ANALYSIS I

9 The Cauchy Criterion

9.1 Cauchy's insight

Our difficulty in proving "$a_n \to \ell$" is this: What is ℓ? Cauchy saw that it was enough to show that if the terms of the sequence got sufficiently close to each other. then completeness will guarantee convergence.

Remark. *In fact Cauchy's insight would let us* construct $\mathbb{R}$ *out of* $\mathbb{Q}$ *if we had time.*

9.2 Definition

Let (a_n) be a sequence [$\mathbb{R}$ or $\mathbb{C}$]. We say that (a_n) is a **Cauchy sequence** if, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$m, n \geqslant N \Longrightarrow |a_m - a_n| < \varepsilon.$$

[Is that all? Yes, it is!]

9.3 Cauchy $\Longrightarrow$ Bounded

Theorem. *Every Cauchy sequence is bounded* [$\mathbb{R}$ *or* $\mathbb{C}$].

Proof. $1 > 0$ so there exists N such that $m, n \geqslant N \Longrightarrow |a_m - a_n| < 1$. So for $m \geqslant N$, $|a_m| \leqslant 1 + |a_N|$ by the Δ law. So for all m

$$|a_m| \leqslant 1 + |a_1| + |a_2| + \cdots + |a_N|.$$

□

9.4 Convergent $\Longrightarrow$ Cauchy [$\mathbb{R}$ or $\mathbb{C}$]

Theorem. *Every convergent sequence is Cauchy.*

Proof. Let $a_n \to l$ and let $\varepsilon > 0$. Then there exists N such that

$$k \geqslant N \Longrightarrow |a_k - l| < \varepsilon/2$$

For $m, n \geqslant N$ we have

$$\begin{aligned} |a_m - l| &< \varepsilon/2 \\ |a_n - l| &< \varepsilon/2 \end{aligned}$$

So

$$\begin{aligned} |a_m - a_n| &\leqslant |a_m - l| + |a_n - l| && \text{by the } \Delta \text{ law} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

□

9.5 Cauchy $\implies$ Convergent $[\mathbb{R}]$

Theorem. *Every real Cauchy sequence is convergent.*

Proof. Let the sequence be (a_n). By the above, (a_n) is bounded. By Bolzano-Weierstrass (a_n) has a convergent subsequence $(a_{n_k}) \to l$, say. So let $\varepsilon > 0$. Then

$$\begin{array}{lllll} \exists N_1 & \text{such that} & r \geqslant N_1 & \implies & |a_{n_r} - l| & < \varepsilon/2 \\ \exists N_2 & \text{such that} & m, n \geqslant N_2 & \implies & |a_m - a_n| & < \varepsilon/2 \end{array}$$

Put $s := \min\{r | n_r \geqslant N_2\}$ and put $N = n_s$. Then

$$\begin{array}{rcccccc} m, n \geqslant N & \implies & |a_m - a_n| & & & & \\ & \leqslant & |a_m - a_{n_s}| & + & |a_{n_s} - l| & & \\ & < & \varepsilon/2 & + & \varepsilon/2 & = \varepsilon \end{array}$$

□

9.6 Cauchy $\implies$ Convergent $[\mathbb{C}]$

Theorem. *Every complex Cauchy sequence is convergent.*

Proof. Put $z_n = x + \mathrm{i}y$. Then x_n is Cauchy: $|x_x - x_m| \leqslant |z_n - z_m|$ (as $|\Re w| \leqslant |w|$). So $x_n \to x$, $y_n \to y$ and so $z_n \to x + \mathrm{i}y$. □

