Linear Algebra 2: Direct sums of vector spaces
Thursday 3 November 2005
Lectures for Part A of Oxford FHS in Mathematics and Joint Schools

- Direct sums of vector spaces
- Projection operators
- Idempotent transformations
- Two theorems
- Direct sums and partitions of the identity

Important note: Throughout this lecture $F$ is a field and $V$ is a vector space over $F$. 
Direct sum decompositions, I

Definition: Let $U$, $W$ be subspaces of $V$. Then $V$ is said to be the direct sum of $U$ and $W$, and we write $V = U \oplus W$, if $V = U + W$ and $U \cap W = \{0\}$.

Lemma: Let $U$, $W$ be subspaces of $V$. Then $V = U \oplus W$ if and only if for every $v \in V$ there exist unique vectors $u \in U$ and $w \in W$ such that $v = u + w$.

Proof.
**Projection operators**

Suppose that $V = U \oplus W$. Define $P : V \to V$ as follows. For $v \in V$ write $v = u + w$ where $u \in U$ and $w \in W$: then define $P(v) := u$.

**Observations:**

1. $P$ is well-defined;
2. $P$ is linear;
3. $\text{Im} P = U$, $\text{Ker} P = W$;
4. $P^2 = P$.

**Proofs.**

**Terminology:** $P$ is called the projection of $V$ onto $U$ along $W$. 
Notes on projection operators

Note 1. Suppose that $V$ is finite-dimensional. Choose a basis $u_1, \ldots, u_r$ for $U$ and a basis $w_1, \ldots, w_m$ for $U$. Then the matrix of $P$ with respect to the basis $u_1, \ldots, u_r, w_1, \ldots, w_m$ of $V$ is

$$
\begin{pmatrix}
I_r & 0 \\
0 & 0
\end{pmatrix}.
$$

Note 2. If $P$ is the projection onto $U$ along $W$ then $I - P$ is the projection onto $W$ along $U$. 
Idempotent operators: a theorem

Terminology: An operator $T$ such that $T^2 = T$ is said to be idempotent.

Theorem. Every idempotent operator is a projection operator.

Proof.
A theorem about projections

Theorem. Let $P : V \to V$ be the projection onto $U$ along $W$. Let $T : V \to V$ be a linear transformation. Then $PT = TP$ if and only if $U$ and $W$ are $T$-invariant (that is $TU \subseteq U$ and $TW \subseteq W$).

Proof.
Direct sum decompositions, II

Definition: \( V \) is said to be **direct sum** of subspaces \( U_1, \ldots, U_k \), and we write \( V = U_1 \oplus \cdots \oplus U_k \), if for every \( v \in V \) there exist unique vectors \( u_i \in U_i \) for \( 1 \leq i \leq k \) such that \( v = u_1 + \cdots + u_k \).

Note: \( U_1 \oplus U_2 \oplus \cdots \oplus U_k = (\cdots ((U_1 \oplus U_2) \oplus U_3) \oplus \cdots \oplus U_k) \).

Note: If \( U_i \subseteq V \) for \( 1 \leq i \leq k \) then \( V = U_1 \oplus \cdots \oplus U_k \) if and only if \( V = U_1 + U_2 + \cdots + U_k \) and \( U_r \cap \sum_{i \neq r} U_i = \{0\} \) for \( 1 \leq r \leq k \).

It is **NOT** sufficient that \( U_i \cap U_j = \{0\} \) whenever \( i \neq j \).

Note: If \( V = U_1 \oplus U_2 \oplus \cdots \oplus U_k \) and \( B_i \) is a basis of \( U_i \) then \( B_1 \cup B_2 \cup \cdots \cup B_k \) is a basis of \( V \). In particular, \( \dim V = \sum_{i=1}^{k} \dim U_i \).
Partitions of the identity

Let $P_1, \ldots, P_k$ be linear mappings $V \to V$ such that $P_i^2 = P_i$ for all $i$ and $P_i P_j = 0$ whenever $i \neq j$. If $P_1 + \cdots + P_k = I$ then \{$P_1, \ldots, P_k$\} is known as a partition of the identity on $V$.

Example: If $P$ is a projection then \{$P, I - P$\} is a partition of the identity.

Theorem. Suppose that $V = U_1 \oplus \cdots \oplus U_k$. Let $P_i$ be the projection of $V$ onto $U_i$ along $\bigoplus_{j \neq i} U_j$. Then \{$P_1, \ldots, P_k$\} is a partition of the identity on $V$. Conversely, if \{$P_1, \ldots, P_k$\} is a partition of the identity on $V$ and $U_i := \text{Im} \ P_i$ then $V = U_1 \oplus \cdots \oplus U_k$.

Proof.