta$.

We simplify the shape of the loop under study in three steps and in the fourth step we prove the estimate on the filling area.

Step 1: Suppose $\mathfrak{C}(\mathbb{S}^1)$ is entirely contained in $\mathscr{M}_\vartheta$. For every point $x \in \mathfrak{C}(\mathbb{S}^1)$ let x_0 be its projection on $\mathscr{P}$ and x_1 the intersection point between $\Phi_{x_0}^{\perp}$ and Π_1. Let $\mathfrak{a}_{xx_1}$ be the arc of circle joining x and x_1 in $\partial\mathscr{N}_R(\mathscr{P})$ and x' the intersection point between $\mathfrak{a}_{xx_1}$ and $\partial\Pi_1^\vartheta$. For two points x and y we have $d(x_1, y_1) = d(x_0, y_0) \leqslant d(x, y)$. This implies that the arc of circle $\mathfrak{a}_{xx_1}$ varies continuously in x and that $d_H(\mathfrak{a}_{xx_1}, \mathfrak{a}_{yy_1}) \ll d(x, y)$. In particular, when x varies on $\mathfrak{C}(\mathbb{S}^1)$ the point x' describes a curve $\mathfrak{C}'$ on $\partial\Pi_1^\vartheta$ of length $\ll \ell$ and $\mathfrak{a}_{xx_1} \cap \mathscr{M}_\vartheta$ describes a filling cylinder between $\mathfrak{C}$ and $\mathfrak{C}'$ of area $\ll \ell R$, hence $\ll \ell^2$.

Suppose $\mathfrak{C}(\mathbb{S}^1)$ contains a point x outside $\mathscr{M}_\vartheta$, for instance $x \in \breve{\Pi}_1^\vartheta$. We endow $\mathbb{S}^1$ and correspondingly $\mathfrak{C}(\mathbb{S}^1)$ with an orientation. Let $y_1, \bar{y}_1, \ldots, y_s, \bar{y}_s$ be points on $\mathfrak{C}(\mathbb{S}^1)$ in the sense of the given orientation such that the arcs of $\mathfrak{C}(\mathbb{S}^1)$ between the pairs of points $(x, y_1), (\bar{y}_s, x)$ and $(\bar{y}_i, y_{i+1})_{i \in \{1,2,\ldots,s-1\}}$ are contained in $\Pi_1^\vartheta \cup \Pi_2^\vartheta$ while the arcs between $(y_i, \bar{y}_i)_{i \in \{1,2,\ldots,s\}}$ are contained in $\breve{\mathscr{M}}_\vartheta$. We consider the arc γ_i of endpoints $y_i, \bar{y}_i$ and of length ℓ_i.

Suppose $y_i, \bar{y}_i$ are both in $\partial\Pi_k^\vartheta$, $k \in \{1, 2\}$. For every point $y \in \gamma_i$ let y_0 be its projection on $\mathscr{P}$, y_1 the intersection point between $\mathscr{K}_{y_0}(\mathscr{P})$ and Π_k and y' the intersection point between the arc of circle joining y to y_1 and $\partial\Pi_k^\vartheta$. As y varies, y' describes an arc γ_i' in $\partial\Pi_k^\vartheta$ of length $\ll \ell_i$. The loop $\gamma_i \cup \gamma_i'$ can be filled by means of the arcs joining y and y' with an area of order $\ell_i R$.

Suppose $y_i \in \partial\Pi_k^\vartheta, \bar{y}_i \in \partial\Pi_l^\vartheta$, $\{k, l\} = \{1, 2\}$. We use the same notations and arguments as previously and thus to every $y \in \gamma_i$ we associate a point y' in $\partial\Pi_k^\vartheta$. Let γ_i' be the curve described by the points y' in $\partial\Pi_k^\vartheta$. The loop composed of γ_i, γ_i' and the arc of circle joining $\bar{y}_i$ to $\bar{y}_i'$ has length $\ll \ell_i$ and it can be filled with an area of order $\ell_i R$.

