The Kinetic Formulation of the Skeleton Equation

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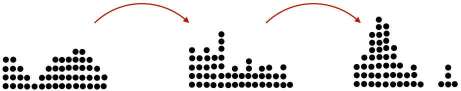
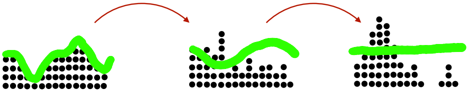
I. The skeleton equation

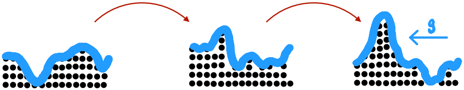
The zero range process: with random clocks $T(k) \sim ke^{-kt}$,



The heat equation: the hydrodynamic limit $\partial_t \overline{\rho} = \Delta \overline{\rho}$,



The skeleton equation: the controlled equation $\partial_t \rho = \Delta \rho - \nabla \cdot (\sqrt{\rho} g)$,



The rate function: we have $\mathbb{P}(\mu^N \simeq \rho) \simeq \exp(-NI(\rho))$ for

$$I(\rho) = \frac{1}{2} \inf \left\{ \|g\|^2_{L^2_{t,x}} : \partial_t \rho = \Delta \rho - \nabla \cdot (\sqrt{\rho} g) \right\}.$$

I. The skeleton equation

The skeleton equation: in the case of the zero range process,

$$\partial_t \rho = \frac{1}{2}\Delta\Phi(\rho) - \nabla\cdot(\Phi^{\frac{1}{2}}(\rho)g) = \nabla\cdot\left(\Phi^{\frac{1}{2}}(\rho)(\nabla\Phi^{\frac{1}{2}}(\rho) - g)\right),$$

for $g \in (L^2_{t,x})^d$ and $\nabla\Phi = 2\Phi^{\frac{1}{2}}(\rho)\nabla\Phi^{\frac{1}{2}}(\rho)$.

Fast diffusion and porous media: for $\alpha \in (0,\infty)$,

$$\partial_t \rho = \Delta\rho^{\alpha} - \nabla\cdot(\rho^{\frac{\alpha}{2}} g).$$

The rescaling: for $\lambda, \eta, \tau \to 0$ the rescaling $\tilde{\rho}(x,t) = \lambda\rho(\eta x, \tau t)$ solves

$$\partial_t \tilde{\rho} = \left(\frac{\tau}{\eta^2\lambda^{\alpha-1}}\right)\Delta\left(\tilde{\rho}^{\alpha}\right) - \nabla\cdot(\tilde{\rho}^{\frac{\alpha}{2}}\tilde{g})$$

for $\tilde{g}(x,t) = \left(\frac{\tau}{\eta\lambda^{\frac{\alpha}{2}-1}}\right)g(\eta x, \tau t)$.

Preserve diffusivity and L^r-norm: fix $\frac{\tau}{\eta^2\lambda^{\alpha-1}} = 1$ and $\lambda = \eta^{\frac{d}{r}}$.

Energy criticality: the Ladyzhenskaya–Prodi–Serrin (LPS) condition yields

$$\|\tilde{g}\|_{L^2_t L^2_x} = \eta^{-d\left(\frac{1}{2}-\frac{1}{2r}\right)}\|g\|_{L^2_t L^2_x}.$$

Energy critical if $r = 1$ and supercritical if $r \in (1,\infty)$.

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A formal uniqueness proof: if ρ_1 and ρ_2 solve

$$\partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g),$$

we have using the distributional equalities

$$|\xi|' = \operatorname{sgn}(\xi) \text{ and } \operatorname{sgn}'(\xi) = 2\delta_0(\xi),$$

that, integrating on the torus $\mathbb{T}^d$,

$$\begin{aligned}
\partial_t \Big(\int |\rho_1 - \rho_2| \Big) &= \int \operatorname{sgn}(\rho_1 - \rho_2) \Delta(\rho_1^\alpha - \rho_2^\alpha) - \int \operatorname{sgn}(\rho^1 - \rho^2) \nabla \cdot \big((\rho_1^{\frac{\alpha}{2}} - \rho_2^{\frac{\alpha}{2}}) g \big) \\
&= \int \operatorname{sgn}(\rho_1^\alpha - \rho_2^\alpha) \Delta(\rho_1^\alpha - \rho_2^\alpha) - \int \operatorname{sgn}(\rho_1^{\frac{\alpha}{2}} - \rho_2^{\frac{\alpha}{2}}) \nabla \cdot \big((\rho_1^{\frac{\alpha}{2}} - \rho_2^{\frac{\alpha}{2}}) g \big) \\
&= - \int 2\delta_0(\rho_1^\alpha - \rho_2^\alpha) |\nabla \rho_1^\alpha - \nabla \rho_2^\alpha|^2 - \int \nabla \cdot \big(|\rho_1^{\frac{\alpha}{2}} - \rho_2^{\frac{\alpha}{2}}| g \big) \\
&= - \int 2\delta_0(\rho_1^\alpha - \rho_2^\alpha) |\nabla \rho_1^\alpha - \nabla \rho_2^\alpha|^2 - 0 \\
&\leq 0.
\end{aligned}$$

We therefore have that

$$\max_{t \in [0,T]} \|\rho_1 - \rho_2\|_{L^1(\mathbb{T}^d)} = \|\rho_{1,0} - \rho_{2,0}\|_{L^1(\mathbb{T}^d)}.$$

A slightly less naive uniqueness proof: for $f^\delta = (f * \kappa^\delta)$, and ρ_1, ρ_2 solving

$$\partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g),$$

we have that

$$\partial_t \Big(\int |\rho_1 - \rho_2|^\delta \Big) = \int \mathrm{sgn}^\delta(\rho_1 - \rho_2) \Delta(\rho_1^\alpha - \rho_2^\alpha) - \int \mathrm{sgn}^\delta(\rho^1 - \rho^2) \nabla \cdot \big((\rho_1^{\frac{\alpha}{2}} - \rho_2^{\frac{\alpha}{2}}) g \big)$$
$$= -\int 2\delta_0^\delta(\rho_1 - \rho_2) \nabla(\rho_1 - \rho_2) \cdot \nabla(\rho_1^\alpha - \rho_2^\alpha) + \int 2\delta_0^\delta(\rho_1 - \rho_2)(\rho_1^{\frac{\alpha}{2}} - \rho_2^{\frac{\alpha}{2}}) \nabla(\rho_1 - \rho_2) \cdot g.$$

The nondegenerate case: if $\alpha = 1$, using Hölder's and Young's inequalities,

$$\partial_t \Big(\int |\rho_1 - \rho_2|^\delta \Big) + \int 2\delta_0^\delta(\rho_1 - \rho_2) |\nabla(\rho_1 - \rho_2)|^2$$
$$\leq \varepsilon \int \delta_0^\delta(\rho_1 - \rho_2) |\nabla(\rho_1 - \rho_2)|^2 + \frac{1}{\varepsilon} \int \delta_0^\delta(\rho_1 - \rho_2)(\sqrt{\rho_1} - \sqrt{\rho_2})^2 |g|^2.$$

