Stochastic PDEs of fluctuating hydrodynamics type

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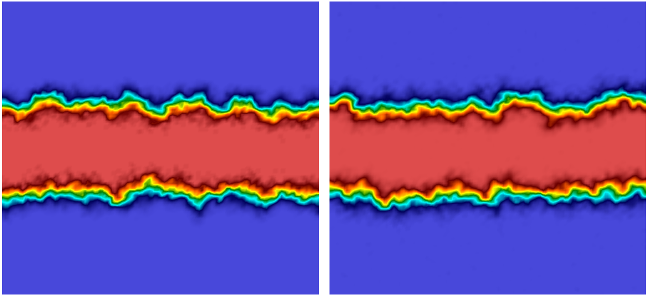
I. Fluctuating hydrodynamics



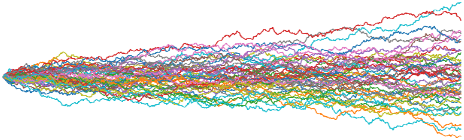
thermal fluctuations create a rough diffusive interface in miscible fluids [Donev; 2018]

- A general conservative stochastic PDE

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\sigma(\rho)\,\mathrm{d}\xi) = \nabla \cdot (\Phi'(\rho)\nabla\rho - \sigma(\rho)\,\mathrm{d}\xi),$$

  for a d-dimensional space-time noise ξ.
  — fluctuating hydrodynamics, [Spohn; 1991]
  — macroscopic fluctuation theory, [Bertini et al.; 2014]

Brownian motion: continuous paths with independent, Gaussian increments



Itô integration: we have that, strongly in $L^2(\Omega)$,

$$\int_0^t f(B_s)\,\mathrm{d}B_s = \lim_{|\mathcal{P}|\to 0} \sum f(B_{t_i})(B_{t_{i+1}} - B_{t_i}).$$

Alternate integration theories: for $\theta \in (0,1)$,

$$\begin{aligned}\int_0^t f(B_s)\circ_\theta\ \mathrm{d}B_s &= \lim_{|\mathcal{P}|\to 0} \sum \big(\theta f(B_{t_{i+1}}) + (1-\theta) f(B_{t_i})\big)(B_{t_{i+1}} - B_{t_i}) \\ &\simeq \lim_{|\mathcal{P}|\to 0} \sum \theta f'(B_{t_i})(B_{t_{i+1}} - B_{t_i})^2 + \lim_{|\mathcal{P}|\to 0} \sum f(B_{t_i})\big)(B_{t_{i+1}} - B_{t_i}) \\ &= \theta \int_0^t f'(B_s)\,\mathrm{d}s + \int_0^t f(B_s)\,\mathrm{d}B_s,\end{aligned}$$

where $\theta = 0$ is Itô, $\theta = {}^1\!/_2$ is Stratonovich, and $\theta = 1$ is Klimontovich.

Itô's formula: for a Brownian motion B_t,

$$f(B_{t_{i+1}}) - f(B_{t_i}) \simeq f'(B_{t_i})(B_{t_{i+1}} - B_{t_i}) + \frac{1}{2} f''(B_{t_i})(B_{t_{i+1}} - B_{t_i})^2.$$

After passing to the limit $\mathcal{P} \to 0$, for $\circ = \circ_{\frac{1}{2}}$,

$$f(B_t) - f(B_0) = \int_0^t f'(B_s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(B_s)\,\mathrm{d}s = \int_0^t f'(B_s) \circ \mathrm{d}B_s,$$

or, in differential form,

$$\mathrm{d}f(B_s) = f'(B_s)\,\mathrm{d}B_s + \frac{1}{2} f''(B_s)\,\mathrm{d}s = f'(B_s) \circ \mathrm{d}B_s.$$

