Honour Moderations: Linear Algebra
Problem Sheet 3
(To be done in Fifth Week)

Michaelmas Term 2005

1. (a) Is it true that if u, v and w are linearly independent vectors in $\mathbb{R}^n$, then so are $u+v$, $v+w$ and $w+u$?

   (b) Show that the functions $f(t) = \sin(t)$, $g(t) = \cos(t)$ and $h(t) = t$ are linearly independent in the vector space V of all real-valued functions on $\mathbb{R}$.

2. Let $\mathbb{R}_2[x]$ be the vector space of all real polynomials of degree at most 2.

   (a) Do $f(x) = 1 + 2x + x^2$ and $g(x) = 2 + x^2$ span $\mathbb{R}_2[x]$?

   (b) Determine which of the following subsets of $\mathbb{R}_2[x]$ are linearly dependent. For those that are, express one vector as a linear combination of the others.

   (i) $\{x,\ 3 + x^2,\ x + 2x^2\}$;

   (ii) $\{-2 + x,\ 3 + x,\ 1 + x^2\}$;

   (iii) $\{-5 + x + 3x^2,\ 13 + x,\ 1 + x + 2x^2\}$.

3. Let S and T be non-empty subsets of the vector space V such that $S \subseteq T$. Prove that

   (a) If T is linearly independent then so is S;

   (b) If S is linearly dependent then so is T.

4. Defining $M_{n\times n}(\mathbb{R})$ to be the set of all $n \times n$ real matrices, compute the dimensions of the following vector spaces:

   (a) the subspace of $M_{n\times n}(\mathbb{R})$ consisting of all diagonal matrices (a matrix $\{a_{i,j}\}$ is *diagonal* if $a_{i,j} = 0$ for $i \neq j$);

   (b) the subspace of $M_{n\times n}(\mathbb{R})$ consisting of all matrices of zero trace (that is, where the sum of the diagonal entries is zero).

5. Show that

   (i) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ form a basis for $M_{2\times 2}(\mathbb{R})$.

   (ii) $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ form a basis of $\mathbb{R}^3$.

   (iii) $1,\ 1+x,\ 1+x+x^2,\ \ldots,\ 1+x+\cdots+x^n$ form a basis for $\mathbb{R}_n[x]$, the polynomials of degree at most n in one variable x.

Optional:
6. Let S be a non-empty subset of the vector space V, and define

$$\mathrm{Sp}(S) \;=\; \{v \,|\, v = \Sigma_{i=1}^{k} a_i v_i \text{for some } a_i \in \mathbb{R} \text{ and } v_i \in S\}.$$

Show that

(a) $\mathrm{Sp}(S)$ is a subspace of V and $S \subseteq \mathrm{Sp}(S)$;

(b) If W is a subspace of V such that $S \subseteq W$, then $\mathrm{Sp}(S) \subseteq W$.

Let S and T be subsets of V. Which of the following statements are true? Give reasons.

(c) $\mathrm{Sp}(S \cap T) = \mathrm{Sp}(S) \cap \mathrm{Sp}(T)$;

(d) $\mathrm{Sp}(S \cup T) = \mathrm{Sp}(S) \cup \mathrm{Sp}(T)$;

(c) $\mathrm{Sp}(S \cup T) = \mathrm{Sp}(S) + \mathrm{Sp}(T)$.

[Here $S + T \;=\; \{s + t \,|\, s \in S, t \in T\}$.]

G.A.S.