We conclude that, by eventually adding an area of order ℓR, so ℓ^2, we may suppose that the intersection of $\mathfrak{C}(\mathbb{S}^1)$ with $\breve{\mathscr{M}}_\vartheta$, if non-empty, is composed of arcs of circle. Thus, we may suppose that either $\mathfrak{C}(\mathbb{S}^1) \subset \Pi_1^\vartheta \cup \Pi_2^\vartheta$ or the points $y_1, \bar{y}_1, \ldots, y_s, \bar{y}_s$ on $\mathfrak{C}(\mathbb{S}^1)$ considered previously are such that moreover $y_i \in \partial\Pi_k^\vartheta, \bar{y}_i \in \partial\Pi_l^\vartheta$, $\{k, l\} = \{1, 2\}$, y_i and $\bar{y}_i$ have the same projection y_0^i on $\mathscr{P}$ and the arc γ_i of $\mathfrak{C}(\mathbb{S}^1)$ between y_i and $\bar{y}_i$ is an arc of circle in $\mathscr{K}_{y_0^i}(\mathscr{P}) \cap \partial\mathscr{N}_R(\mathscr{P})$.

Step 2: Let $\mathfrak{g}$ be either $\mathfrak{C}(\mathbb{S}^1)$, if $\mathfrak{C}(\mathbb{S}^1) \subset \Pi_1^\vartheta \cup \Pi_2^\vartheta$, or one of the sub-arcs of $\mathfrak{C}(\mathbb{S}^1)$ of endpoints $(p, q) = (\bar{y}_i, y_{i+1})$ or $(\bar{y}_s, y_1)$. The curve $\mathfrak{g}$ is in both cases contained in $\Pi_k^\vartheta, k \in \{1, 2\}$. For every y on $\mathfrak{g}$ let y_0 be its projection on $\mathscr{P}$ and y' the intersection point between $\Phi_{y_0}^{\perp}$ and Π_k. The points y' describe a curve $\mathfrak{g}'$ in Π_k of length at most equal to the length of $\mathfrak{g}$ and if $\mathfrak{a}_{yy'}$ denotes the arc of circle joining y and y' in $\mathscr{K}_{y_0}(\mathscr{P}) \cap \partial\mathscr{N}_R(\mathscr{P})$, $d_H(\mathfrak{a}_{yy'}, \mathfrak{a}_{zz'}) \leqslant d_\ell(y, z)$ for every $y, z \in \mathfrak{g}$. If

$\mathfrak{g} = \mathfrak{C}(\mathbb{S}^1) \subset \Pi_k^{\vartheta}$ then $\mathfrak{g}'$ is a loop in Π_k and $\mathfrak{a}_{yy'}, y \in \mathfrak{g}$, compose a filling cylinder of area $\ll \ell R$ between $\mathfrak{g}$ and $\mathfrak{g}'$. Since $\mathfrak{g}'$ can be filled in Π_k with an area $\ll \ell^2$, this ends the argument in this case, and by the first part of Step 1 also in the case when $\mathfrak{C}(\mathbb{S}^1) \subset \mathscr{M}_{\vartheta}$. If $\mathfrak{g} \neq \mathfrak{C}(\mathbb{S}^1)$, in the loop $\mathfrak{C}$ we replace each sub-arc of type $\mathfrak{g}$ with $\mathfrak{a}_{pp'} \cup \mathfrak{g}' \cup \mathfrak{a}_{qq'}$ and we obtain a new loop $\mathfrak{C}'$ and a filling cylinder between $\mathfrak{C}$ and $\mathfrak{C}'$ of area $\ll \ell^2$. The length of the loop $\mathfrak{C}'$ is $\ll \ell$ since instead of each arc $\mathfrak{g}$ we have $\mathfrak{g}'$ of smaller length and instead of each arc γ_i we have $\mathfrak{a}_{y_i' y_i} \cup \gamma_i \cup \mathfrak{a}_{\bar{y}_i \bar{y}_i'}$ of length $\ll R$, while the length of γ_i is at least $2\vartheta R$.

We may moreover replace in $\mathfrak{C}'$ each curve $\mathfrak{a}_{y_i' y_i} \cup \gamma_i \cup \mathfrak{a}_{\bar{y}_i \bar{y}_i'}$ with an arc of circle γ_i' joining the opposite points $y_i', \bar{y}_i'$ in $\partial \mathscr{N}_R(\mathscr{P})$. For each such replacement we have to add an area of order $O(R^2) = O((\text{length}\,\gamma_i)^2)$, therefore on the whole we add an area of order $O(\ell^2)$.

In the end we may therefore suppose that our loop $\mathfrak{C}$ is composed of curves in Π_1, curves in Π_2 and arcs of circle joining points in Π_1 to points in Π_2.