— Stratonovich integration satisfies the chain rule

I. Fluctuating hydrodynamics

The empirical density: let m_n denote the measure

$$m_n = \frac{1}{n}\sum_{k=1}^n \delta_{B_t^k}.$$

The derivation: for every $f \in \mathrm{C}^\infty(\mathbb{T}^d)$,

$$\partial_t\Big(\int_{\mathbb{T}^d} f(x)m_n\Big) = \partial_t\Big(\frac{1}{n}\sum_{k=1}^n f(B_t^k)\Big) = \frac{1}{n}\sum_{k=1}^n \frac{1}{2}\Delta f(B_t^k) + \frac{1}{n}\sum_{k=1}^n \int_{\mathbb{T}^d} \nabla f(x)\delta_{B^k}\,\mathrm{d}B_t^k.$$

The covariance: for points (x,t) and (y,s),

$$\mathbb{E}\Big[\frac{1}{n}\sum_{k=1}^n \delta_0(x-B_t^k)\,\mathrm{d}B_t^k \cdot \frac{1}{n}\sum_{j=1}^n \delta_0(y-B_s^j)\,\mathrm{d}B_s^j\Big] = \frac{1}{n^2}\delta_0(x-B_t^k)\delta_0(x-y)\delta_0(s-t)$$

$$= \frac{1}{n}m_n(x,t)\delta_0(x-y)\delta_0(s-t).$$

The Dean–Kawasaki equation: for $\mathrm{d}\xi$ an $\mathbb{R}^d$-valued space-time white noise,

$$\partial_t\Big(\int_{\mathbb{T}^d} f(x)m_n\Big) = \frac{1}{2}\int_{\mathbb{T}^d}\Delta f m_n + \frac{1}{\sqrt{n}}\sum_{k=1}^n \int_{\mathbb{T}^d} \nabla f(x)\cdot\sqrt{m_n}\,\mathrm{d}\xi.$$

Or, in the sense of distributions,

$$\partial_t m_n = \frac{1}{2}\Delta m_n - \frac{1}{\sqrt{n}}\nabla\cdot(\sqrt{m_n}\,\mathrm{d}\xi).$$

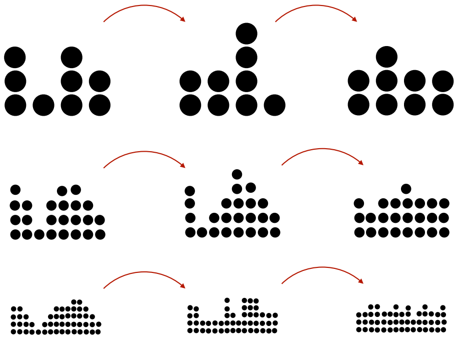
I. Fluctuating hydrodynamics

A zero range process: on $\mathbb{T}^d_N = (\mathbb{Z}^d / N\mathbb{Z}^d)$ with generator

$$(\mathcal{L}_N f)(\eta) = \sum_{x \in \mathbb{T}^d_N} \sum_{z \in \mathbb{T}^d_N} p_N(z) g(\eta(x)) \big(f(\eta^{x,x+z}) - f(\eta) \big),$$

The transition kernel: $p_N(z) = \sum_{y \in \mathbb{Z}^d} p(z + y_N)$ for a compactly supported p with zero mean $\sum_{z \in \mathbb{T}^d} z p(z) = 0$.

Parabolic rescalings: for $N = 4, 8, 15$,



I. Fluctuating hydrodynamics

Hydrodynamic limit: for a parabolically rescaled, centered particle process, as the particle number $N \to \infty$ [Ferrari et al.; 1988],

$$\mu_t^N \rightharpoonup \overline{\rho}\,\mathrm{d}x \;\text{ for }\; \partial_t \overline{\rho} = \Delta\Phi(\overline{\rho}).$$

Sensitivity to fluctuations: if the jumps are biased in direction $\gamma \in \mathbb{R}^d$, the hyperbolically rescaled densities ν_t^N satisfy, as $N \to \infty$,

$$\nu_t^N \rightharpoonup \overline{v}\,\mathrm{d}x \;\text{ for }\; \partial_t \overline{v} = \nabla \cdot (m(\overline{v})\gamma),$$

for the mobility $m(\overline{v})$ [Rezakhonlou; 1991].