9.7 Example

Let

$$a_n = 1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{(-1)^{n+1}}{n}$$

Then with $m \geqslant n$, and $m - n$ odd we have

$$\begin{aligned} |a_m - a_n| &= \left| \overbrace{\frac{1}{n+1} - \frac{1}{n+2}} + \overbrace{\frac{1}{n+3} - \frac{1}{n+4}} + \ldots + \overbrace{\frac{1}{m-1} - \frac{1}{m}} \right| \\ &= \underbrace{\frac{1}{n+1}} - \underbrace{\frac{1}{n+2} + \frac{1}{n+3}} - \ldots - \underbrace{\frac{1}{m-2} + \frac{1}{m-1}} - \underbrace{\frac{1}{m}} \\ &\leqslant \tfrac{1}{n+1} \leqslant \tfrac{1}{n} \end{aligned}$$

If $m - n$ is even, we write

$$|a_m - a_n| = |a_m - a_{m-1} + a_m| \leqslant |a_m - a_{m-1}| + |a_m| \leqslant \frac{1}{N+1} + \frac{1}{m} \leqslant \frac{2}{n}$$

Let $\varepsilon > 0$. Chose $N > \frac{1}{2\varepsilon}$ and convergence follows.
Question: What is $\lim_{n\to\infty} a_n$? Later we'll see it is $\log 2$.

10 Series

10.1 Definition

Note. *So far we have always used sequences defined by functions* $a : \mathbb{N} \to \mathbb{R}$*. It is convenient from now on to start off at* a_0*, that is to work with functions* $a : \mathbb{N} \cup \{0\} \to \mathbb{R}$*.*

So let $(a_n)_{n=0}^{\infty}$ be a sequence of real (complex) numbers. Define $s_m := \sum_{n=0}^{m} a_n$, a well defined real (complex) number.
Consider the sequence $(s_n)_{n=0}^{\infty}$. We call this the series defined by the sequence (a_n), and we denote it by $\sum_{n=0}^{\infty} a_n$.

Note. *This is a very odd name indeed! Don't let it mislead you:* $\sum_{n=0}^{\infty} a_n$ *is just the sequence* $a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots$*.*

10.2 Examples

(i) **The Geometric Series** Let $a_n := x^n$ Then $\sum x^n$ is

$$(1,\ 1+x,\ 1+x+x^2,\ \ldots\ ,\ 1+x+x^2+\cdots+x^n,\ \ldots)$$

(ii) **The Harmonic Series** Let $a_n := \frac{1}{n+1}$. Then $\sum \frac{1}{n+1}$ is

$$\left(1,\ 1+\frac{1}{2},\ 1+\frac{1}{2}+\frac{1}{3},\ \ldots\right)$$

(iii) **The Exponential Series** Let $a_n := x^n/n!$. Then $\sum x^n/n!$ is

$$\left(1,\ 1+x,\ 1+x,\ 1+x+\frac{x^2}{2!},\ \ldots\right)$$

(iv) **The Cosine Series** Let

$$a_n = \begin{cases} \frac{x^{2m}}{2m!}(-1)^m & \text{if } n = 2m \\ 0 & \text{otherwise} \end{cases}$$

Then $\sum a_n$ is

$$\left(1,\ 1,\ 1-\frac{x^2}{2!},\ 1-\frac{x^2}{2!},\ 1-\frac{x^2}{2!}+\frac{x^4}{4!},\ \ldots\right)$$

10.3 Convergence and Divergence

With the setup above if (s_n) converges, then we say "the series $\sum_{n=0}^{\infty}$ is **convergent**". Otherwise we say that the series is **divergent**.

10.4 More examples

(i) Let $a_n = x^n$.

(a) If $x = 1$ then $\sum x^n$ is divergent.

Proof. $s_0 = 1$, $s_1 = 2$, $\ldots$, $s_n = n + 1$ by induction. $\square$

(b) If $x \neq 1$, then $s_n = \frac{1-x^{n+1}}{1-x}$.

Proof. Pure algebra. $\square$

(c) If $|x| < 1$ then $\sum x^n$ is convergent.

(d) If $|x| > 1$ then $\sum x^n$ is divergent.