Therefore, using that $\delta_0^\delta(\rho_1 - \rho_2) \lesssim \delta^{-1} \mathbf{1}_{\{|\rho_1 - \rho_2| < \delta\}}$,

$$\partial_t \Big(\int |\rho_1 - \rho_2|^\delta \Big) + \int \delta_0^\delta(\rho_1 - \rho_2) |\nabla(\rho_1 - \rho_2)|^2 \lesssim \int \mathbf{1}_{\{0 < |\rho_1 - \rho_2| < \delta\}} |g|^2.$$

I. The skeleton equation

Conservation of mass: solutions preserve mass if $\rho_0 \geq 0$,

$$\partial_t\Big(\int_{\mathbb{T}^d}\rho(x,t)\Big) = \int_{\mathbb{T}^d}\partial_t\rho = \int_{\mathbb{T}^d}\nabla\cdot\big(2\rho^{\frac{\alpha}{2}}(2\nabla\rho^{\frac{\alpha}{2}} - g)\big) = 0.$$

An a priori estimate: for an arbitrary nonlinearity Ψ with $\psi = \Psi'$,

$$\partial_t\Big(\int_{\mathbb{T}^d}\Psi(\rho)\Big) = -\alpha\int_{\mathbb{T}^d}\rho^{\alpha-1}\psi'(\rho)|\nabla\rho|^2 + \int_{\mathbb{T}^d}\rho^{\frac{\alpha}{2}}g\cdot\psi'(\rho)\nabla\rho.$$

Therefore, using Hölder's and Young's inequalities, for every $\varepsilon \in (0,1)$,

$$\partial_t\Big(\int_{\mathbb{T}^d}\Psi(\rho)\Big) + \alpha\int_{\mathbb{T}^d}\rho^{\alpha-1}\psi'(\rho)|\nabla\rho|^2 \leq \frac{1}{2\varepsilon}\int_{\mathbb{T}^d}|g|^2 + \frac{\varepsilon}{2}\int_{\mathbb{T}^d}\rho^{\alpha}\psi'(\rho)^2|\nabla\rho|^2.$$

To close the estimate,

$$\rho^{\alpha}\psi'(\rho)^2 \lesssim \psi'(\rho)\rho^{\alpha-1} \;\text{ so }\; \psi'(\xi) \lesssim \frac{1}{\xi}.$$

Entropy dissipation: if $\psi(\xi) = \log(\xi)$ then $\Psi(\xi) = \xi\log(\xi) - \xi$ and using

$$\rho^{\alpha-2}|\nabla\rho|^2 = |\rho^{\frac{\alpha-2}{2}}\nabla\rho|^2 = \frac{4}{\alpha^2}|\nabla\rho^{\frac{\alpha}{2}}|^2,$$

we have that (for nonnegative solutions!)

$$\max_{t\in[0,T]}\int_{\mathbb{T}^d}\rho\log(\rho) + \int_0^T\int_{\mathbb{T}^d}|\nabla\rho^{\frac{\alpha}{2}}|^2 \lesssim \int_{\mathbb{T}^d}\rho_0\log(\rho_0) + \int_0^T\int_{\mathbb{T}^d}|g|^2.$$

I. The skeleton equation

The equation: we have that

$$\partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g).$$

A local H^1-estimate: for $M \in (0, \infty)$, $\psi_M' = \mathbf{1}_{\{M<\xi<M+1\}}$, and $\Psi_M' = \psi_M$,

$$\max_{t\in[0,T]} \int_{\mathbb{T}^d} \Psi_M(\rho) + \int_0^T \int_{\mathbb{T}^d} \mathbf{1}_{\{M<\rho<M+1\}} \rho^{\alpha-1} |\nabla \rho|^2$$
$$\lesssim \int_{\mathbb{T}^d} \Psi_M(\rho_0) + \int_0^T \int_{\mathbb{T}^d} |g|^2 \rho^{\frac{1}{2}} \rho^{\frac{\alpha-1}{2}} \nabla \rho \mathbf{1}_{\{M<\rho<M+1\}},$$

Since $\Psi_M(\xi) \leq (\xi - M)_+$, we have from Hölder's and Young's inequalities that

$$\int_0^T \int_{\mathbb{T}^d} \mathbf{1}_{\{M<\rho<M+1\}} \rho^{\alpha-1} |\nabla \rho|^2 \lesssim \int_{\mathbb{T}^d} (\rho_0 - M)_+ + \int_0^T \int_{\mathbb{T}^d} \rho |g|^2 \rho \mathbf{1}_{\{M<\rho<M+1\}}$$
$$\lesssim \int_{\mathbb{T}^d} (\rho_0 - M)_+ + \int_0^T \int_{\mathbb{T}^d} (M+1) |g|^2 \mathbf{1}_{\{M<\rho<M+1\}}.$$

Local regularity: for every $K \in (1, \infty)$,

$$\left((\rho \wedge K) \vee K^{-1}\right) \in L_t^2 H_x^1.$$

I. The skeleton equation

The skeleton equation: for an $(L^2_{t,x})^d$-valued control g,

$$\partial_t \rho = \Delta \rho^{\alpha} - \nabla \cdot (\rho^{\frac{\alpha}{2}} g).$$

L^1**-contraction**: if ρ_1 and ρ_2 are solutions,

$$\max_{t\in[0,T]} \|\rho_1 - \rho_2\|_{L^1(\mathbb{T}^d)} = \|\rho_{1,0} - \rho_{2,0}\|_{L^1(\mathbb{T}^d)} .$$

Preservation of nonnegativity and mass: if $\rho_0 \geq 0$ then $\rho \geq 0$ with

$$\|\rho(x,t)\|_{L^1(\mathbb{T}^d)} = \|\rho_0\|_{L^1(\mathbb{T}^d)} .$$

The entropy estimate: if ρ_0 is nonnegative then

$$\max_{t\in[0,T]} \int_{\mathbb{T}^d} \rho \log(\rho) + \int_0^T \int_{\mathbb{T}^d} |\nabla \rho^{\frac{\alpha}{2}}|^2 \lesssim \int_{\mathbb{T}^d} \rho_0 \log(\rho_0) + \int_0^T \int_{\mathbb{T}^d} |g|^2.$$

The local H^1**-estimate**: for every $M \in (0,\infty)$,

$$\int_0^T \int_{\mathbb{T}^d} \mathbf{1}_{\{M<\rho<M+1\}} \rho^{\alpha-1} |\nabla \rho|^2 \lesssim \int_{\mathbb{T}^d} (\rho_0 - M)_+ + \int_0^T \int_{\mathbb{T}^d} (M+1)|g|^2 \mathbf{1}_{\{M<\rho<M+1\}}.$$

The entropy space: nonnegative functions with finite entropy

$$\mathrm{Ent}(\mathbb{T}^d) = \{\rho \in L^1(\mathbb{T}^d) \colon \rho \text{ is nonnegative and measurable with } \int_{\mathbb{T}^d} \rho \log(\rho) < \infty\}.$$