The SPDE: the following formal SPDE, for a finite particle number N,

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot \alpha \;\text{ for }\; \mathbb{E}\big[\alpha(x,t)\alpha(y,s)\big] = \frac{m(\rho)}{N^d}\delta_0(x-y)\delta_0(t-s).$$

Formally, for a space time white noise $\mathrm{d}\xi$,

$$\partial_t \rho = \Delta\Phi(\rho) - N^{-\frac{d}{2}}\nabla \cdot (\sqrt{m(\rho)}\,\mathrm{d}\xi).$$

Examples: for the zero range and symmetric simple exclusion processes

$$\partial_t \rho = \Delta\Phi(\rho) - N^{-\frac{d}{2}}\nabla \cdot (\Phi^{\frac{1}{2}}(\rho)\,\mathrm{d}\xi) \;\text{ and }\; \partial_t \rho = \Delta\rho - N^{-\frac{d}{2}}\nabla \cdot (\sqrt{\rho(1-\rho)}\,\mathrm{d}\xi).$$

I. Fluctuating hydrodynamics

The zero range process: for the zero range process with finite particle number N,

$$\partial_t \rho = \Delta\Phi(\rho) - N^{-\frac{d}{2}} \nabla \cdot (\Phi^{\frac{1}{2}}(\rho)\,\mathrm{d}\xi).$$

The Dean–Kawasaki equation: in the case of independent particles,

$$\partial_t \rho = \Delta\rho - N^{-\frac{d}{2}} \nabla \cdot (\sqrt{\rho}\,\mathrm{d}\xi).$$

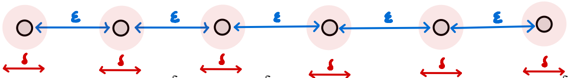
— supercritical in the language of singular SPDE

— formally derived by [Dean; 1996] and [Kawasaki; 1998] as the SPDE satisfied by

$$m_N(x,t) = \frac{1}{N^d} \sum_{i=1}^{N^d} \delta_0(B_t^i - x).$$

— ill-posedness vs. triviality [Konarovskyi, Lehmann, von Renesse; 2018]

- White noise is too singular (particles systems, course graining):



- Spatially correlated noise: $\xi^\delta = \xi * \kappa^\delta$ for convolution kernels $\{\kappa^\delta\}_{\delta\in(0,1)}$.

The Dean–Kawasaki equation: with a spatially correlated noise ξ^δ, for ε the inverse particle number,

$$\begin{aligned}\partial_t \rho &= \Delta\rho - \varepsilon^{\frac{d}{2}} \nabla \cdot (\sqrt{\rho}\, \mathrm{d}\xi^\delta) \\ &= \Delta\rho - \varepsilon^{\frac{d}{2}} (\nabla\sqrt{\rho}) \cdot \mathrm{d}\xi^\delta - \varepsilon^{\frac{d}{2}} \sqrt{\rho}(\nabla \cdot \mathrm{d}\xi^\delta) \\ &= \Delta\rho - \frac{\varepsilon^{\frac{d}{2}}}{2\sqrt{\rho}} \nabla\rho \cdot \mathrm{d}\xi^\delta - \varepsilon^{\frac{d}{2}} \sqrt{\rho}(\nabla \cdot \mathrm{d}\xi^\delta).\end{aligned}$$

Parabolicity and stochastic transport: for a Brownian motion B_t and $a, b \in \mathbb{R}$,

$$\partial_t v = a\Delta v + b\nabla v \cdot \mathrm{d}B_t.$$

If $\tilde{v}$ solves

$$\partial_t \tilde{v} = \big(a - \frac{b^2}{2}\big)\Delta\tilde{v} \quad \text{then} \quad v(x,t) = \tilde{v}(x + bB_t, t).$$

Parabolic and well-posed if and only if $\frac{b^2}{2} < a$.