Proof. If $|x| < 0$ then $x^n \to 0$ as $n \to \infty$. So by AOL $s_n \to 1/(1-x)$. If $s_n \to s$ then $x^{n+1} \to 11 - (1-x)s$ by AOL. But if $|x| > 1$, x^n is not convergent. $\square$

(ii) Let $a_n = \frac{1}{n}$. Then $\sum \frac{1}{n}$ is divergent.

Proof. $s_{2^n - 1} = 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} =$

$$\begin{aligned} &= 1 + \left(\tfrac{1}{2}\right) + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) + \ldots \\ &\geqslant 1 + \tfrac{1}{2} + \tfrac{2}{4} + \tfrac{4}{8} + \ldots + \tfrac{2^{n-1}}{2^n} \\ &\geqslant \tfrac{n}{2} \end{aligned}$$

so (s_n) has a subsequence which is not convergent. $\square$

(iii) Let $a_n = \frac{1}{(n+1)^2}$. Then $\sum \frac{1}{n^2}$ is convergent.

Proof.

$$\begin{aligned} s_n &= 1 + \tfrac{1}{4} + \tfrac{1}{9} + \ldots + \tfrac{1}{n^2} + \tfrac{1}{(n+1)^2} \\ &\leqslant 1 + \tfrac{1}{1\cdot 2} + \tfrac{1}{2\cdot 3} + \ldots + \tfrac{1}{n(n+1)} \\ &= 1 + \left(\tfrac{1}{1} - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \ldots + \left(\tfrac{1}{n} - \tfrac{1}{n+1}\right) \\ &= 2 - \tfrac{1}{n+1} \\ &\leqslant 2 \end{aligned}$$

So (s_n) is monotone increasing $[s_{n+1} = s_n + \frac{1}{(n+1)^2} \geqslant s_n]$ and bounded by 2. So (s_n) is convergent by Bolzano-Weierstrass (**??**). $\square$

10.5 Notation and its abuse

More notation: if the series $\sum_{n=0}^{\infty} a_n$ is convergent then we often denote the limit by $\sum_{n=0}^{\infty} a_n$, and call it the **sum**.

Note. *We must take great care, but this double use is traditional. I will try to distinguish the two uses, and say "The series $\sum a_n$" and "The sum $\sum a_n$". I suggest that you do the same for a bit.*

Note. *Just attaching the label "sum" to the symbol $\sum_{n=0}^{\infty} a_n$ does not turn it into a proper mathematical sum. Look at our axioms for $\mathbb{R}$. They only speak of adding pairs of real numbers, which we can extend using axiom A2 to finite sets of real numbers. But given an infinite set of real numbers we can't simply "add" them and get a "sum". Instead we have to talk about sequences ...*

10.6 Tails

Let (a_n) be a sequence and (s_n) be the corresponding series.
Sometimes we want to look at $(a_k, a_{k+1}, a_{k+2}, \dots)$. We write this series $\sum_{n=k}^{\infty} a_n$. We put $S_n = a_k + a_{k+1} + \cdots + a_n$ and note that what we said about Tails of sequences, and adding constants, ensures that (s_n) is convergent if and only if S_n is convergent.

10.7 Cauchy's criterion

We rewrite Cauchy Criterion for series.

Theorem. *The series $\sum_{n=0}^{\infty} a_n$ is convergent if and only if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$l \geqslant k > N \implies \Bigg| \underbrace{\sum_{n=k}^{l} a_n}_{\substack{\textit{A genuine} \\ \textbf{\textit{sum}}}} \Bigg| < \varepsilon$$

Note. *Clearly in practice when we estimate the sum we'll use the Δ law when we can.*

10.8 Absolute Convergence

Let a_n be a sequence. Then we say that $\sum a_n$ is **absolutely convergent** if the series $\sum |a_n|$ is convergent.