Heat equation: for the skeleton equation with $\alpha = 1$, $g = 0$, and $\rho_0 \in \mathrm{Ent}(\mathbb{T}^d)$,

$$\partial_t \rho = \Delta \rho - 0 \text{ in } \mathbb{T}^d \times [0, T] \text{ with } \rho(\cdot, 0) = \rho_0,$$

we have that

$$\max_{t \in [0,T]} \int_{\mathbb{T}^d} \rho \log(\rho) + \int_0^T \int_{\mathbb{T}^d} |\nabla \rho^{\frac{1}{2}}|^2 \lesssim \int_{\mathbb{T}^d} \rho_0 \log(\rho_0).$$

Time reversibility: if $\tilde{\rho}(x,t) = \rho(x, T-t)$ then, since $\nabla \tilde{\rho} = 2\tilde{\rho}^{\frac{1}{2}} \nabla \tilde{\rho}^{\frac{1}{2}}$,

$$\begin{aligned} \partial_t \tilde{\rho} &= -\Delta \tilde{\rho} = -2\nabla \cdot \left(\tilde{\rho}^{\frac{1}{2}} \nabla \tilde{\rho}^{\frac{1}{2}}\right) \\ &= 2\nabla \cdot \left(\tilde{\rho}^{\frac{1}{2}} \nabla \tilde{\rho}^{\frac{1}{2}} - 2\tilde{\rho}^{\frac{1}{2}} \nabla \tilde{\rho}^{\frac{1}{2}}\right) \\ &= \Delta \tilde{\rho} - \nabla \cdot (\tilde{\rho}^{\frac{1}{2}} \tilde{g}), \end{aligned}$$

for the L^2-valued control $\tilde{g}(x,t) = 4\nabla \tilde{\rho}^{\frac{1}{2}}(x,t) = 4\nabla \rho^{\frac{1}{2}}(x, T-t)$.

II. The kinetic formulation of the skeleton equation

A renormalized equation: for $\eta \in (0,1)$ we consider the regularized equation

$$\partial_t \rho_\eta = \Delta \rho_\eta^\alpha + \eta \Delta \rho_\eta - \nabla \cdot (\rho_\eta^{\frac{\alpha}{2}} g).$$

Then, for a smooth $S \colon \mathbb{R} \to \mathbb{R}$ and $\phi \in \mathrm{C}^\infty(\mathbb{T}^d)$,

$$\begin{aligned}
\partial_t \Big(\int_{\mathbb{T}^d} S(\rho_\eta)\phi(x) \Big) &= \int S'(\rho_\eta)\phi(x) \big(\Delta \rho_\eta^\alpha + \eta \Delta \rho_\eta - \nabla \cdot (\rho_\eta^{\frac{\alpha}{2}} g) \big) \\
&= - \int S'(\rho_\eta) \nabla \phi(x) \cdot (\alpha \rho_\eta^{\alpha-1} \nabla \rho_\eta + \eta \nabla \rho_\eta) \\
&\quad - \int S''(\rho_\eta)\phi(x) \big(\alpha \rho_\eta^{\alpha-1} |\nabla \rho_\eta|^2 + \eta |\nabla \rho_\eta|^2 \big) \\
&\quad + \int S''(\rho_\eta)\phi(x) \rho_\eta^{\frac{\alpha}{2}} g \cdot \nabla \rho_\eta + S'(\rho_\eta) \nabla \phi(x) \cdot (\rho_\eta^{\frac{\alpha}{2}} g).
\end{aligned}$$

The entropy formulation: if $S'' \geq 0$, $\phi \geq 0$, and $\rho_\eta \to \rho$ as $\eta \to 0$,

$$\begin{aligned}
\partial_t \Big(\int_{\mathbb{T}^d} S(\rho)\phi(x) \Big) \leq & - \int S'(\rho) \nabla \phi(x) \cdot (\alpha \rho^{\alpha-1} \nabla \rho) - \int S''(\rho)\phi(x) (\alpha \rho^{\alpha-1} |\nabla \rho|^2) \\
& + \int S''(\rho)\phi(x) \rho^{\frac{\alpha}{2}} g \cdot \nabla \rho + S'(\rho) \nabla \phi(x) \cdot (\rho^{\frac{\alpha}{2}} g).
\end{aligned}$$

II. The kinetic formulation of the skeleton equation

The kinetic function: for $\xi \in \mathbb{R}$ and $\chi_\eta(x,\xi,t) = \mathbf{1}_{\{0<\xi<\rho_\eta(x,t)\}} - \mathbf{1}_{\{\rho_\eta(x,t)<\xi<0\}}$,

$$\partial_\xi \chi_\eta = \delta_0 - \delta_{\rho_\eta} \text{ and } \nabla_x \chi_\eta = \delta_{\rho_\eta} \nabla \rho_\eta,$$

for $\delta_{\rho_\eta} = \delta_0(\xi - \rho_\eta(x,t))$.

A renormalized equation: since we have that

$$\begin{aligned}\partial_t\Big(\int_{\mathbb{T}^d} S(\rho_\eta)\phi(x)\Big) = & -\int S'(\rho_\eta)\nabla\phi(x)\cdot(\alpha\rho_\eta^{\alpha-1}\nabla\rho_\eta + \eta\nabla\rho_\eta) \\ & -\int S''(\rho_\eta)\phi(x)\big(\alpha\rho^{\alpha-1}|\nabla\rho_\eta|^2 + \eta|\nabla\rho_\eta|^2\big) \\ & +\int S''(\rho_\eta)\phi(x)\rho_\eta^{\frac{\alpha}{2}} g\cdot\nabla\rho_\eta + S'(\rho_\eta)\nabla\phi(x)\cdot(\rho_\eta^{\frac{\alpha}{2}} g),\end{aligned}$$

we have using the equality $\int_{\mathbb{R}} S'(\xi)\chi_\eta(x,\xi,t)\,\mathrm{d}\xi = S(\rho_\eta)$ that

$$\begin{aligned}\partial_t\Big(\int_{\mathbb{T}^d}\int_{\mathbb{R}} \chi_\eta S'(\xi)\phi(x)\Big) = & -\int\int \nabla_x(S'(\xi)\phi(x))\cdot(\alpha\xi^{\alpha-1}\nabla\chi_\eta + \eta\nabla\chi_\eta) \\ & -\int\int \partial_\xi(S'(\xi)\phi(x))\delta_{\rho_\eta}\big(\alpha\xi^{\alpha-1}|\nabla\rho_\eta|^2 + \eta|\nabla\rho_\eta|^2\big) \\ & +\int\int \partial_\xi(S'(\xi)\phi(x))\xi^{\frac{\alpha}{2}} g\cdot\nabla\chi_\eta - \nabla_x(S'(\rho)\phi(x))\cdot(\xi^{\frac{\alpha}{2}}\partial_\xi\chi_\eta g).\end{aligned}$$