I. Fluctuating hydrodynamics

The noise: if ξ^δ takes the form

$$\xi^\delta = \sum_{k=1}^{\infty} f_k(x) B_t^k \ \text{ then } \ \langle \xi^\delta \rangle_1 = \sum_{k=1}^{\infty} f_k^2.$$

Probabilistic stationarity: we assume that

$$\langle \xi^\delta \rangle_1 \ \text{ is spatially constant with } \ \langle \nabla \cdot \xi^d \rangle_1 = \sum_{k=1}^{\infty} |\nabla f_k|^2 < \infty.$$

The Dean–Kawasaki equation: analyzing the parabolicity yields that

$$\partial_t \rho = \Delta \rho - \frac{\varepsilon^{\frac{d}{2}}}{2\sqrt{\rho}} \nabla \rho \cdot \mathrm{d}\xi^\delta - \varepsilon^{\frac{d}{2}} \sqrt{\rho} (\nabla \cdot \mathrm{d}\xi^\delta)$$

$$\text{``} = \Delta\left(\rho - \frac{\varepsilon \langle \xi^\delta \rangle_1}{8} \log(\rho)\right) - \varepsilon^{\frac{d}{2}} \sqrt{\rho} (\nabla \cdot \mathrm{d}\xi^\delta)\text{''}$$

The corrected equation: for $\theta \in [0, \infty)$, we consider the corrected equation

$$\partial_t \rho = \Delta \rho - \nabla \cdot (\sqrt{\rho}\, \mathrm{d}\xi^\delta) + \frac{\theta \langle \xi^\delta \rangle_1}{4} \Delta \log(\rho).$$

Itô integration is $\theta = 0$, Stratonovich is $\theta = 1/2$, and Kilmontovich is $\theta = 1$.

I. Fluctuating hydrodynamics

The Dean–Kawasaki equation with correlated Stratonovich noise:

$$\partial_t \rho = \Delta\rho - \nabla\cdot(\sqrt{\rho}\,\mathrm{d}\xi^\delta) + \frac{\langle \xi^\delta\rangle_1}{8}\Delta\log(\rho).$$

The entropy formulation: for a convex S and a nonnegative ψ,

$$\int_{\mathbb{T}^d} S(\rho)\psi\Big|_{s=0}^{s=t} \leq -\int_0^t\int_{\mathbb{T}^d}(\nabla\rho + \frac{\langle\xi^\delta\rangle_1}{8}\nabla\log(\rho))\cdot S'(\rho)\nabla\psi - \int_0^t\int_{\mathbb{T}^d} S''(\rho)\psi|\nabla\rho|^2$$
$$-\int_0^t\int_{\mathbb{T}^d} S'(\rho)\psi\nabla\cdot(\sqrt{\rho}\,\mathrm{d}\xi^\delta) + \frac{1}{2}\int_0^t\int_{\mathbb{T}^d}\langle\nabla\cdot\xi^\delta\rangle_1\rho S''(\rho)\psi.$$

The kinetic formulation: for $\eta\in\mathbb{R}$ and a nonnegative measure q,

$$\chi(x,t,\eta) = \mathbf{1}_{\{0<\eta<\rho(x,t)\}} - \mathbf{1}_{\{\rho(x,t)<\eta<0\}} \text{ and } \delta_\rho|\nabla\rho|^2 \leq q,$$

for $\delta_\rho = \delta_0(\eta-\rho)$, we have the distributional equalities

$$\nabla\chi = \delta_0(\eta-\rho)\nabla\rho \quad\text{and}\quad \partial_\eta\chi = \delta_0 - \delta_\rho \quad\text{and}\quad \int_{\mathbb{R}}\chi S'(\eta) = S(\rho),$$

and we have

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi S'\psi\Big|_{s=0}^{s=t} = -\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}(1+\frac{\langle\xi^\delta\rangle_1}{8\eta})\nabla\chi\cdot\nabla_x(S'\psi) - \int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}\partial_\eta(S'\psi)\,\mathrm{d}q$$
$$-\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}} S'(\rho)\psi\delta_\rho\nabla\cdot(\sqrt{\rho}\,\mathrm{d}\xi^\delta) + \frac{1}{2}\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}\langle\nabla\cdot\xi^\delta\rangle_1\eta\delta_\rho\partial_\eta(S'\psi).$$

