Honour Moderations: Linear Algebra
Problem Sheet 4
(To be done in Sixth Week)

Michaelmas Term 2005

1. For each of the following statements about subspaces X, Y, Z of a vector space V **either** give a proof of the statement, **or** find a counterexample. [$\mathbb{R}^2$ and $\mathbb{R}^3$ will provide all the counterexamples required.]

(a) $V \setminus X$ is never a subspace of V;

(b) $(X \cap Y) + (X \cap Z) = X \cap (Y + Z)$;

(c) $(X + Y) \cap (X + Z) = X + (Y \cap Z)$;

(d) if $Y \subseteq X$, then $Y + (X \cap Z) = X \cap (Y + Z)$.

2. Let $V = M_{2\times 2}(\mathbb{R})$, the set of all 2×2 real matrices, and let

$$U = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| \text{ for all } a, b \in \mathbb{R} \right\}, \quad W = \left\{ \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \middle| \text{ for all } a, c \in \mathbb{R} \right\}.$$

Describe $U + W$ and $U \cap W$. Find bases for the subspaces U, W, $U + W$ and $U \cap W$ and hence find the dimensions of these four subspaces.

3. Let X and Y be subspaces of a finite-dimensional vector space V.

(a) Show that if X is a subspace of Y and if $\dim X = \dim Y$, then $X = Y$.

(b) Suppose that $\dim V = \dim X + \dim Y$. Show that $V = X + Y$ if and only if $X \cap Y = \{0\}$.

(c) Suppose that S and T are bases for X and Y respectively, and that $V = X + Y$, $X \cap Y = \{0\}$. Show that $S \cup T$ is a basis for V. [When $V = X + Y$ and $X \cap Y = \{0\}$, then we say that V is the *direct sum* of X and Y, and write $V = X \oplus Y$.]

(d) Show that $M_{n\times n}(\mathbb{R})$ has a basis with the property that each matrix in the basis is either symmetric or skew-symmetric.

4. Let V be a vector space of dimension n over $\mathbb{R}$. Prove that for each r such that $0 \leq r \leq n$, V contains a subspace of dimension r.

5. (a) Decide which of the following mappings $T : \mathbb{R}^3 \to \mathbb{R}^3$ are linear transformations:

(i) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z \\ 0 \end{pmatrix}$;

(ii) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} |x| \\ -z \\ 0 \end{pmatrix}$;

(iii) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-1 \\ x \\ y \end{pmatrix}$;

(iv) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ y-2 \\ 4y \end{pmatrix}$.

(b) Let $V = M_{n\times n}(\mathbb{R})$, and let X be an arbitrary but fixed matrix in V. Define $T : V \to V$ by $T(A) = AX + XA$, where A is any matrix in V. Show that T is a linear transformation of V.

Optional:
6. Let V be a vector space over $\mathbb{R}$ and U and W be subspaces of V. Define

$$U + W = \{v \in V \mid v = u + w \text{ for some } u \in U \text{ and } w \in W\}.$$

Show that $U+W$ is a subspace of V. In the case where V has dimension n ($n \geq 2$) and U and W have dimension $n-1$, show that **either** $U = W$ **or** $\dim(U \cap W) = n-2$.

G.A.S.