— the test function $\psi(x,\xi) = S'(\xi)\phi(x)$

A renormalized equation: for the regularized skeleton equation

$$\partial_t \rho_\eta = \Delta \rho_\eta^\alpha + \eta \Delta \rho_\eta - \nabla \cdot (\rho_\eta^{\frac{\alpha}{2}} g),$$

we have for $\chi_\eta = \mathbf{1}_{\{0<\xi<\rho_\eta(x,t)\}}$, for every $\psi \in \mathrm{C}_c^\infty(\mathbb{T}^d \times (0,\infty))$,

$$\begin{aligned}\partial_t\Big(\int_{\mathbb{T}^d}\int_{\mathbb{R}} \chi_\eta \psi\Big) = &-\int\int \nabla\psi \cdot (\alpha \xi^{\alpha-1} \nabla \chi_\eta + \eta \nabla \chi_\eta) \\ &- \int\int \partial_\xi \psi \delta_{\rho_\eta}\left(\alpha \xi^{\alpha-1} |\nabla \rho_\eta|^2 + \eta |\nabla \rho_\eta|^2\right) \\ &+ \int\int \partial_\xi \psi \xi^{\frac{\alpha}{2}} g \cdot \nabla \chi_\eta - \nabla \psi \cdot (\xi^{\frac{\alpha}{2}} \partial_\xi \chi_\eta g),\end{aligned}$$

or, distributionally, for the measure $q_\eta = \delta_{\rho_\eta}\left(\alpha \xi^{\alpha-1} |\nabla \rho_\eta|^2 + \eta |\nabla \rho_\eta|^2\right)$,

$$\partial_t \chi_\eta = \alpha \xi^{\alpha-1} \Delta_x \chi_\eta + \eta \Delta_x \chi_\eta + \partial_\xi q_\eta - \partial_\xi(\xi^{\frac{\alpha}{2}} g \cdot \nabla_x \chi_\eta) + \nabla_x \cdot (\xi^{\frac{\alpha}{2}} \partial_\xi \chi_\eta g).$$

Using the distributional equalities $\nabla \chi_\eta = \delta_{\rho_\eta} \nabla \rho_\eta$ and $\partial_\xi \chi_\eta = \delta_0 - \delta_{\rho_\eta}$,

$$\begin{aligned}\partial_t\Big(\int_{\mathbb{T}^d}\int_{\mathbb{R}} \chi_\eta \psi\Big) = &-\int (\alpha \rho_\eta^{\alpha-1} + \eta)(\nabla \psi)(x,\rho_\eta) \cdot \nabla \rho_\eta \\ &- \int\int \partial_\xi \psi(x,\xi) q_\eta \\ &+ \int (\partial_\xi \psi)(x,\rho_\eta) \rho_\eta^{\frac{\alpha}{2}} g \cdot \nabla \rho_\eta + \rho_\eta^{\frac{\alpha}{2}} g \cdot (\nabla \psi)(x,\rho_\eta).\end{aligned}$$

The kinetic formulation of the skeleton equation: for the regularized equation

$$\partial_t \rho_\eta = \Delta \rho_\eta^\alpha + \eta \Delta \rho_\eta - \nabla \cdot (\rho_\eta^{\frac{\alpha}{2}} g),$$

for the defect measure $q_\eta = \delta_{\rho_\eta}(\alpha \xi^{\alpha-1} |\nabla \rho_\eta|^2 + \eta |\nabla \rho_\eta|^2)$, the kinetic formulation is

$$\partial_t \chi_\eta = \alpha \xi^{\alpha-1} \Delta \chi_\eta + \eta \Delta \chi_\eta + \partial_\xi q_\eta - \partial_\xi (\xi^{\frac{\alpha}{2}} g \cdot \nabla \chi_\eta) + \nabla \cdot (\xi^{\frac{\alpha}{2}} (\partial_\xi \chi_\eta) g).$$

If $\rho_\eta \to \rho$ as $\eta \to 0$ then,

$$q_\eta \rightharpoonup q \geq \delta_\rho (\alpha \xi^{\alpha-1} |\nabla \rho|^2),$$

and, for the kinetic function $\chi = \mathbf{1}_{\{0<\xi<\rho(x,t)\}}$ of ρ,

$$\partial_t \chi = \alpha \xi^{\alpha-1} \Delta \chi + \partial_\xi q - \partial_\xi (\xi^{\frac{\alpha}{2}} g \cdot \nabla \chi) + \nabla \cdot (\xi^{\frac{\alpha}{2}} (\partial_\xi \chi) g),$$

for a locally finite, nonnegative measure q on $\mathbb{T}^d \times \mathbb{R} \times [0, T]$ with

$$q \geq \delta_\rho (\alpha \xi^{\alpha-1} |\nabla \rho|^2).$$

— the kinetic formulation exactly quantifies this “entropy inequality”

— for example, [Perthame; 1998], [Chen, Perthame; 2003]

II. The kinetic formulation of the skeleton equation

The skeleton equation: for $g \in (L^2_{t,x})^d$ and $\alpha \in (0,\infty)$,

$$\partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g)$$

for $\chi = \mathbf{1}_{\{0<\xi<\rho(x,t)\}} - \mathbf{1}_{\{\rho(x,t)<\xi<0\}}$ the kinetic formulation is

$$\partial_t \chi = \alpha \xi^{\alpha-1} \Delta \chi + \partial_\xi q - \partial_\xi (\xi^{\frac{\alpha}{2}} g \cdot \nabla \chi) + \nabla \cdot (\xi^{\frac{\alpha}{2}} (\partial_\xi \chi) g),$$

for a locally finite, nonnegative measure $q \geq \delta_\rho(\alpha \xi^{\alpha-1} |\nabla \rho|^2)$.

The entropy estimate: for the test function $\psi(\xi) = \log(\xi)$,

$$\int_{\mathbb{T}^d} \int_{\mathbb{R}} \chi \log(\xi) \Big|_{s=0}^{s=T} = -\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}} \frac{1}{\xi} q + \int_0^T \rho^{\frac{\alpha}{2}-1} g \cdot \nabla \rho$$
$$= -\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}} \frac{1}{\xi} q + \frac{2}{\alpha} \int_0^T g \cdot \nabla \rho^{\frac{\alpha}{2}}.$$

Regularity from the measure: we have that

$$\frac{1}{\xi} q \geq \frac{1}{\xi} \cdot \delta_\rho \left(\xi^{\alpha-1} |\nabla \rho|^2 \right) = \rho^{\alpha-2} |\nabla \rho|^2 \simeq |\nabla \rho^{\frac{\alpha}{2}}|^2,$$

and that, by the preservation of mass and $\int_{\mathbb{R}} \chi \log(\xi) = \rho \log(\rho) - \rho$,

$$\max_{t\in[0,T]} \int_{\mathbb{T}^d} \rho \log(\rho) + \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}} \frac{1}{\xi} q \lesssim \int_{\mathbb{T}^d} \rho_0 \log(\rho_0) + \int_0^T \int_{\mathbb{T}^d} |g|^2.$$

II. The kinetic formulation of the skeleton equation

The kinetic formulation of the skeleton equation: for $q \geq \delta_\rho \alpha \xi^{\alpha-1}|\nabla\rho|^2$,

$$\partial_t \chi = \alpha\xi^{\alpha-1}\Delta\chi + \partial_\xi q - \partial_\xi(\xi^{\frac{\alpha}{2}} g \cdot \nabla\chi) + \nabla \cdot (\xi^{\frac{\alpha}{2}}(\partial_\xi \chi) g).$$

Preservation of nonnegativity and mass: if $\rho_0 \geq 0$ then $\rho \geq 0$ with

$$\|\rho(x,t)\|_{L^1(\mathbb{T}^d)} = \|\chi\|_{L^1(\mathbb{T}^d \times \mathbb{R})} = \|\rho_0\|_{L^1(\mathbb{T}^d)}.$$