Renormalized kinetic solutions [F., Gess; 2024]

Let $\rho_0 \in L^1(\Omega; L^1(\mathbb{T}^d))$ be nonnegative and $\mathcal{F}_0$-measurable. A *renormalized kinetic solution* is a nonnegative, $\mathbb{P}$-a.s. continuous $L^1(\mathbb{T}^d)$-valued, and $\mathcal{F}_t$-predictable function $\rho \in L^\infty(\Omega; L^1(\mathbb{T}^d \times [0,T])$ that satisfies:

— *Preservation of mass*: $\mathbb{P}$-a.s. for every time t, $\|\rho(x,t)\|_{L^1(\mathbb{T}^d)} = \|\rho_0\|_{L^1(\mathbb{T}^d)}$.

— *Local regularity*: $\mathbb{P}$-a.s. for every $K \in [1,\infty)$,

$$\left((\rho \wedge K) \vee K^{-1}\right) \in L^2([0,T]; H^1(\mathbb{T}^d)).$$

Furthermore, there exists a nonnegative parabolic defect measure q that satisfies:

— *Regularity*: $\mathbb{P}$-a.s. in the sense of measures, $\delta_\rho |\nabla \rho|^2 \le q$.

— *Vanishing at infinity*: $\mathbb{P}$-a.s. $\liminf_{M\to\infty} q(\mathbb{T}^d \times [0,T] \times [M, M+1]) = 0$.

— *The equation*: $\mathbb{P}$-a.s. for every $\psi \in \mathrm{C}_c^\infty(\mathbb{T}^d \times (0,\infty))$,

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}} \chi\psi\Big|_{s=0}^{s=t} = -\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}(1 + \frac{\langle \xi^\delta\rangle_1}{8\eta})\nabla\chi\cdot\nabla_x\psi - \int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}\partial_\eta\psi\,\mathrm{d}q$$
$$-\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}\psi\delta_\rho\nabla\cdot(\sqrt{\rho}\,\mathrm{d}\xi^\delta) + \frac{1}{2}\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}\partial_\eta\psi\langle\nabla\cdot\xi^\delta\rangle_1\eta\delta_\rho.$$

A useful equality: for two solutions ρ_1 and ρ_2,

$$\begin{aligned}\int_{\mathbb{T}^d} |\rho_1 - \rho_2| = \int_{\mathbb{T}^d}\int_{\mathbb{R}} |\chi_1 - \chi_2|^2 = \int_{\mathbb{T}^d}\int_{\mathbb{R}} |\chi_1| + |\chi_2| - 2\chi_1\chi_2 \\ = \int_{\mathbb{T}^d}\int_{\mathbb{R}} \operatorname{sgn}(\eta)\chi_1 + \operatorname{sgn}(\eta)\chi_2 - 2\chi_1\chi_2.\end{aligned}$$

The stochastic integral: since

$$\partial_t \rho_i = \Delta \rho_i - \nabla \cdot (\sqrt{\rho_i}\,\mathrm{d}\xi^\delta) + \frac{\langle \xi^\delta \rangle_1}{8} \Delta \log(\rho_i),$$

