Honour Moderations: Linear Algebra
Problem Sheet 7 (Vacation Sheet)

Michaelmas Term 2005

1. Find the row rank and the column rank of

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

2. Show that if the $n \times n$ matrix A with real entries is in reduced echelon form, then **either** A has at least one zero row **or** $A = I_n$, the $n \times n$ identity matrix.

3. Let U be the space of all 3×3 symmetric matrices, and W the space of all 3×3 lower triangular matrices (that is, matrices of the form $C = (c_{ij})$ with $c_{ij} = 0$ unless $i \geq j$). What is $U \cap W$? Find a basis for $U + W$ which contains a basis for U and a basis for W.

4. Let V and W be finite-dimension real vector spaces, and $T : V \to W$ be a linear transformation. Define the *kernel*, $\ker(T)$, and the *image*, $\operatorname{Im}(T)$, of T.
Prove that T is one-to-one if and only if $\ker(T) = \{0\}$.
Suppose that $\dim V = \dim W$. Prove that T maps V onto W if and only if T is one-to-one.

5. Let A be an element of $M_{n\times n}(\mathbb{R})$, the $n \times n$ matrices with real entries. Prove that there is a polynomial $f(t) = a_r t^r + \cdots + a_1 t + a_0$, where $a_i \in \mathbb{R}$, which has A as a root and is not identically zero; that is,

$$a_r A^r + a_{r-1} A^{r-1} + \cdots + a_1 A + a_0 = 0.$$

[**Hint:** consider the matrices $I,\ A,\ A^2, \ldots$ in $M_{n\times n}(\mathbb{R})$.]

6. Let A be an $m \times n$ matrix in $M_{m\times n}(\mathbb{R})$. Prove that

(i) the space of solutions of the system $A\mathbf{x} = \mathbf{0}$ has dimension at least $n - m$;

(ii) the equation $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} \in \mathbb{R}^n$ if and only if $\operatorname{rank} A = \operatorname{rank}[A \mid \mathbf{b}]$, where $[A \mid \mathbf{b}]$ denotes the augmented matrix with $\mathbf{b}$ as the final column.

7. (Mods 2001) Consider the system of equations

$$\begin{array}{rcrcrcrcl} x & + & 2y & + & 4z & + & 3t & = & a \\ 3x & + & y & + & 2z & + & 4t & = & b \\ -x & + & 3y & + & \alpha z & + & 2t & = & c \end{array}$$

where a, b and c are real numbers. Determine the values of α for which the equations have infinitely many solutions, and find the value of α for which there is no solution unless a, b and c satisfy a certain relation which you should give. If $\alpha = 6$, $a = 2$, $b = -1$ and $c = 5$, verify that this relation is satisfied and find the most general solution.

8. (Mods 2004) Let V and W be vector spaces over $\mathbb{R}$, let $T : V \to V$ be a linear transformation such that $T^2 = T$, and let I be the identity transformation on V. Prove that

(i) $\ker T = \operatorname{Im}(I - T)$, and $\ker(I - T) = \operatorname{Im} T$;

(ii) $\ker T \cap \operatorname{Im} T = \{0\}$;

(iii) every $v \in V$ can be uniquely expressed in the form $v = v_1 + v_2$, where $v_1 \in \ker T$ and $v_2 \in \operatorname{Im} T$.

G.A.S.