The entropy estimate: if ρ_0 is nonnegative with finite entropy then

$$\begin{aligned}
&\max_{t\in[0,T]} \int_{\mathbb{T}^d} \rho\log(\rho) + \int_0^T \int_{\mathbb{T}^d} |\nabla \rho^{\frac{\alpha}{2}}|^2 \\
&\lesssim \max_{t\in[0,T]} \int_{\mathbb{T}^d} \rho\log(\rho) + \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}} \frac{1}{\xi} q \\
&\lesssim \int_{\mathbb{T}^d} \rho_0\log(\rho_0) + \int_0^T \int_{\mathbb{T}^d} |g|^2.
\end{aligned}$$

The local H^1-estimate: for every $M \in (0,\infty)$,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^d} \mathbf{1}_{\{M<\rho<M+1\}} |\nabla \rho^{\frac{\alpha+1}{2}}|^2 \\
&\lesssim \int_0^T \int_{\mathbb{T}^d} \int_M^{M+1} q \\
&\lesssim \int_{\mathbb{T}^d} (\rho_0 - M)_+ + (M+1)\int_0^T \int_{\mathbb{T}^d} \mathbf{1}_{\{M<\rho<M+1\}} |g|^2.
\end{aligned}$$

II. The kinetic formulation of the skeleton equation

The local H^1-estimate: if $\psi'(\xi) = \mathbf{1}_{\{M<\xi<M+1\}}$,

$$\int_0^T \int_{\mathbb{T}^d} \int_M^{M+1} q \lesssim \int_{\mathbb{T}^d} (\rho_0 - M)_+ + (M+1) \int_0^T \int_{\mathbb{T}^d} \mathbf{1}_{\{M<\rho<M+1\}} |g|^2.$$

A real analysis lemma: if a_k are nonnegative with $\sum_{k=1}^\infty a_k < \infty$ then

$$\liminf_{k\to\infty} k a_k = 0.$$

Initial data: if $\rho_0 \in L^1(\mathbb{T}^d)$ then $\lim_{M\to\infty} \int_{\mathbb{T}^d} (\rho_0 - M)_+ = 0$.

The control: if for $k \in \mathbb{N}$,

$$a_k = \int_0^T \int_{\mathbb{T}^d} \mathbf{1}_{\{k-1<\rho<k\}} |g|^2,$$

then $\sum_{k=1}^\infty a_k \le \int_0^T \int_{\mathbb{T}^d} |g|^2 < \infty$ and

$$\liminf_{k\to\infty} k a_k = \liminf_{M\to\infty} (M+1) \int_0^T \int_{\mathbb{T}^d} \mathbf{1}_{\{M<\rho<M+1\}} |g|^2 = 0.$$

Vanishing of the defect measure at infinity: from the local H^1-estimate,

$$\liminf_{M\to\infty} \int_0^T \int_{\mathbb{T}^d} \int_M^{M+1} q = 0.$$

II. The kinetic formulation of the skeleton equation

A renormalized kinetic solution of the skeleton equation [F., Gess; 2023]

Let $\rho_0 \in L^1(\mathbb{T}^d)$ be nonnegative and $g \in (L^2_{t,x})^d$. A renormalized kinetic solution of the skeleton equation is a nonnegative $\rho \in \mathrm{C}([0,T];L^1(\mathbb{T}^d))$ that satisfies:

— *Preservation of mass*: $\|\rho(x,t)\|_{L^1(\mathbb{T}^d)} = \|\rho_0\|_{L^1(\mathbb{T}^d)}$ for every $t \in [0,T]$.

— *Local H^1-regularity*: $\left((\rho \wedge K) \vee \frac{1}{K}\right) \in L^2([0,T];H^1(\mathbb{T}^d))$ for every $K \in \mathbb{N}$.

Furthermore, there exists a nonnegative, locally finite measure q on $\mathbb{T}^d \times \mathbb{R} \times [0,T]$ such that:

— *Regularity and vanishing of the measure at infinity*: we have that

$$\delta_\rho\left(\alpha\xi^{\alpha-1}|\nabla\rho|^2\right) \le q \quad \text{and} \quad \liminf_{M\to\infty} q(\mathbb{T}^d \times [M,M+1] \times [0,T]) = 0.$$

— *The equation*: for every $\psi \in \mathrm{C}^\infty_c(\mathbb{T}^d \times (0,\infty))$ and $t \in [0,T]$,

$$\begin{aligned}\int_{\mathbb{T}^d} \chi\psi\Big|_{s=0}^{s=t} = &-\int_0^t\int_{\mathbb{T}^d} \alpha\rho^{\alpha-1}\nabla\rho\cdot(\nabla\psi)(x,\rho) - \int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}(\partial_\xi\psi)(x,\xi)q \\ &+\int_0^t\int_{\mathbb{T}^d}(\partial_\xi\psi)(x,\rho)\rho^{\frac{\alpha}{2}}g\cdot\nabla\rho + \int_0^t\int_{\mathbb{T}^d}\rho^{\frac{\alpha}{2}}g\cdot(\nabla\psi)(x,\rho).\end{aligned}$$

— the equation is not enforced on the set $\{\rho = 0\}$! Why are solutions unique?

Vanishing of the defect measure: for the equation

$$\partial_t \chi = \alpha \xi^{\alpha-1}\Delta\chi + \partial_\xi q - \partial_\xi(\xi^{\frac{\alpha}{2}} g\cdot\nabla\chi) + \nabla\cdot(\xi^{\frac{\alpha}{2}}(\partial_\xi\chi)g),$$

for the test functions $\psi'_\beta = \frac{2}{\beta}\mathbf{1}_{\{\frac{\beta}{2}<\xi<\beta\}}$ and $\zeta'_M = -\mathbf{1}_{\{M<\xi<M+1\}}$,

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi\psi_\beta\zeta_M\Big|_{s=0}^{s=t} = -\frac{2}{\beta}\int_0^t\int_{\mathbb{T}^d}\int_{\frac{\beta}{2}}^{\beta} q + \int_0^t\int_{\mathbb{T}^d}\int_M^{M+1} q$$

$$+\frac{2}{\beta}\int_0^t\int_{\mathbb{T}^d}\rho^{\frac{\alpha}{2}} g\cdot\nabla\rho\mathbf{1}_{\{\frac{\beta}{2}<\rho<\beta\}} + \int_0^t\int_{\mathbb{T}^d}\rho^{\frac{\alpha}{2}} g\cdot\nabla\rho\mathbf{1}_{\{M<\rho<M+1\}}.$$

We have using $\rho^{\frac{\alpha}{2}}\nabla\rho = \rho^{\frac{1}{2}}\cdot\rho^{\frac{\alpha-1}{2}}\nabla\rho$ and Hölder's and Young's inequalities that

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi\psi_\beta\zeta_M\Big|_{s=0}^{s=t} + \frac{2}{\beta}\int_0^t\int_{\mathbb{T}^d}\int_{\frac{\beta}{2}}^{\beta} q$$

$$\lesssim \int_0^t\int_{\mathbb{T}^d}\int_M^{M+1} q + \frac{1}{\beta}\int_0^t\int_{\mathbb{T}^d}\rho|g|^2\mathbf{1}_{\{\frac{\beta}{2}<\rho<\beta\}} + \int_0^t\int_{\mathbb{T}^d}\rho|g|^2\mathbf{1}_{\{M<\rho<M+1\}}.$$

The righthand side vanishes as $M\to\infty$ and $\beta\to 0$. Therefore,

$$\text{“}\int_0^t\int_{\mathbb{T}^d} q(x,0,s)\text{”} = \lim_{\beta\to 0}\frac{2}{\beta}\int_0^t\int_{\mathbb{T}^d}\int_{\frac{\beta}{2}}^{\beta} q = \int_{\mathbb{T}^d}\big(\rho_0(x)-\rho(x,t)\big) = 0.$$

A renormalized kinetic solution of the skeleton equation [F., Gess; 2023]

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— *Preservation of mass*: $\|\rho(x,t)\|_{L^1(\mathbb{T}^d)} = \|\rho_0\|_{L^1(\mathbb{T}^d)}$ for every $t \in [0,T]$.