formally differentiating the above,

$$\begin{aligned}\partial_t \Big(\int_{\mathbb{T}^d} |\rho_1 - \rho_2| \Big) &= \ldots - \int_{\mathbb{T}^d} \operatorname{sgn}(\rho_1 - \rho_2) \nabla \cdot \big((\sqrt{\rho_1} - \sqrt{\rho_2})\,\mathrm{d}\xi^\delta \big) + \ldots \\ &= \ldots + \int_{\mathbb{T}^d} (\sqrt{\rho_1} - \sqrt{\rho_2}) \delta_0(\rho_1 - \rho_2) \nabla(\rho_1 - \rho_2) \cdot \mathrm{d}\xi^\delta + \ldots \\ &\text{“} \simeq \ldots + \int_{\mathbb{T}^d} (\sqrt{\rho_1} - \sqrt{\rho_2}) \delta_0^\kappa(\rho_1 - \rho_2) \nabla(\rho_1 - \rho_2) \cdot \mathrm{d}\xi^\delta + \ldots \text{”} \\ &\text{“} \simeq \ldots + \int_{\mathbb{T}^d} \kappa^{\frac{1}{2}} \cdot \kappa^{-1} \mathbf{1}_{\{0 < |\rho_1 - \rho_2| < \kappa\}} \nabla(\rho_1 - \rho_2) \cdot \mathrm{d}\xi^\delta + \ldots \text{”}\end{aligned}$$

Renormalized kinetic solutions [F. Gess; 2024]

Let $\rho_0 \in L^1(\Omega; L^1(\mathbb{T}^d))$ be nonnegative and $\mathcal{F}_0$-measurable. Then, there exists a unique renormalized kinetic solution of the equation

$$\partial_t \rho = \Delta\rho - \nabla \cdot (\sqrt{\rho}\, \mathrm{d}\xi^\delta) + \frac{\langle \xi^\delta \rangle_1}{8} \Delta \log(\rho),$$

with initial data ρ_0. Furthermore, two solutions ρ_1 and ρ_2 satisfy $\mathbb{P}$-a.s. that

$$\max_{t\in[0,T]} \|\rho_1 - \rho_t\|_{L^1(\mathbb{T}^d)} \leq \|\rho_{1,0} - \rho_{2,0}\|_{L^1(\mathbb{T}^d)}.$$

Extensions: general equations of the type

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot \big(\sigma(\rho) \circ \xi^\delta + \nu(\rho)\big) + \lambda(\rho) + \phi(\rho)\xi^\delta,$$

including $\Phi(\rho) = \rho^m$ for every $m \in (0, \infty)$, $\sigma(\rho) = \rho^{\frac{m}{2}}$, and $\phi(\rho) = \sqrt{\rho}$.

Flux through zero: if ρ is a nonnegative solution of the heat equation,

$$\lim_{\beta\to 0} \int_0^t \int_{\mathbb{T}^d} \beta^{-1} |\nabla\rho|^2 \mathbf{1}_{\{\beta/2<\rho<\beta\}} = 0.$$

I. Fluctuating hydrodynamics

Random dynamical system [F., Gess, Gvalani; 2024]

Let $\rho(t,\rho_0,\omega)$ denote the solution

$$\partial_t\rho = \Delta\rho - \varepsilon^{\frac{d}{2}}\nabla\cdot(\sqrt{\rho}\,\mathrm{d}\xi^\delta(\omega)) + \frac{\varepsilon^d\langle\xi^\delta\rangle_1}{8}\Delta\log(\rho)\ \text{ in }\ \mathbb{T}^d\times(t,\infty),$$

with $\rho(t,\rho_0,\omega)(x,t)=\rho_0$. The solutions $\rho(t,\rho_0,\omega)$ generate a random dynamical system on with respect to the usual Borel σ-algebra on the space of nonnegative, integrable functions on $\mathbb{T}^d$.

— the pathwise contraction of solutions controls zero sets

— new estimates for the initial time continuity of the flow

Invariant measures [F., Gess, Gvalani; 2024]

The Markov process $\rho(t,\cdot,\cdot)$ is uniquely ergodic: it has a unique invariant probability measure and is strongly mixing.