Step 3: Let $p_1, \bar{p}_1, \ldots, p_s, \bar{p}_s$ be points on $\mathfrak{C}(\mathbb{S}^1)$ in the sense of the given orientation such that:

- the arcs of $\mathfrak{C}(\mathbb{S}^1)$ of endpoints $p_i, \bar{p}_i$ are in $\Pi_1 \cup \Pi_2$;
- the pairs of points $(\bar{p}_i, p_{i+1})$ or $(\bar{p}_s, p_1)$ have the same projection q_i or q_s, respectively, on $\mathscr{P}$ and the arc of $\mathfrak{C}(\mathbb{S}^1)$ of endpoints $(\bar{p}_i, p_{i+1})$ or $(\bar{p}_s, p_1)$ is an arc of circle of length πR in $\mathscr{K}_{q_i}(\mathscr{P}) \cap \partial \mathscr{N}_R(\mathscr{P})$ or in $\mathscr{K}_{q_s}(\mathscr{P}) \cap \partial \mathscr{N}_R(\mathscr{P})$, respectively.

It follows $s = 2\tau$. Suppose $p_1, \bar{p}_1$ are in Π_1. Then $p_i, \bar{p}_i$ are in Π_1 if i is odd and in Π_2 if i is even. If we join with segments in Π_1 the pairs of points $\bar{p}_i, p_{i+2}$ for all i odd, $1 \leqslant i \leqslant s-3$, and $\bar{p}_{s-1}, p_1$, and we also join with segments in Π_2 the pairs of points $p_j, \bar{p}_j$ for all j even, we cut the loop into one loop contained in Π_1, τ loops contained in Π_2 and τ "quadrangles" composed of two segments and two arcs of circle. The filling area is quadratic for the loops contained in $\Pi_1 \cup \Pi_2$. We show that each "quadrangle" of length λ_i has a filling area of order $\lambda_i^2 + \lambda_i \delta$, thereby ending the proof.

Step 4: Let $\mathfrak{C}$ be a curve of length $\ell \geqslant R$ and such that there exist $p, q, \bar{q}, \bar{p}$ points on it in the sense of the chosen orientation such that the arcs of endpoints p, q and $\bar{p}, \bar{q}$ are segments contained in Π_1 and Π_2, respectively, and the arcs γ of endpoints $\bar{p}, p$ and γ' of endpoints $q, \bar{q}$ are arcs of circle in $\mathscr{K}_{p_0}(\mathscr{P}) \cap \partial \mathscr{N}_R(\mathscr{P})$ and in $\mathscr{K}_{q_0}(\mathscr{P}) \cap \partial \mathscr{N}_R(\mathscr{P})$, respectively, where $p_0, q_0 \in \partial \mathscr{P}$. By hypothesis $\mathscr{P}$ is contained in a codimension two (ϵ, δ)-strip $\mathscr{S}$ in Φ.

If p_0 and q_0 are in the same codimension one face $\mathfrak{f}$ of $\mathscr{P}$ then, modulo an area $\ll R^2$, we may suppose $\mathfrak{C}(\mathbb{S}^1)$ is contained in $\mathscr{K}_{\mathfrak{f}} \cap \partial \mathscr{N}_R(\mathscr{P})$, in which it can be filled with a quadratic area.

Suppose p_0 and q_0 are in the interiors of two codimension one faces of $\mathscr{P}$, of parallel affine spans Ψ and Ψ'. There exists a hyperplane H bounding $\mathscr{S}$ not parallel to Ψ and Ψ'. Let p_0^1 and q_0^1 be the projections of p_0 and q_0, respectively, onto H along Ψ and Ψ'. Let γ_1 be the image of γ by translation of vector $\overrightarrow{p_0 p_0^1}$ and γ_1' the image of γ' by translation of vector $\overrightarrow{q_0 q_0^1}$. Let $p_1, \bar{p}_1$ be the endpoints of γ_1 and $q_1, \bar{q}_1$ the endpoints of γ_1'. By hypothesis Ψ and H form dihedral angles greater than ϵ and smaller than $\pi - \epsilon$ and likewise for Ψ' and H. It follows that $d(p_0^1, q_0^1) \ll d(p_0, q_0)$ and therefore that the loop $\mathfrak{C}' = [p_1, q_1] \cup \gamma_1' \cup [\bar{q}_1, \bar{p}_1] \cup \gamma_1$ has length of order ℓ. It also follows that $d(p_0, p_0^1), d(q_0, q_0^1) \ll \delta$.