— *Local H^1-regularity*: $\left((\rho \wedge K) \vee \frac{1}{K}\right) \in L^2([0,T]; H^1(\mathbb{T}^d))$ for every $K \in \mathbb{N}$.

Furthermore, there exists a nonnegative, locally finite measure q on $\mathbb{T}^d \times \mathbb{R} \times [0,T]$ such that:

— *Regularity and vanishing of the measure at infinity*: we have that

$$\delta_\rho\left(\alpha\xi^{\alpha-1}|\nabla\rho|^2\right) \leq q \;\text{ and }\; \liminf_{M\to\infty} q(\mathbb{T}^d \times [M, M+1] \times [0,T]) = 0.$$

— *The equation*: for every $\psi \in \mathrm{C}^\infty_c(\mathbb{T}^d \times (0,\infty))$ and $t \in [0,T]$,

$$\begin{aligned}\int_{\mathbb{T}^d} \chi\psi\Big|_{s=0}^{s=t} = &-\int_0^t \int_{\mathbb{T}^d} \alpha\rho^{\alpha-1}\nabla\rho \cdot (\nabla\psi)(x,\rho) - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}} (\partial_\xi\psi)(x,\xi) q \\ &+ \int_0^t \int_{\mathbb{T}^d} (\partial_\xi\psi)(x,\rho)\rho^{\frac{\alpha}{2}} g \cdot \nabla\rho + \int_0^t \int_{\mathbb{T}^d} \rho^{\frac{\alpha}{2}} g \cdot (\nabla\psi)(x,\rho).\end{aligned}$$

— we have that $\lim_{\beta\to 0}\left(\beta^{-1} q\left(\mathbb{T}^d \times (\frac{\beta}{2}, \beta) \times [0,T]\right)\right) = 0$.

II. The kinetic formulation of the skeleton equation

A useful identity: if ρ_1 and ρ_2 are nonnegative kinetic solutions, for

$$\chi_i(x,\xi,t) = \mathbf{1}_{\{0<\xi<\rho_i(x,t)\}} - \mathbf{1}_{\{\rho_i(x,t)<\xi<0\}},$$

we have

$$\begin{aligned}\int_{\mathbb{T}^d} |\rho_1 - \rho_2| = \int_{\mathbb{T}^d}\int_{\mathbb{R}} |\chi_1 - \chi_2|^2 &= \int_{\mathbb{T}^d}\int_{\mathbb{R}} (\chi_1)^2 + (\chi_2)^2 - 2\chi_1\chi_2 \\ &= \int_{\mathbb{T}^d}\int_{\mathbb{R}} \chi_1 \mathrm{sgn}(\xi) + \chi_2 \mathrm{sgn}(\xi) - 2\chi_1\chi_2 \\ &= \int_{\mathbb{T}^d}\int_{\mathbb{R}} \chi_1 + \chi_2 - 2\chi_1\chi_2.\end{aligned}$$

The cutoff functions: the cutoff at zero, for $\beta \in (0,1)$,

$$\psi_\beta(0) = 0 \ \text{ and } \ \psi'_\beta = \frac{2}{\beta}\mathbf{1}_{\{\frac{\beta}{2}<\xi<\beta\}},$$

and the cutoff at infinity, for $M \in (1,\infty)$,

$$\zeta_M(0) = 1 \ \text{ and } \ \zeta'_M = -\mathbf{1}_{\{M<\xi<M+1\}}.$$

The essential identity: we will use that

$$\int_{\mathbb{T}^d} |\rho_1 - \rho_2| = \lim_{\beta\to 0}\lim_{M\to\infty}\left(\int_{\mathbb{T}^d}\int_{\mathbb{R}} (\chi_1 + \chi_2 - 2\chi_1\chi_2)\psi_\beta\zeta_M\right).$$

The equation: we have that

$$\partial_t \chi = \alpha \xi^{\alpha-1} \Delta \chi + \partial_\xi q - \partial_\xi(\xi^{\frac{\alpha}{2}} g \cdot \nabla \chi) + \nabla \cdot (\xi^{\frac{\alpha}{2}} (\partial_\xi \chi) g),$$

and we use that

$$\int_{\mathbb{T}^d} |\rho_1 - \rho_2| = \lim_{\beta \to 0} \lim_{M \to \infty} \Big(\int_{\mathbb{T}^d} \int_{\mathbb{R}} \big(\chi_1 + \chi_2 - 2\chi_1\chi_2\big) \psi_\beta \zeta_M \Big).$$

The singletons: we have that

$$\begin{aligned}
&\partial_t \Big(\int_{\mathbb{T}^d} \int_{\mathbb{R}} \chi_i \psi_\beta \zeta_M \Big) = -\int_{\mathbb{T}^d} \int_{\mathbb{R}} q_i \partial_\xi (\psi_\beta \zeta_M) + \int_{\mathbb{T}^d} \big(\partial_\xi (\psi_\beta \zeta_M)\big)(\rho_i) \rho_i^{\frac{\alpha}{2}} g \cdot \nabla \rho_i \\
&= -\frac{2}{\beta} q_i \Big(\mathbb{T}^d \times (\frac{\beta}{2}, \beta) \times (0,t)\Big) + q_i \Big(\mathbb{T}^d \times (M, M+1) \times (0,t)\Big) \\
&\quad + \frac{2}{\beta} \int_{\mathbb{T}^d} \mathbf{1}_{\{\frac{\beta}{2} < \rho_i < \beta\}} \zeta_M(\rho_i) \rho_i^{\frac{1}{2}} g \cdot \rho_i^{\frac{\alpha-1}{2}} \nabla \rho_i + \int_{\mathbb{T}^d} \mathbf{1}_{\{M < \rho_i < M+1\}} \psi_\beta(\rho_i) \rho_i^{\frac{1}{2}} g \cdot \rho_i^{\frac{\alpha-1}{2}} \nabla \rho_i \\
&\lesssim \int_0^t \int_{\mathbb{T}^d} \int_M^{M+1} q + \frac{1}{\beta} \int_0^t \int_{\mathbb{T}^d} \rho |g|^2 \mathbf{1}_{\{\frac{\beta}{2} < \rho < \beta\}} + \int_0^t \int_{\mathbb{T}^d} \rho |g|^2 \mathbf{1}_{\{M < \rho < M+1\}}.
\end{aligned}$$

These terms vanish in the limit $M \to \infty$ and $\beta \to 0$.