10.9 Absolute Convergence $\implies$ Convergence

Theorem. *If $\sum a_n$ is absolutely convergent then it is convergent.*

Proof. Let $\varepsilon > 0$ Then there exists $N \in \mathbb{N}$ such that

$$l \geqslant k \geqslant N \implies \left| \sum_{k}^{l} |a_n| \right| < \varepsilon$$

So

$$l \geqslant k \geqslant N \Longrightarrow \underbrace{\left|\sum_k^l a_n\right| \leqslant \sum_k^l |a_n|}_{\text{By the } \Delta \text{ law}} = \left|\sum_k^l |a_n|\right| < \varepsilon$$

□

Note. *Why is absolute convergence a good thing to have? Because it makes use of Cauchy criterion easy!*

10.10 Examples

(i) $\sum x^n$ absolutely convergent if $|x| < 1$

(ii) $\sum \frac{(-1)^n}{n^2}$ is absolutely convergent

(iii) $\sum \frac{\sin n}{n^3}$ is absolutely convergent

(iv) $\sum \frac{(-1)^n}{n+1}$ is convergent, but not absolutely convergent.

10.11 Re-arrangements

Let $p : \mathbb{N} \longrightarrow \mathbb{N}$ one-to-one and onto. We can then put $b_n = a_{p(n)}$ and consider $\sum b_n$, which we call a **rearrangement** of the series $\sum a_n$.
Funny this can happen! Later on we will be able to prove that

Example.

$$\begin{array}{lcl} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots & \rightarrow & \log 2 \\ 1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{4} + \dots & \rightarrow & \log(2/3) \end{array}$$

Theorem. *If (a_n) is absolutely convergent and (b_n) is a rearrangement of (a_n) then $\sum b_n$ is absolutely convergent too.*

10.12 Multiplication of series

Theorem. *Suppose $\sum a_n$, $\sum b_n$ are absolutely convergent. Suppose that*

$$c_n := \sum_{r=0}^{n} a_r b_{n-r}$$

Then

(i) $\sum c_n$ *is absolutely convergent*

(ii) $\sum c_n = \sum a_n \sum b_n$

Proof.

(i) By pure algebra

$$\sum_0^N |a_n| \sum_0^N |b_n| \leqslant \sum_0^2 N|c_r| \leqslant \sum_0^{2N} |a_n| \sum_0^{2N} |b_n|$$

So by Sandwich Rule, $\sum |c_n|$ is convergent to $\sum |a_n| \sum |b_n|$.

(ii) Hence $\sum c_n$ is convergent.

(iii) Given $\varepsilon > 0$ there exist N such that

$$l \geqslant k \geqslant N \Longrightarrow \sum_k^l |c_n| < \varepsilon$$

Then for $k \geqslant N$

$$\left| \sum_{n=0}^{2k} c_n - \sum_{n=0}^{k} a_n \sum_{n=0}^{k} b_n \right| = \left| \sum_{r+s \leqslant 2k,\ r \geqslant k \text{ or } s \geqslant k} a_r b_s \right| \leqslant \sum_{ditto} |a_r b_s| \leqslant \sum_k^{2k} |c_n| < \varepsilon$$

□

Note. *Draw a diagram in the (r, s) plane marking out which $a_r b_s$ are included in the various sums.*

10.13 Two Applications

(i)

$$\sum (n+1)x^n = \frac{1}{(1-x)^2}$$

Proof. $a_n := x^n$, $b_n := x^n$, therefore

$$c_n = \sum_{r+s=n} x^r x^s = (n+1)x^n$$

□

(ii)

$$\sum \frac{x^n}{n!} \sum \frac{y^n}{n!} = \sum \frac{(x+y)^n}{n!}$$

Proof. $a_n := \frac{x^n}{n!}$, $b_n := \frac{y^n}{n!}$ therefore

$$c_n = \sum_{r+s=n} \frac{x^r}{r!} \frac{y^s}{s!} = \frac{(x+y)^n}{n!}$$

□