Between the loops $\mathfrak{C}$ and $\mathfrak{C}'$ can be constructed a filling cylinder of area $\ll \delta \ell$. Modulo an area of order R^2 we may moreover suppose that γ_1 and γ_1' are in $\partial \mathscr{N}_R(H) \cap D$, where D is the half-space of boundary $Span(H \cup \Phi_{p_0^1}^{\perp})$ whose interior is disjoint of $\mathscr{P}$. In $\partial \mathscr{N}_R(H) \cap D$ the loop $\mathfrak{C}'$ thus modified can be filled with an area $\ll \ell^2$. We thus construct a filling disk for $\mathfrak{C}$ of area $\ll \ell^2 + \ell\delta$ in $Ext_R(\mathscr{P})$.

Suppose p_0 and q_0 are contained into two nonparallel codimension one faces $\mathfrak{f}, \mathfrak{f}'$ of $\mathscr{P}$, of affine spans Ψ and Ψ'. Modulo an area of order R^2 we may suppose $\gamma \subset \mathscr{K}^{\mathfrak{f}}_{p_0}(\mathscr{P})$ and $\gamma' \subset \mathscr{K}^{\mathfrak{f}'}_{q_0}(\mathscr{P})$. Let $\Upsilon = \Psi \cap \Psi'$. By hypothesis Ψ and Ψ' form dihedral angles greater than ϵ and smaller than $\pi - \epsilon$. This implies that there exists $z_0 \in \Upsilon$ such that $d(p_0, z_0) + d(z_0, q_0) \ll d(p_0, q_0)$. Let $\tilde{\gamma}$ be the image of γ by translation of vector $\overrightarrow{p_0 z_0}$ and $\tilde{\gamma}'$ the image of γ' by translation of vector $\overrightarrow{q_0 z_0}$. Between the loop $\tilde{\mathfrak{C}} = \tilde{\gamma} \cup \tilde{\gamma}'$ and $\mathfrak{C}$ there is a filling cylinder of area $\ll \ell^2$. Let $\mathscr{Q}$ be the skew quadrant in Φ determined by Ψ and Ψ' and containing $\mathscr{P}$. In $\mathscr{K}_\Upsilon(\mathscr{Q}) \cap \partial \mathscr{N}_R(\mathscr{Q})$ the loop $\widetilde{\mathfrak{C}}$ can be filled with an area $\ll R^2$. We thus obtain a filling disk for $\mathfrak{C}$ in $Ext_R(\mathscr{P})$ of area $\ll \ell^2$. □