The mixed term: we have that

$$\partial_t \chi = \alpha \xi^{\alpha-1} \Delta \chi + \partial_\xi q - \partial_\xi (\xi^{\frac{\alpha}{2}} g \cdot \nabla \chi) + \nabla \cdot (\xi^{\frac{\alpha}{2}} (\partial_\xi \chi) g),$$

and, therefore,

$$\begin{aligned}
&\partial_t \Big(\int_{\mathbb{T}^d} \int_{\mathbb{R}} \chi_1 \chi_2 \psi_\beta \zeta_M \Big) \\
&= - \int_{\mathbb{T}^d} \int_{\mathbb{R}} q_1 (\partial_\xi \chi_2) \psi_\beta \zeta_M - \int_{\mathbb{T}^d} \int_{\mathbb{R}} q_2 (\partial_\xi \chi_1) \psi_\beta \zeta_M - 2 \int_{\mathbb{T}^d} \int_{\mathbb{R}} \alpha \xi^{\alpha-1} \nabla \chi_1 \cdot \nabla \chi_2 \\
&\quad + \int_{\mathbb{T}^d} \int_{\mathbb{R}} \xi^{\frac{\alpha}{2}} \partial_\xi \chi_2 g \cdot \nabla \chi_1 \psi_\beta \zeta_M + \int_{\mathbb{T}^d} \int_{\mathbb{R}} \xi^{\frac{\alpha}{2}} \partial_\xi \chi_1 g \cdot \nabla \chi_2 \psi_\beta \zeta_M \\
&\quad - \int_{\mathbb{T}^d} \int_{\mathbb{R}} \xi^{\frac{\alpha}{2}} \partial_\xi \chi_1 g \cdot \nabla \chi_2 \psi_\beta \zeta_M - \int_{\mathbb{T}^d} \int_{\mathbb{R}} \xi^{\frac{\alpha}{2}} \partial_\xi \chi_2 g \cdot \nabla \chi_1 \psi_\beta \zeta_M \\
&\quad + \int_{\mathbb{T}^d} \int_{\mathbb{R}} \partial_\xi (\psi_\beta \zeta_M) \chi_2 \xi^{\frac{\alpha}{2}} g \cdot \nabla \chi_1 + \int_{\mathbb{T}^d} \int_{\mathbb{R}} \partial_\xi (\psi_\beta \zeta_M) \chi_1 \xi^{\frac{\alpha}{2}} g \cdot \nabla \chi_2 \\
&\quad - \int_{\mathbb{T}^d} \int_{\mathbb{R}} q_1 \chi_2 \partial_\xi (\psi_\beta \zeta_M) - \int_{\mathbb{T}^d} \int_{\mathbb{R}} q_2 \chi_1 \partial_\xi (\psi_\beta \zeta_M).
\end{aligned}$$

In comparison to the skeleton equation

$$\partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g) + \text{“cutoff error”}.$$

The dissipative error: for $\partial_t\rho = \Delta\rho^\alpha - \nabla\cdot(\rho^{\frac{\alpha}{2}}g) +$ "cutoff error",

$$\begin{aligned}
&\partial_t\Big(\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi_1\chi_2\psi_\beta\zeta_M\Big)\\
&= -\int_{\mathbb{T}^d}\int_{\mathbb{R}} q_1(\partial_\xi\chi_2)\psi_\beta\zeta_M - \int_{\mathbb{T}^d}\int_{\mathbb{R}} q_2(\partial_\xi\chi_1)\psi_\beta\zeta_M - 2\int_{\mathbb{T}^d}\int_{\mathbb{R}}\alpha\xi^{\alpha-1}\nabla\chi_1\cdot\nabla\chi_2\psi_\beta\zeta_M\\
&= \int_{\mathbb{T}^d}\int_{\mathbb{R}}\delta_{\rho_2}q_1\psi_\beta\zeta_M + \int_{\mathbb{T}^d}\int_{\mathbb{R}}\delta_{\rho_1}q_2\psi_\beta\zeta_M - 2\int_{\mathbb{T}^d}\int_{\mathbb{R}}\alpha\xi^{\alpha-1}\nabla\rho_1\cdot\nabla\rho_2\delta_{\rho_1}\delta_{\rho_2}\psi_\beta\zeta_M\\
&\geq \int_{\mathbb{T}^d}\int_{\mathbb{R}}\delta_{\rho_1}\delta_{\rho_2}\alpha\xi^{\alpha-1}\big(|\nabla\rho_1|^2+|\nabla\rho_2|^2\big)\psi_\beta\zeta_M - 2\int_{\mathbb{T}^d}\int_{\mathbb{R}}\alpha\xi^{\alpha-1}\nabla\rho^1\cdot\nabla\rho^2\delta_{\rho_1}\delta_{\rho_2}\psi_\beta\zeta_M\\
&\geq 0.
\end{aligned}$$

Local regularity: after regularizing $\chi_i^\delta = (\chi * \kappa^\delta)$, for $\overline{\kappa}_i^\delta = \kappa^\delta(\rho_i - \xi)$,

$$\begin{aligned}
2\int_{\mathbb{T}^d}\int_{\mathbb{R}}\alpha\xi^{\alpha-1}\nabla\chi_1^\delta\cdot\nabla\chi_2^\delta\psi_\beta\zeta_M &= \int_{\mathbb{T}^d}\int_{\mathbb{R}}(\alpha\rho_1^{\alpha-1}+\alpha\rho_2^{\alpha-1})\nabla\rho_1\cdot\nabla\rho_2\delta_{\rho_1}^\delta\delta_{\rho_2}^\delta\psi_\beta\zeta_M\\
&= \int_{\mathbb{T}^d}\int_{\mathbb{R}}\alpha\Big(\rho_1^{\frac{\alpha-1}{2}}-\rho_2^{\frac{\alpha-1}{2}}\Big)^2\nabla\rho_1\cdot\nabla\rho_2\overline{\kappa}_1^\delta\overline{\kappa}_2^\delta\psi_\beta\zeta_M\\
&\quad + 2\int_{\mathbb{T}^d}\int_{\mathbb{R}}\alpha\rho_1^{\frac{\alpha-1}{2}}\rho_2^{\frac{\alpha-1}{2}}\nabla\rho_1\cdot\nabla\rho_2\overline{\kappa}_1^\delta\overline{\kappa}_2^\delta\psi_\beta\zeta_M.
\end{aligned}$$

The conservative error: for $\partial_t\rho = \Delta\rho^{\alpha} - \nabla\cdot(\rho^{\frac{\alpha}{2}}g) +$ "cutoff error",

$$\partial_t\Big(\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi_1\chi_2\psi_\beta\zeta_M\Big) \geq \int_{\mathbb{T}^d}\int_{\mathbb{R}}\xi^{\frac{\alpha}{2}}\partial_\xi\chi_2 g\cdot\nabla\chi_1\psi_\beta\zeta_M + \int_{\mathbb{T}^d}\int_{\mathbb{R}}\xi^{\frac{\alpha}{2}}\partial_\xi\chi_1 g\cdot\nabla\chi_2\psi_\beta\zeta_M$$
$$- \int_{\mathbb{T}^d}\int_{\mathbb{R}}\xi^{\frac{\alpha}{2}}\partial_\xi\chi_1 g\cdot\nabla\chi_2\psi_\beta\zeta_M - \int_{\mathbb{T}^d}\int_{\mathbb{R}}\xi^{\frac{\alpha}{2}}\partial_\xi\chi_2 g\cdot\nabla\chi_1\psi_\beta\zeta_M$$