— the entropy dissipation estimate:

$$\mathbb{E}\Big[\max_{t\in[0,T]}\int_{\mathbb{T}^d}\rho\log(\rho)\Big]+\mathbb{E}\Big[\int_0^T\int_{\mathbb{T}^d}|\nabla\sqrt{\rho}|^2\Big]\leq\mathbb{E}\Big[\int_{\mathbb{T}^d}\rho_0\log(\rho_0)\Big]+c\big(\langle\xi^\delta\rangle_1+T\langle\nabla\cdot\xi^\delta\rangle_1\big).$$

Quantitative LLN and Central Limit Theorem [Clini, F.; 2025]

The scaling limit: let $\delta(\varepsilon)$ be any sequence satisfying, as $\varepsilon \to 0$,

$$\varepsilon\delta(\varepsilon)^{-(1+2/d)} \to 0 \text{ and } \delta(\varepsilon) \to 0,$$

and for every $\varepsilon \in (0,1)$ let ρ^ε be the solution

$$\partial_t \rho^\varepsilon = \Delta\Phi(\rho^\varepsilon) - \varepsilon^{\frac{d}{2}} \nabla \cdot (\Phi^{\frac{1}{2}}(\rho^\varepsilon) \circ \xi^{\delta(\varepsilon)}).$$

Law of Large Numbers: then almost surely with a quantitative rate, as $\varepsilon \to 0$,

$$\rho^\varepsilon \to \overline{\rho} \text{ for } \partial_t \overline{\rho} = \Delta\Phi(\overline{\rho}).$$

Central Limit Theorem: then as distributions, as $\varepsilon \to 0$,

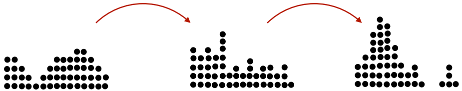
$$\varepsilon^{-\frac{d}{2}}(\rho^\varepsilon - \overline{\rho}) \to v \text{ for } \partial_t v = \Delta(\Phi'(\rho)v) - \nabla \cdot (\Phi^{\frac{1}{2}}(\overline{\rho})\xi).$$

The central limit theorem is a statement about "universal" behavior:

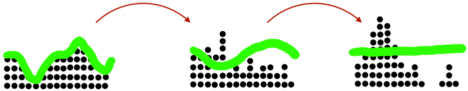
$$\partial_t \tilde{\rho}^\varepsilon = \Delta\Phi(\tilde{\rho}^\varepsilon) - \varepsilon^{\frac{d}{2}} \nabla \cdot (\Phi^{\frac{1}{2}}(\overline{\rho})\xi^{\delta(\varepsilon)}).$$

II. Macroscopic fluctuation theory

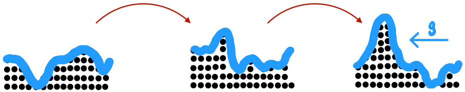
The zero range process: μ^N on $\mathbb{T}^1 \times [0,T]$ for $N = 15$ and $T(k) \sim ke^{-kt}$,



The heat equation: the hydrodynamic limit $\partial_t \overline{p} = \Delta \overline{\rho}$,



The skeleton equation: the controlled equation $\partial_t \rho = \Delta \rho - \nabla \cdot (\sqrt{\rho} \cdot g)$,



The rate function: we have $\mathbb{P}(\mu^N \simeq \rho) \simeq \exp(-N^d I(\rho))$ for

$$I(\rho) = \frac{1}{2} \inf\{\|g\|^2_{L^2_{t,x}} : \partial_t \rho = \Delta \rho - \nabla \cdot (\sqrt{\rho} g)\}.$$

The hydrodynamic limit: for the parabolically rescaled, mean zero particle process μ_t^N on $\mathbb{T}_N^d$, as $N \to \infty$, for $J(\overline{\rho}) = \nabla\sigma(\overline{\rho})$,

$$\mu_t^N \rightharpoonup \overline{\rho}\,\mathrm{d}x \ \text{ for } \ \partial_t\overline{\rho} = \Delta\sigma(\overline{\rho}) = \nabla\cdot J(\overline{\rho}).$$

Macroscopic fluctuation theory: the probability of observing a space-time fluctuation (ρ, j) satisfying

$$\partial_t\rho = \nabla\cdot j,$$

satisfies the large deviations bound [Bertini et al.; 2014]

$$\mathbb{P}[\mu^N \simeq \rho] \simeq \exp\left(-N^d I(\rho)\right) \ \text{ for } \ I(\rho) = \int_0^T\int_{\mathbb{T}^d}(j - J(\rho))\cdot m(\rho)^{-1}(j - J(\rho)).$$