References

[1] D. Allcock, An isoperimetric inequality for the Heisenberg groups, Geom. Funct. Anal. 8 (1998) 219–233.
[2] J. Alonso, Inégalités isopérimétriques et quasi-isométries, C. R. Acad. Sci. Paris, Série 1 311 (1991) 761–764.
[3] G. Arzhantseva, D. Osin, Solvable groups with polynomial Dehn functions, Trans. Amer. Math. Soc. 354 (8) (2002) 3329–3348 (electronic).
[4] W. Ballmann, M. Gromov, V. Schroeder, Manifolds of Nonpositive curvature, Birkhäuser, Basel, 1985.
[5] G. Baumslag, C.F. Miller, H. Short, Isoperimetric inequalities and the homology of groups, Invent. Math. 113 (1993) 531–560.
[6] A. Borel, Introduction Aux Groupes Arithmétiques, Hermann, Paris, 1969.
[7] N. Bourbaki, Topologie Générale, 4th Edition, Hermann, Paris, 1965.
[8] M.R. Bridson, A. Haefliger, Metric Spaces of Non-positive Curvature, Springer, Berlin, 1999.
[9] K.S. Brown, Buildings, Springer, Paris, 1989.
[10] C. Druţu, Nondistorsion des horosphères dans des immeubles euclidiens et dans des espaces symétriques, Geom. Funct. Anal. 7 (5) (1997) 712–754.
[11] C. Druţu, Remplissage dans des réseaux de Q-rang 1 et dans des groupes résolubles, Pacific J. Math. 185 (2) (1998) 269–305.
[12] A. Dyubina, Instability of the virtual solvability and the property of being virtually torsion-free for quasi-isometric groups, Internat. Math. Res. Notices 21 (2000) 1097–1101.
[13] D.B.A. Epstein, J.W. Cannon, D.F. Holt, M.S. Paterson, W.P. Thurston, Word Processing and Group Theory, Jones and Bartlett Publications, 1992.
[14] B. Farb, L. Mosher, A rigidity theorem for the solvable Baumslag–Solitar groups, With an appendix by Daryl Cooper. Invent. Math. 131 (2) (1998) 419–451.
[15] B. Farb, L. Mosher, Quasi-isometric rigidity for the solvable Baumslag–Solitar groups. II, Invent. Math. 137 (3) (1999) 613–649.
[16] B. Farb, L. Mosher, On the asymptotic geometry of Abelian-by-cyclic groups, Acta Math. 184 (2) (2000) 145–202.
[17] S.M. Gersten, Isoperimetric and isodiametric functions of finite presentations, in: G.A. Niblo, M.A. Roller (Eds.), Geometric Group Theory, Vol. 1, Proceedings of the Symposium held in Sussex, LMS Lecture Notes Series, Vol. 181, Cambridge University Press, Cambridge, 1991.
[18] S.M. Gersten, D.F. Holt, T.R. Riley, Isoperimetric inequalities for nilpotent groups, preprint 2002.
[19] M. Gromov, Groups of polynomial growth and expanding maps, Publ. Math. IHES 53 (1981) 53–73.
[20] M. Gromov, Asymptotic invariants of infinite groups, in: G.A. Niblo, M.A. Roller (Eds.), Geometric Group Theory, Vol. 2, Proceedings of the Symposium held in Sussex, LMS Lecture Notes Series, Vol. 181, Cambridge University Press, Cambridge 1991.
[21] S. Helgason, Differential Geometry, Lie Groups and Symmetric Spaces, Academic Press, New York, 1978.
[22] B. Kleiner, B. Leeb, Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings, Publ. Math. IHES 86 (1997) 115–197.

[23] E. Leuzinger, Ch. Pittet, Isoperimetric Inequalities for Lattices in Semisimple Lie Groups of Rank 2, Geom. Funct. Anal. 6 (3) (1996) 489–511.
[24] E. Leuzinger, Ch. Pittet, Quadratic Dehn functions for solvable groups, preprint 2000.
[25] A. Lubotzky, Sh. Mozes, M.S. Raghunathan, Cyclic subgroups of exponential growth and metrics on discrete groups, C.R. Acad. Sci. Paris, Série 1 317 (1993) 735–740.
[26] A. Lubotzky, Sh. Mozes, M.S. Raghunathan, The word and Riemannian metrics on lattices of semisimple groups, Publ. Math. IHES 91 (2000) 5–53.
[27] G.A. Margulis, Discrete Subgroups of Semisimple Lie, Groups, Springer, Berlin, 1991.
[28] G.D. Mostow, Strong Rigidity of Locally Symmetric Spaces, AMS Studies, Vol. 78, Princeton University Press, Princeton, NJ, 1973.
[29] A.Yu. Olshanskii, M.V. Sapir, Quadratic isoperimetric functions of the Heisenberg groups, A combinatorial proof, Algebra, 11, J. Math. Sci. (New York) 93 (6) (1999) 921–927.
[30] G. Prasad, Strong rigidity of Q-rank one lattices, Invent. Math. 21 (1973) 255–286.
[31] M.S. Raghunathan, Discrete Subgroups of Lie Groups, Springer, Berlin, 1972.
[32] J. Tits, Classification of algebraic semisimple groups, Proc. Symp. Pure Math., Vol. 9, Algebraic Groups and Discontinuous Subgroups, Boulder, 1965, Amer. Math. Soc., Providence, RI, 1966, pp. 33-62.
[33] J. Tits, Buildings of Spherical Type and Finite BN-Pairs, in: Lecture Notes, Vol. 386, Springer, Berlin, 1974.
[34] L. Van Den Dries, A.J. Wilkie, On Gromov's theorem concerning groups of polynomial growth and elementary logic, J. of Algebra 89 (1984) 349–374.
[35] D. Witte Morris, Introduction to Arithmetic groups, http://www.math.okstate.edu/∼dwitte.