Local regularity of $\xi^{\frac{\alpha}{2}}$: after regularizing $\chi_i^\delta = (\chi * \kappa^\delta)$, for $\overline{\kappa}_i^\delta = \kappa^\delta(\rho_i - \xi)$,

$$\partial_\xi\chi_i^\delta(x,\xi,t) = (\partial_\xi\chi * \kappa^\delta)(x,\xi,t) = \kappa^\delta(\xi) - \kappa^\delta(\rho_i - \xi),$$

and

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}}\xi^{\frac{\alpha}{2}}\partial_\xi\chi_2 g\cdot\nabla\chi_1\psi_\beta\zeta_M + \int_{\mathbb{T}^d}\int_{\mathbb{R}}\xi^{\frac{\alpha}{2}}\partial_\xi\chi_1 g\cdot\nabla\chi_2\psi_\beta\zeta_M$$
$$- \int_{\mathbb{T}^d}\int_{\mathbb{R}}\xi^{\frac{\alpha}{2}}\partial_\xi\chi_1 g\cdot\nabla\chi_2\psi_\beta\zeta_M - \int_{\mathbb{T}^d}\int_{\mathbb{R}}\xi^{\frac{\alpha}{2}}\partial_\xi\chi_2 g\cdot\nabla\chi_1\psi_\beta\zeta_M$$
$$= \int_{\mathbb{T}^d}\int_{\mathbb{R}}(\rho_2^{\frac{\alpha}{2}} - \rho_1^{\frac{\alpha}{2}})g\cdot\nabla\rho_1\overline{\kappa}_1^\delta\overline{\kappa}_2^\delta\psi_\beta\zeta_M + \int_{\mathbb{T}^d}\int_{\mathbb{R}}(\rho_1^{\frac{\alpha}{2}} - \rho_2^{\frac{\alpha}{2}})g\cdot\nabla\rho_2\overline{\kappa}_1^\delta\overline{\kappa}_2^\delta\psi_\beta\zeta_M$$
$$\simeq \int_{\mathbb{T}^d}\int_{\mathbb{R}}\mathbf{1}_{\{0<|\rho_1-\rho_2|<\delta\}}|g|\big(|\nabla\rho_1| + |\nabla\rho_2|\big)\psi_\beta(\rho_1)\zeta_M(\rho_1).$$

The cutoff error: for $\partial_t\rho = \Delta\rho^\alpha - \nabla\cdot(\rho^{\frac{\alpha}{2}}g) +$ "cutoff error",

$$\partial_t\Big(\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi_1\chi_2\psi_\beta\zeta_M\Big) \geq \dots$$
$$+\int_{\mathbb{T}^d}\int_{\mathbb{R}}\partial_\xi(\psi_\beta\zeta_M)\chi_2\xi^{\frac{\alpha}{2}}g\cdot\nabla\chi_1 + \int_{\mathbb{T}^d}\int_{\mathbb{R}}\partial_\xi(\psi_\beta\zeta_M)\chi_1\xi^{\frac{\alpha}{2}}g\cdot\nabla\chi_2$$
$$-\int_{\mathbb{T}^d}\int_{\mathbb{R}}q_1\chi_2\partial_\xi(\psi_\beta\zeta_M) - \int_{\mathbb{T}^d}\int_{\mathbb{R}}q_2\chi_1\partial_\xi(\psi_\beta\zeta_M).$$

We have that

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}}\partial_\xi(\psi_\beta\zeta_M)\chi_2\xi^{\frac{\alpha}{2}}g\cdot\nabla\chi_1 = \int_{\mathbb{T}^d}\rho_1^{\frac{\alpha}{2}}g\cdot\nabla\rho_1\chi_2(x,\rho_1,t)\partial_\xi(\psi_\beta\zeta_M)(\rho_1)$$
$$\lesssim \int_{\mathbb{T}^d}\rho_1^{\frac{1}{2}}g\cdot\rho_1^{\frac{\alpha-1}{2}}\nabla\rho_1\Big(\frac{2}{\beta}(\mathbf{1}_{\{\frac{\beta}{2}<\rho_1<\beta\}} + \mathbf{1}_{\{M<\rho_1<M+1\}}\Big)$$
$$\lesssim \int_{\mathbb{T}^d}\rho_1|g|^2\Big(\frac{2}{\beta}(\mathbf{1}_{\{\frac{\beta}{2}<\rho_1<\beta\}} + \mathbf{1}_{\{M<\rho_1<M+1\}}\Big) + \int_{\mathbb{T}^d}\int_{\frac{\beta}{2}}^{\beta}\frac{1}{\xi}q_1 + \int_{\mathbb{T}^d}\int_M^{M+1}q_1.$$

Conclusion: we have that $\partial_t\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi_1\chi_2 \geq 0$ and, therefore,

$$\partial_t\int_{\mathbb{T}^d}|\rho_1-\rho_2| = \partial_t\int_{\mathbb{T}^d}\int_{\mathbb{R}}\big(\chi_1+\chi_2-2\chi_1\chi_2\big) \leq 0.$$

II. The kinetic formulation of the skeleton equation

Well-posedness of renormalized kinetic solutions [F., Gess; 2023]

Let $T \in (0,\infty)$, $d \in \mathbb{N}$, and let $\Phi \in \mathrm{C}^1_{\mathrm{loc}}((0,\infty)) \cap \mathrm{C}([0,\infty))$ satisfy that

- $\Phi(0) = 0$ with $\Phi' > 0$ on $(0,\infty)$,
- Φ' is locally $1/2$-Hölder continuous on $(0,\infty)$,
- and $\max_{\{0<\xi\leq M\}} \frac{\Phi(\xi)}{\Phi'(\xi)} \leq cM$.

Then for every nonnegative $\rho_0 \in L^1(\mathbb{T}^d)$ and $g \in (L^2_{t,x})^d$ there exists a unique renormalized kinetic solution of the equation

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla\cdot(\Phi^{\frac{1}{2}}(\rho)g) \ \text{ in } \ \mathbb{T}^d \times (0,T) \ \text{ with } \ \rho(\cdot,0) = \rho_0.$$

Furthermore, if ρ_1 and ρ_2 are two solutions with initial data $\rho_{1,0}$ and $\rho_{2,0}$, then

$$\max_{t\in[0,T]} \|\rho_1 - \rho_2\|_{L^1(\mathbb{T}^d)} = \|\rho_{1,0} - \rho_{2,0}\|_{L^1(\mathbb{T}^d)}.$$

- including $\Phi(\xi) = \xi^\alpha$ for every $\alpha \in (0,\infty)$, for which

$$\partial_t \rho = \Delta\rho^\alpha - \nabla\cdot(\rho^{\frac{\alpha}{2}} g).$$

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