The skeleton equation: if $(j - J(\rho)) = \sqrt{m(\rho)}g$ then $I(\rho) = \int_0^T\int_{\mathbb{T}^d}|g|^2$ and

$$\partial_t\rho = \nabla\cdot\left(J(\rho) + (j - J(\rho))\right) = \Delta\sigma(\rho) - \nabla\cdot(\sqrt{m(\rho)}g).$$

The zero range process: we have that, for $\Phi(\rho) = \rho^m$ for any $m \in [1,\infty)$,

$$\partial_t\rho = \Delta\Phi(\rho) - \nabla\cdot(\Phi^{\frac{1}{2}}(\rho)g).$$

The exclusion process: we have that

$$\partial_t\rho = \Delta\rho - \nabla\cdot(\sqrt{\rho(1-\rho)}g).$$

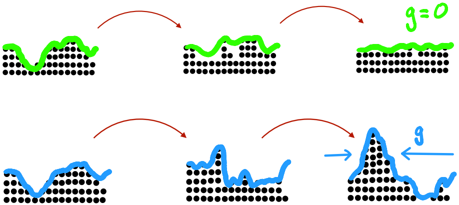
The large deviations rate function: for every $m \in L^1(\mathbb{T}^d \times [0,T])$,

$$I(m) = \frac{1}{2} \inf \left\{ \|g\|^2_{L^2(\mathbb{T}^d \times [0,T])^d} : \partial_t m = \Delta m - \nabla \cdot \left(\sqrt{m} g\right) \right\}.$$

Large Deviations Principle [Benois, Kipnis, Landim; 1995], [F., Gess; 2023]

For every measurable $A \subseteq L^1([0,T]; L^1(\mathbb{T}^d))$,

$$\mathbb{P}\left[\mu^N \in A\right] \simeq e^{-N^d \left(\inf_{\pi \in A} I(\pi) \right)}.$$



The Large Deviations Principle [F., Gess; 2024]

The scaling limit: let $\delta(\varepsilon)$ be any sequence satisfying, as $\varepsilon\delta(\varepsilon)^{-(1+2/d)} \to 0$ and $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, and for every $\varepsilon \in (0,1)$ let ρ_ε be the solution

$$\partial_t \rho_\varepsilon = \Delta\rho_\varepsilon - \varepsilon^{\frac{d}{2}} \nabla \cdot (\sqrt{\rho_\varepsilon}\, \mathrm{d}\xi^{\delta(\varepsilon)}) + \frac{\varepsilon^d \langle \xi^\delta \rangle_1}{8} \Delta \log(\rho_\varepsilon).$$

The large deviations principle: the solutions ρ_ε satisfy a large deviations principle in $L^1_{t,x}$ and $\mathrm{C}([0,T];\mathcal{M})$ with rate function

$$I(\rho) = \frac{1}{2} \inf \left\{ \|g\|^2_{L^2} : \partial_t \rho = \Delta\rho - \nabla \cdot (\sqrt{\rho} g) \right\}.$$

The linear fluctuating hydrodynamics: the linear fluctuating hydrodynamics

$$\partial_t \tilde{\rho}_\varepsilon = \Delta \tilde{\rho}_\varepsilon - \sqrt{\varepsilon} \nabla \cdot (\sqrt{\overline{\rho}}\, \mathrm{d}\xi^{\delta(\varepsilon)}),$$

for the hydrodynamics limit $\partial_t \overline{\rho} = \Delta\Phi(\overline{\rho})$ satisfy an LDP with rate function

$$\tilde{I}(\rho) = \frac{1}{2} \inf \left\{ \|g\|^2_{L^2} : \partial_t \rho = \Delta\rho - \nabla \cdot (\sqrt{\overline{\rho}} g) \right\}.$$

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