Honour Moderations: Linear Algebra
Problem Sheet 2
(To be done in Fourth Week)

Michaelmas Term 2005

1. If A and B are two matrices, we say that A and B *commute* if $AB = BA$. Now let A be a 2×2 matrix.

(a) Show that A commutes with $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ if and only if A is diagonal.

(b) Show that A commutes with $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ if and only if A is diagonal.

(c) Which 2×2 matrices A commute with $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$?

(d) Deduce that A commutes with *all* 2×2 matrices if and only if $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ for some scalar λ.

2. (a) In each of the following cases, **either** give a careful proof that V is a vector space over $\mathbb{R}$, **or** give a reason why it is not:

(a) V is the set of all polynomials over $\mathbb{R}$ which have a non-zero constant term, with the usual addition of polynomials and the usual scalar multiplication.

(b) V is the set of all functions $f : X \to \mathbb{R}$ (for some fixed set X), and if $f, g \in V$, $\alpha \in \mathbb{R}$, then the functions $f + g$, αf are defined by setting

$$(f + g)(x) = f(x) + g(x), \quad (\alpha f)(x) = \alpha f(x).$$

(c) V is the set of all symmetric $n \times n$ matrices over $\mathbb{R}$.

(d) V is the set of all skew-symmetric $n \times n$ matrices over $\mathbb{R}$.

(e) V is the set of all invertible $n \times n$ matrices over $\mathbb{R}$ (that is, the set of all matrices A such that A^{-1} exists).

3. Determine which of the following subsets of $\mathbb{R}^n$ are subspaces:

(a) All vectors $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ such that $x_1 = 1$;

(b) All vectors $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ such that $x_1 + 2x_2 = 0$;

(c) All vectors $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ such that $x_1 + x_2 + \cdots + x_n = 1$;

(d) All vectors $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ such that $x_1^2 = x_2$.

4. If A is a real $m \times n$ matrix, prove that the solutions of the system $A\mathbf{x} = \mathbf{0}$ form a subspace of $\mathbb{R}^n$.

5. Let $V = \mathbb{R}[x]$, the vector space of all real polynomials in one variable x. Determine whether or not U is a subspace of V when:

(a) U consists of all polynomials with degree $\geq k$ for fixed k, together with the zero polynomial;

(b) U consists of all polynomials with only even powers of x;

(c) U consists of all polynomials with integral coefficients.

Optional:
6. Let ω be a complex cube root of 1 ($\omega \neq 1$). Prove that $1 + \omega + \omega^2 = 0$. Letting A be the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

find A^2 and A^{-1}.

G.A.S.